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LIGHTFACE Σ^1_2-INDESCRIBABLE CARDINALS

DAVID SCHRITTESSER

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ABSTRACT. Σ^1_3-absoluteness for ccc forcing means that for any ccc forcing P, ${H_{\omega_1}}^V \prec_{\Sigma_2} {H_{\omega_1}}^{V^P}$. "$\omega_1$ inaccessible to reals" means that for any real r, ${\omega_1}^{L[r]} < \omega_1$. To measure the exact consistency strength of "Σ^1_3-absoluteness for ccc forcing and ω_1 is inaccessible to reals", we introduce a weak version of a weakly compact cardinal, namely, a (lightface) Σ^1_2-indescribable cardinal; κ has this property exactly if it is inaccessible and $H_\kappa \prec_{\Sigma_2} H_{\kappa^+}$.

1. INTRODUCTION

The result presented in this paper contributes to the study of principles of generic absoluteness (see [Bag] for a survey of this area). These principles can be seen as generalizations of the bounded forcing axioms, like MA, $BPFA$, $BSPFA$ and BMM (see [Bag00]). Here is the general form of such a principle.

1.1. **Definition.** Let W be a definable subclass of V, Φ a class of formulas with parameters and Γ a class of forcing notions. $\mathcal{A}(W, \Phi, \Gamma)$ is the statement that for any formula ϕ that belongs to Φ and for any $P \in \Gamma$, $W^V \vDash \phi \iff W^{V^P} \vDash \phi$.

We denote $\mathcal{A}(H_{\omega_1}, \mathbf{\Sigma_2}, \Gamma)$ by Σ^1_3-Abs(Γ), pronounced "Σ^1_3-absoluteness (for Γ)", as Σ_2 formulas over H_{ω_1} are equivalent to Σ^1_3 formulas. In this special case, the consistency strengths for various classes Γ are known. See [BF01] and [Fri04] for proofs (in the table, we denote by ccc, *proper*, *semi* $-$ *proper* the obvious classes of forcing notions; *set* denotes the class of all set-sized forcing notions).

Σ^1_3-Abs(ccc)	Σ^1_3-Abs($proper$)	Σ^1_3-Abs($semi-proper$)	Σ^1_3-Abs(set)
ZFC	ZFC	ZFC	reflecting

1.2. **Definition** ([GS95])**.** A regular cardinal κ is *reflecting iff* $V_\kappa \prec_{\Sigma_2} V$, or, equivalently, *iff* for any regular θ and any formula ϕ with parameters from V_κ such that $H_\theta \vDash \phi$, there is a regular $\gamma < \kappa$ such that $H_\gamma \vDash \phi$.

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Forcing axioms are often considered with respect to their interaction with other interesting propositions. For example, knowing how to construct a model of MA, one may ask what is needed to obtain a model of $MA \wedge$ "every projective set of reals is Lebesgue measurable" or $MA \wedge$ "ω_1 is inaccessible to reals" (as in [HS85]).

1.3. **Definition.** We say ω_1 is *inaccessible to reals iff* for any real r, ${\omega_1}^{L[r]} < \omega_1$.

The fact that Σ^1_3-Abs(set) implies that ω_1 is inaccessible to reals (in fact Σ^1_3-absoluteness for ω_1-preserving forcing suffices) is pivotal in setting it apart from the weaker axioms, those that are equiconsistent with ZFC. One can show that Σ^1_3-Abs($proper$) together with the assumption that ω_1 is inaccessible to reals has the full strength of a reflecting cardinal ([Fri04]; see also [Sch04]). The question is: how does the additional assumption that ω_1 is inaccessible to reals interact with those forcing axioms that do not directly imply it? The answer is made available in the following table, where the additional hypothesis is indicated by "Ω". We present a proof of the result on the far left.[1] For this we introduce (in section 3) a large cardinal property, "lightface Σ^1_2-indescribable" (denoted by "lf-Σ^1_2-id" in the table).

Σ^1_3-Abs(ccc) $\wedge\ \Omega$	Σ^1_3-Abs($proper$) $\wedge\ \Omega$	Σ^1_3-Abs($semi-proper$) $\wedge\ \Omega$	Σ^1_3-Abs(set) $\wedge\ \Omega$
lf-Σ^1_2-id	reflecting	reflecting	reflecting

The proof uses (among other things) a coding technique developed in [HS85, theorem C], which we adapt to our needs in section 4. Also, the proof uses a bit of fine structure theory; as we are making an effort to make this paper very accessible, we will review some of the facts we rely on in the next section.

2. Notation, facts, definitions

To define the large cardinal property hinted at, we need a bit of second-order logic. We differentiate second-order from first-order variables or constant symbols by using uppercase for the former and lowercase for the latter. We remind the reader that $\Phi(X_0, \dots X_k)$ is a Σ^1_n-formula if Φ starts with a block of existential quantifiers over second-order variables, with n changes of quantifier, followed by an arbitrary number of first-order quantifiers. Π^1_n means negation of Σ^1_n. Remember that for a first-order formula Φ (mentioning some second-order variables $Y_0, \dots, Y_r, X_0, \dots, X_r$)

$$\langle M, X_0, \dots, X_k\rangle \vDash \exists Y_0 \dots QY_r \Phi(Y_0, \dots, Y_r, X_0, \dots, X_r)$$

(where Q denotes $\exists$ or $\forall$) exactly if

$$\exists Y_0 \in \mathcal{P}(M) \dots QY_r \in \mathcal{P}(M)$$
$$\text{such that } \langle M, X_0, \dots, X_k, Y_0, \dots, Y_r\rangle \vDash \Phi(Y_0, \dots, Y_r, X_0, \dots, X_r).$$

For an elaborate definition, see [Kan03, p. 7ff].

Let $(\phi_i)_{i\in\omega}$ enumerate all Δ_0 formulas. Now we define the value of the i-th Σ_1-Skolem function for L_α (α a limit ordinal), denoted by $h_i^{L_\alpha}$: if $L_\alpha \vDash \exists v \exists w \phi_i(v, w, x)$ we say $h_i^{L_\alpha}(x) = y$ just if (y, z) is the $\leq_L$-least pair such that $L_\alpha \vDash \phi_i(y, z, x)$;

[1] This material was part of the author's thesis [Sch04] and was sketched during a talk at the Summer Workshop in Fine Structure Theory (SWIFT) in Bonn, July 2003, and was the content of a talk given in Münster in February 2004. The author thanks Ralph Schindler for his kind invitation and hospitality.

$h_i^{L_\alpha}(x) = \emptyset$ otherwise. By the Σ_1-Skolem hull of M inside L_α, denoted by $h_{\Sigma_1}^{L_\alpha}(M)$, we mean the least set containing M and closed under all $h_i^{L_\alpha}$. In this case, $h_{\Sigma_1}^{L_\alpha}(M) = \bigcup_{i\in\omega} h_i^{L_\alpha}[M]$. Most importantly, Σ_1-Skolem functions are uniformly Σ_1-definable, i.e. there is a Δ_0 formula Φ such that $h_i^{L_\alpha}(x) = y$ if and only if $L_\alpha \models \exists z \Phi(i,x,y,z)$. We will make use of simple facts about Skolem hulls, such as:

2.1. **Fact.** If $\langle L_\alpha, x\rangle$ is isomorphic to $\langle h_{\Sigma_1}^{L_{\bar\alpha}}(M \cup \{\bar x\}), \bar x\rangle$ and M is transitive, then $\langle L_\alpha, x\rangle = \langle h_{\Sigma_1}^{L_\alpha}(M \cup \{x\}), x\rangle$.

2.2. **Fact.** If $L_\alpha = h_{\Sigma_1}^{L_\alpha}(M)$ and $\sigma : L_\alpha \to_{\Sigma_1} L_\beta$, then $\mathrm{ran}(\sigma) = h_{\Sigma_1}^{L_\beta}(\sigma[M])$.

For details, see [Dev84, II, 6]).

3. Lightface Σ^1_2-indescribable cardinals

3.1. **Definition.** We say that a cardinal κ has the Σ^1_2 reflection property if whenever

$$V_\kappa \models \exists X \forall Y \, \Phi(X,Y,p),$$

where Φ is first order in the language of set theory with X and Y as additional predicates, and $p \in V_\kappa$, then there is $\xi < \kappa$ such that $V_\xi \models \exists X \forall Y \, \Phi(X,Y,p)$. We say κ is (lightface) Σ^1_2-indescribable if in addition κ is inaccessible.

3.2. **Fact.** κ is lightface Σ^1_2-indescribable $\iff$ κ is inaccessible and $H_\kappa \prec_{\Sigma_2} H_{\kappa^+}$.

Proof. First assume indescribability. Let $H_{\kappa^+} \models \exists x \forall y \phi(x,y,p)$, where $p \in H_\kappa$. Pick a witness x_0 in H_{κ^+}. For any transitive $M \in H_\kappa$ containing x_0 and p, we have $M \models \forall y \phi(x_0, y, p)$. So $H_{\kappa^+} \models \exists x \forall y \phi(x,y,p)$ is equivalent to a Σ^1_2 assertion over H_κ, and thus is reflected by some H_ξ, for inaccessible $\xi < \kappa$. Thus $H_{\xi^+} \models \exists x \forall y \phi(x,y,p)$, and as Σ_2 formulas are upward absolute for members of the H-hierarchy,

$$H_\kappa \models \exists x \forall y \phi(x,y,p).$$

For the other direction, let $H_\kappa \models \exists X \forall Y \, \phi(X,Y,p)$. Then H_{κ^+} thinks "there is an ordinal θ such that $\exists x \subseteq V_\theta \forall y \subseteq V_\theta \langle V_\theta, x, y\rangle \models \phi(x,y,p)$". As "$z = V_\theta$" is Π_1 in z and θ, this is seen to be a Σ_2 statement in the parameter p and so holds in H_κ. $\square$

3.3. **Fact.** (1) If κ has the Σ^1_2 reflection property, it is a limit cardinal and is not equal to 2^λ for any $\lambda < \kappa$.

(2) There is a stationary set of Σ^1_2-indescribable cardinals below any Mahlo cardinal. The least Mahlo is not Σ^1_2-indescribable.

(3) Reflecting implies Σ^1_2-indescribable, which in turn implies the existence of many inaccessibles.

(4) If P is a partial ordering of size less than κ, then forcing with P preserves the Σ^1_2-indescribability of κ.

Proof. (1) If $\kappa = \lambda^+$, the Π^1_1 sentence (with λ as a parameter) "there is no function from λ onto the ordinals" holds in V_κ. But this sentence cannot hold in any V_ξ containing λ, $\xi < \kappa$.

If $\kappa = 2^\delta$, look at the sentence "there is a bijection between $\mathcal{P}(\delta)$ and the ordinals". Argue as above.

(2) Let κ be Mahlo. Consider the function that assigns to each ordinal $\eta < \kappa$ the least ξ such that: if Φ is Σ^1_2 with a parameter from V_η and there is $\alpha < \kappa$ such that $V_\alpha \models \Phi$, then there is $\alpha < \xi$ such that $V_\alpha \models \Phi$. Any closure point under

this function has the Σ^1_2 reflection property, and by replacement in V_κ, the closure points under this function form a *cub* subset of κ. This proves the first assertion. To see that the least Mahlo cannot have the reflection property, observe that κ being Mahlo is expressible by a Π^1_1 statement over V_κ.

(3) The first assertion follows from Fact 3.3, as Σ_2 sentences are upward absolute for members of the H-hierarchy. Second, being inaccessible is expressible as a Π^1_1 statement (for example the power set of any set exists and can be mapped injectively into some ordinal, and there is no function with a set as domain but unbounded range).

(4) By Fact 3.2, it suffices to proove $H_\kappa \prec_{\Sigma_2} H_{\kappa^+}$ holds in the extension, assuming it holds in the ground model. Observe that for a partial order P and a regular α such that $P \in H_\alpha$, every element of $(H_\alpha)^{V^P}$ has a P-name in H_α (straightforwardly check, constructing names by induction on the rank; in fact, it suffices to assume that P is a subset of H_κ and has the κ-cc). Recall that for any given Δ_0 formula ϕ and for any transitive ZF^--model M such that $P \in M$, the forcing relation for ϕ on $P \times (P$-names in $M)$ is uniformly Δ_1-definable over M (i.e. the definition is the same for all such M). Thus $\Vdash_P$"$H_{\kappa^+} \vDash \exists x \forall y \phi(y, y, \dot{p})$" is equivalent to a statement of the form

$$\exists \dot{x} \in H_{\kappa^+} \quad \forall \dot{y} \in H_{\kappa^+} \quad H_{\kappa^+} \vDash \phi'(\dot{x}, \dot{y}, \dot{p}, P),$$

where ϕ' is Δ_1. As $H_\kappa \prec_{\Sigma_2} H_{\kappa^+}$, the above holds with κ^+ replaced by κ. As $P \in H_\kappa$, $\Vdash_P$"$H_\kappa \vDash \exists x \forall y \Phi(x, y, \dot{p})$". □

Lightface Σ^1_2-indescribability does imply a certain fragment of Mahloness, as we observed in [Sch04]. For an application of the following notion see [BB04, §5].

3.4. **Definition.** Let us call a cardinal κ Σ_n-Mahlo (resp. Π_n-Mahlo) *iff* it is inaccessible, and every *cub* subset C of κ with a Σ_n (resp. Π_n) definition in H_κ, with parameters, contains an inaccessible cardinal. Lightface Σ_n-Mahlo (resp. Π_n-Mahlo) is defined analogously, but without allowing parameters in the definition of C.

3.5. **Fact.** If κ is lightface Σ^1_2-indescribable, then it is Σ_2-Mahlo. In fact, κ is an inaccessible limit of Σ_2-Mahlo cardinals. Any lightface Π_2-Mahlo cardinal is an inaccessible limit of Σ^1_2-indescribable cardinals.

Proof. First, assume κ is Σ^1_2-indescribable. To prove that κ is Σ_2-Mahlo, let C be a *cub* subset of κ such that $x \in C \iff H_\kappa \vDash \phi(\xi)$, where $\phi(\xi)$ is Σ_2. $H_{\kappa^+} \vDash$"there is an inaccessible θ such that $H_\theta \vDash \forall \xi \exists \bar{\xi} > \xi \phi(\bar{\xi})$". This statement is itself Σ_2, so it also holds in H_κ. So there is an inaccessible $\theta < \kappa$ such that $H_\theta \vDash \forall \xi \exists \bar{\xi} > \xi \phi(\bar{\xi})$. By upward absoluteness of $\phi(\xi)$ for members of the H-hierarchy and closedness of C, $\theta \in C$. This completes the proof of the first assertion. It is straightforward to check that "κ is Σ_2-Mahlo" is Π^1_1 over V_κ, so since κ is Σ^1_2-indescribable, there are unboundedly many Σ_2-Mahlo cardinals below κ. For the last assertion, assume κ is lightface Π_2-Mahlo. We follow the proof of Fact 3.3(2). Check that the *cub* set mentioned there, consisting of $\xi < \kappa$ having the Σ^1_2 reflection property, has a Π_2 definition, without parameters, over V_κ. So there are unboundedly many Σ^1_2-indescribable cardinals below κ. □

4. Coding using an Aronszajn-tree

We fix the following notation: for a tree T, we denote by $<_T$ (or $\leq_T$) the tree order, T_α denotes the α-th level of T and $T \upharpoonright \alpha$ denotes the subtree of T consisting of all levels of height less than α. By pred(t) we mean of course $\{t' \in T | t' <_T t\}$. The following works for any Aronszajn-tree T, that is, a tree of height ω_1 with countable levels and without any cofinal branches (i.e. linearly ordered sets of type ω_1). Aronszajn trees can be “specialized” by a ccc forcing: that is, one adds an order preserving function from the tree into the rationals (a so-called specializing function). This ensures that one cannot add, by further forcing, cofinal branches without at the same time collapsing ω_1. Applying this forcing to code a subset of ω_1 by a real, [HS85] proves that MA together with “ω_1 is inaccessible to reals” implies ω_1 is weakly compact in L. We present a slight variation.

4.1. **Fact.** Let $S = (s_\alpha)_{\alpha<\omega_1}$ be a sequence of reals. There is a ccc forcing P that adds a real r such that in the extension the following holds: whenever M is a transitive model of ZF^- such that $r \in M$, $\langle T, \leq_T\rangle \in M$, we have $(s_\alpha)_{\alpha<\omega_1} \in M$.

Proof. To achieve this, we iterate the following notion of forcing: fix Q_0, Q_1, two disjoint dense sets whose union is all rational numbers. For any sequence $S = (s_\alpha)_{\alpha<\omega_1}$ consider P^S_T consisting of all conditions f such that

(1) f is a function with domain a finite subset of $T \times \omega$.
(2) For each $n \in \omega$, the function $t \mapsto f(t, n)$ is a partial order preserving mapping from $(T, <_T)$ into the rationals.
(3) For any $\alpha < \omega_1$, t at the α-th level of T and $n \in \omega$, if $(t, n) \in \text{dom}(f)$, then $f(t, n) \in Q_0$ if and only if $n \in s_\alpha$.

4.2. **Lemma.** *Let $F = \bigcup G$, where G is generic. Then F is a function from $T \times \omega$ into the rationals which is order preserving and continuous at limit nodes of T; moreover, for any $\alpha < \omega_1$ and any $t \in T_\alpha$, $\{ n \in \omega \mid F(t, n) \in Q_0 \} = s_\alpha$.*

Proof. Clearly, $D_{(t,n)} := \{ p \in P^S_T \mid (t, n) \in \text{dom}(p) \}$ is dense for any $(t, n) \in T \times \omega$: given a condition p, there is an interval of possible values for p at (t, n) (since p has finite domain), so if t is at level α of T, we can choose a value from Q_0 or Q_1, depending on whether $n \in s_\alpha$ or not. So F is a total, order-preserving function on $T \times \omega$, and the “moreover” clause holds by definition. F is continuous as $D_{(t,n),\epsilon} := \{ p \in P^S_T \mid \exists t' \in T \ |p(t', n) - p(t, n)| < \epsilon \}$ is dense for any $n \in \omega$, $\epsilon > 0$ and t at a limit level of T (again, by the finiteness of the domain of any condition). □

4.3. **Lemma.** *P^S_T is* ccc.

Proof. Assume $(p_\alpha)_{\alpha<\omega_1}$ is an uncountable antichain; then $\{\text{dom}(p_\alpha) | \alpha < \omega_1\}$ is an uncountable subset of $[T \times \omega]^{<\omega}$, so we can apply the delta-systems lemma and assume that for each α, $\text{dom}(p_\alpha) = r \cup d_\alpha$, where $r, (d_\alpha)_{\alpha<\omega_1}$ are pairwise disjoint. Let us also assume that the d_α all have the same cardinality k. There are only countably many possibilities for the values of the p_α on r, so we assume that all the conditions agree on r. So for any $\alpha, \alpha' < \omega_1$, there is $t \in d_\alpha$, $t' \in d_{\alpha'}$ and $n \in \omega$ such that $p_\alpha \cup p_{\alpha'}$ is not order preserving on $\{(t, n), (t', n)\}$, whence in particular t and t' are comparable in the tree order. As any node of the tree has only countably many predecessors in the tree order, by thinning out $(p_\alpha)_{\alpha<\omega_1}$ we can further assume that for all $\alpha < \alpha' < \omega_1$, there are $t \in d_\alpha$, $t' \in d_{\alpha'}$ such that $t <_T t'$. Let us now enumerate the d_α as $t^0_\alpha, \dots, t^{k-1}_\alpha$. We know that all the conditions in the antichain have comparable nodes in their domain; we will now find a sufficiently coherent subset of conditions to get a branch through T. Enlarge (using Zorn's lemma) the filter of co-initial subsets of ω_1 to an ultrafilter U (U contains only sets of size ω_1, i.e. U is uniform). For any $\alpha < \omega_1$, we have $\{ \beta < \omega_1 \mid \exists i, j \;\; t^i_\alpha <_T t^j_\beta \} \in U$. So by finite additivity of U, for each α there are i, j such that $\{ \beta < \omega_1 \mid \;\; t^i_\alpha <_T t^j_\beta \} \in U$. Moreover, there is an uncountable set I and i, j such that the above holds for all $\alpha \in I$ and this particular pair i, j. So for any $\alpha, \alpha' \in I$, as elements of U have non-empty (in fact large) intersection, there is β such that $t^i_\alpha <_T t^j_\beta$ and $t^i_{\alpha'} <_T t^j_\beta$, so t^i_α and $t^i_{\alpha'}$ are comparable and $(t^i_\alpha)_{\alpha \in I}$ is an uncountable branch through T. □

Now we can prove Fact 4.1. We build P as the finite support iteration of $(P_k)_{k\in\omega}$.

Let $s^0_\alpha = s_\alpha$; P_0 is the forcing coding this sequence of reals into a specializing function for T. At stage n, we have added a specializing function F_n; let s^{n+1}_α be a real coding (in some absolute way) F_n restricted to $(T \restriction \alpha + 2) \times \omega$. P_{n+1} is the forcing for coding the sequence $(s^n_\alpha)_{\alpha<\omega_1}$.

Let r be a real coding all reals $(s^k_0)_{k\in\omega}$; we check by induction on $\eta \leq \omega_1$ that r has the property promised in Fact 4.1: assume that for all k, $(s^k_\xi)_{\xi<\eta} \in M$ (this holds by assumption if $\eta = 1$). If $\eta = \zeta + 1$, as s^{k+1}_ζ codes F_k restricted to $T \restriction \zeta + 2$, for an arbitrary $t \in T_{\zeta+1}$, $s^k_{\zeta+1} = \{n | F_k(t, n) \in Q_0\} \in M$. For limit η, using $(s^k_\xi)_{k\in\omega,\xi<\eta}$ we have F_k restricted to $T \restriction \eta$ inside M, and therefore, picking an arbitrary $t \in T_\eta$, $n \in s^k_\eta$ exactly if $sup(\{F(t', n) | t' <_T t\}) \in Q_0$; so $s^k_\eta \in M$ for all k.

5. An equiconsistency

5.1. **Theorem.** *"Σ^1_3-Abs(ccc) and ω_1 inaccessible to reals" has the consistency strength of a Σ^1_2-indescribable.*

Proof. First, observe that in order to prove that an inaccessible cardinal κ has the Σ^1_2 reflection property, it suffices to prove the seemingly weaker property where we treat *all second-order quantifiers as ranging over sets of ordinals*, rather than over arbitrary subsets of a structure. For notational reasons, we shall sometimes identify sets (those denoted by X, $\bar{X}$, $X^\star$, etc.) with their characteristic functions, and therefore write "$X \restriction \xi$" for "$X \cap \xi$". Let κ denote ${\omega_1}^V$, and work in L. Observe that κ is inaccessible. Let Φ be some first-order formula (with parameter in L_κ, which we suppress), and let $X^\star \in L$ be some function from κ into 2, such that

$$\langle L_\kappa, X^\star \rangle \vDash \forall A \Phi(X^\star, A).$$

We may naturally assume that for all $\xi < \kappa$ there is $A \subseteq \xi$ such that

$$\langle L_\xi, X^\star \restriction \xi, A \rangle \vDash \neg\Phi(X^\star \restriction \xi, A),$$

for otherwise, we are done.[2] Varying the well-known construction of an Aronszajn-tree whose height is an inaccessible cardinal which is not weakly compact in L, we now define a tree T and its ordering $\leq_T$:

Elements of T are tuples (β, X), where $\beta < \kappa$ and $X \in {}^\delta 2$, for some δ, and

(1) $L_\beta = h^{L_\beta}_{\Sigma_1}(|X| \cup \{X\})$ (in particular, $X \in L_\beta$),
(2) $X \restriction |X| = X^\star \restriction |X|$,

[2]In other words, we may assume κ is not weakly compact in L, as witnessed by $X^\star$ and Φ.

(3) for all $\xi \leq \mathrm{dom}(X)$, there is $A \in L_\beta$, a subset of ξ, such that

$$\langle L_\xi, X \restriction \xi, A\rangle \vDash \neg\Phi(X \restriction \xi, A).$$

Define $(\beta, X) \leq_T (\bar\beta, \bar X) \iff X \leq_L \bar X$, and there is a Σ_1-elementary embedding $\sigma : L_\beta \to L_{\bar\beta}$ such that $\sigma(X) = \sigma(\bar X)$ and σ is the identity on $|X|$. This can be motivated by observing that branches correspond to a failure of reflection, as will become clear in a moment.

Let us check $\leq_T$ is a tree order. Clearly, $\leq_T$ is transitive and reflexive. Also, $\leq_T$ is antisymmetric: assume $(\beta, X) \leq (\beta', X')$ and $(\beta', X') \leq (\beta, X)$. As $X = X'$, the embedding witnessing $(\beta, X) \leq_T (\beta', X')$ shows that L_β is isomorphic to $h^{L_{\beta'}}_{\Sigma_1}(|X'| \cup \{X'\})$ (by Fact 2.2); but the latter is just $L_{\beta'}$, by item (1) in the definition of T. It remains to check that any two predecessors of a node are comparable: say (β, X), $(\beta', X') \leq_T (\bar\beta, \bar X)$, as witnessed by embeddings σ and σ'. Without loss of generality assume $X \leq_L X'$, whence also $|X| \leq |X'|$ (if not, since X' is a function on an ordinal, $X' \in L_{|X'|^+} \subseteq L_{|X|}$, a contradiction). So (once more using Fact 2.2) $\mathrm{ran}(\sigma) = h^{L_{\bar\beta}}_{\Sigma_1}(|X| \cup \{\bar X\}) \subseteq h^{L_{\bar\beta}}_{\Sigma_1}(|X'| \cup \{\bar X\}) = \mathrm{ran}(\sigma')$, whence $(\sigma')^{-1} \circ \sigma$ is a well-defined elementary embedding and so $(\beta, X) \leq_T (\beta', X')$.

We now show T is a κ-Aronszajn tree. First observe that for a node $(\bar\beta, \bar X)$ of T and a cardinal $\alpha \leq \bar\beta$, there is *exactly one* $t \leq_T (\beta, X)$ of cardinality α. Existence: look at the transitive collapse L_β of $h^{L_{\bar\beta}}_{\Sigma_1}(\alpha \cup \{\bar X\})$ and let X denote the image of $\bar X$ under the collapsing map (let σ denote the inverse of this map). Then $|X| = \alpha$, so $L_\beta = h^{L_\beta}_{\Sigma_1}(|X| \cup \{X\})$, by Fact 2.1. Item (3) holds for $(\bar\beta, \bar X)$, so by a Skolem hull argument, it also holds for (β, X). So $(\beta, X) \in T$. If $\alpha < |\bar X|$, $X \leq_L \bar X$, and σ witnesses $(\beta, X) \leq_T (\bar\beta, \bar X)$. If $\alpha = |\bar X|$, by item (1), $X = \bar X$ and $\beta = \bar\beta$. Uniqueness: say (β, X), $(\beta', X') \leq_T (\bar\beta, \bar X)$, and $\alpha = |\beta| = |\beta'|$. By Fact 2.2, both $\langle L_\beta, X\rangle$ and $\langle L_{\beta'}, X'\rangle$ are isomorphic to $h^{L_{\bar\beta}}_{\Sigma_1}(\alpha \cup \{\bar X\})$, so they are identical. As a corollary we obtain that if $(\beta, X) \in T$ and $|\beta| = \omega_\alpha$, (β, X) has exactly α predecessors in $\leq_T$, i.e. the height of (β, X) in T is α. So $T \restriction \alpha \subseteq L_{\omega_\alpha}$ (T has small levels). T has height at least κ: let any $\alpha < \kappa$ be given. Let $X := X^\star \restriction \omega_\alpha$ and let L_β be the transitive collapse of $H := h^{L_\kappa}_{\Sigma_1}(\omega_\alpha \cup \{X\})$. It is easy to check that $(\beta, X) \in T$ (for item (3), observe that $\mathrm{dom}(X) = \omega_\alpha \in H$), and we have seen its height is exactly α. To conclude that T is Aronszajn (in V), it remains to check:

5.2. **Lemma.** *T does not have a branch of order-type κ in V.*

Proof. Let $(\beta(\alpha), X(\alpha))_{\alpha<\kappa}$ be such a branch. Let $\sigma^{\bar\alpha}_\alpha : L_{\beta(\alpha)} \to_{\Sigma_1} L_{\beta(\bar\alpha)}$ be the embedding witnessing $(\beta(\alpha), X(\alpha)) \leq_T (\beta(\bar\alpha), X(\bar\alpha))$. A straightforward argument involving the Σ_1-definable Skolem functions shows that for $\alpha < \alpha' < \bar\alpha$, $\sigma^{\bar\alpha}_{\alpha'} \circ \sigma^{\alpha'}_\alpha = \sigma^{\bar\alpha}_\alpha$. As κ has uncountable cofinality, the direct limit of this chain of models is well founded and a model of $V = L$, therefore isomorphic to some L_δ. Each $L_{\beta(\alpha)}$ is Σ_1-elementarily embeddable into L_δ via a map that is the identity on $|\beta(\alpha)|^L$, and all the $X(\alpha)$ are mapped to one X_0 which must therefore end-extend $X^\star$ (in the sense that $X_0 \restriction \kappa = X^\star$). So $\delta > \kappa$ (as $X_0 \in L_\delta$). By elementarity (and condition (3) in the definition of T), there is $A \in L_\delta$, a subset of κ, such that $\langle L_\kappa, X_0 \restriction \kappa, A\rangle \vDash \neg\Phi(X_\delta \restriction \kappa, A)$, a contradiction. □

Let us go back to working in L again, for yet a little while. T is not pruned (there are dying branches and branches that do not split), and T need not even have unique limit nodes (in the sense that for t and t' at a limit level T_λ, if t and t' have the same predecessors, then $t = t'$). The latter shortcoming has to be remedied, and this is easily accomplished by replacing T by T', where $T' \restriction \omega = T \restriction \omega$, $T'_{\alpha+1} = T_\alpha$ for any infinite ordinal $\alpha < \kappa$, while for limit ordinals λ we set $T'_\lambda = \{\text{pred}(t) | t \text{ in } T_\lambda\}$. T' carries the obvious order ($t \leq_{T'} t'$ exactly if either $t \subseteq t'$ or $t \in t'$ or $t \subseteq \text{pred}(t')$ or $t \leq_T t'$).

Fix $\delta^\star$ such that $X^\star \in L_{\delta^\star}$. Pick E, a binary relation on κ, such that

(1) $\langle \kappa, E \rangle \cong \langle L_{\delta^\star}, \in \rangle$, and
(2) $X^\star(\xi) = 1 \iff (\xi + 1)\ E\ \emptyset$.

Define $C := \{\xi < \kappa | \xi \text{ is a cardinal and } \langle L_\xi, X^\star \restriction \xi, E \cap (\xi \times \xi)\rangle \prec \langle L_\kappa, X^\star, E \rangle\}$. By inaccessibility of κ this is a *cub* set. Let C be enumerated as $(c_\xi)_{\xi<\kappa}$.

Now we work in V: let s_ξ be a real coding, in some absolute manner, the tuple

$$(T_{\xi+1}, X^\star \restriction c_\xi, E \cap (c_\xi \times c_\xi)).$$

Apply the forcing just described (Fact 4.1) to code the sequence $S = (s_\xi)_{\xi<\omega_1}$ into a single real r, using T.

Consider any $\beta < \kappa$ such that $L_\beta[r]$ is a model of "ZF^- and ω_1 exists". Let α denote $\omega_1^{L_\beta[r]}$. We claim that for some $\xi \leq \alpha$, there is $x \in L_\beta \cap \mathcal{P}(\xi)$ such that for all $a \in L_\beta \cap \mathcal{P}(\xi)$, $\langle L_\xi, x, a \rangle \vDash \Phi(x, a)$, i.e. that from the point of view of L_β, reflection occurs before or at ω_1. Assume otherwise; we show how to recursively reconstruct $(s_\xi)_{\xi<\alpha}$ inside $L_\beta[r]$, and then obtain a contradiction. We construct $(s_\xi)_{\xi<\eta}$ by recursion on $\eta \leq \alpha$. $s_0 \in L_\beta[r]$ is immediate. Now say $\eta = \gamma + 1$: by induction hypothesis $(s_\xi)_{\xi\leq\gamma} \in L_\beta[r]$, so $T \restriction \gamma + 2 \in L_\beta[r]$. As in the proof of Fact 4.1, utilizing the specializing functions on that tree (coded recursively by r), we obtain $s_{\gamma+1} \in L_\beta[r]$.

We shall now consider two cases simultaneously, since the next few steps of the argument are identical for both:

(1) $\eta < \alpha$ is a limit ordinal; in this case, we must show how to continue the construction of $(s_\xi)_{\xi<\alpha}$.
(2) we have constructed $(s_\xi)_{\xi<\alpha}$ and $\eta = \alpha$; this leads to a contradiction.

In any case, we may assume $(s_\xi)_{\xi<\eta} \in L_\beta[r]$, whence $X^\star \restriction c_\eta$, $E \cap (c_\eta \times c_\eta) \in L_\beta[r]$. $E \cap (c_\eta \times c_\eta)$ is of course a well-founded relation, and by the definition of C and elementarity, its transitive collapse is equal to some $L_{\zeta^\star}$ such that $X^\star \restriction c_\eta \in L_{\zeta^\star}$ and $\zeta^\star < \beta$.

To be sure the construction of $(s_\xi)_{\xi<\alpha}$ takes place entirely in $L_\beta[r]$, we feel we should mention the triviality that since $(\omega_\eta)^L < \beta$, $(\omega_\eta)^L = (\omega_\eta)^{L_\beta}$. Work in L. Since $c_\eta \geq \omega_\eta$, $X^\star \restriction \omega_\eta \in L_\beta$. For each $\xi < \eta$, look at the transitive collapse $L_{\beta(\xi)}$ of $h_{\Sigma_1}^{L_\beta}(\omega_\xi \cup \{X^\star \restriction \omega_\eta\})$, and let $X(\xi)$ be the image of $X^\star \restriction \omega_\eta$ under the collapsing map. By definability of the Skolem-hull operator and by replacement in L_β, the sequence $(\beta(\xi), X(\xi))_{\xi<\eta}$ is an element of L_β. Observe that for $\xi < \eta$, c_ξ is countable in $L_\beta[r]$, so $c_\eta \leq \alpha$ and thus $\omega_\eta \leq \alpha$. Hence, by assumption, for each $\xi \leq \omega_\eta$ there is $a \in L_\beta$, $a \subseteq \xi$, such that $\langle L_\xi, X^\star \restriction \xi, a \rangle \vDash \neg\Phi(X^\star \restriction \xi, a)$. This ensures that item (3) in the definition of T holds for each $(\beta(\xi), X(\xi))$, so arguing just as in the proof showing T is Aronszajn, we have $(\beta(\xi), X(\xi)) \in T$ and its height is ξ. So we have found a branch b of order-type η in $T \restriction \eta$, $b \in L_\beta$.

We work in V again. Once more, as in the proof of Fact 4.1, we may recover, for all k, $F_k \restriction (b \times \omega)$ from b and r, inside $L_\beta[r]$ (observe only one node on each level suffices for the construction). Now we proceed to argue by cases: in case (1),

b has order-type $\alpha = {\omega_1}^{L_\beta[r]}$, contradicting $F_0 \restriction (b \times \omega) \in L_\beta[r]$, as F_0 yields an order-preserving function from b into the rationals. For case (2), we must show $s_\eta \in L_\beta[r]$, and indeed, this holds as $n \in s_\eta \iff \sup\{F_0(t,n) | t \in b\} \in Q_0$. This finishes the proof of the claim.

So we have found, after forcing with a ccc partial order, a real r with the Π_1 property

$$\forall \beta < \omega_1, \text{ if } L_\beta[r] \models \text{“}ZF^- \text{ and } \omega_1 \text{ exists”, then}$$
$$\exists \xi \leq (\omega_1)^{L_\beta[r]} \text{ such that } (L_\xi \models \exists X \forall A \Phi(X,A))^{L\beta}.$$

By Σ^1_3-absoluteness, we may assume that r is in the ground model. But since ω_1 is inaccessible to reals, we may look at $\beta := (\omega_2)^{L[r]}$. By the above, for some $\xi \leq (\omega_1)^{L_\beta[r]}$, $(L_\xi \models \exists X \forall A \Phi(X,A))^{L\beta}$, and therefore $(L_\xi \models \exists X \forall A \Phi(X,A))^{L}$. This completes one direction of the proof.

For the other direction, assume κ is Σ^1_2-indescribable. We show that after forcing with the Lévy-collapse of κ, Σ^1_3-absoluteness for ccc forcing holds and κ is inaccessible to reals. The latter is clear, as any real in the extension can be absorbed into an intermediate model where κ is still inaccessible.

In $V^{Coll(\omega,\kappa)}$, let P be a ccc partial order which forces a $\Sigma^1_3(r)$ statement $\phi(r)$, $r \in V^{Coll(\omega,\{\alpha\})}$, for some $\alpha < \kappa$.

First, we can assume that $|P| = \omega_1$: using the tree representation of Σ^1_2 sets, write $\phi(r)$ as “there is a real x such that $T(x)$ is well-founded”. Here, T is a tree on ω_1 which is Δ_0 definable in the parameters r and ω_1. So $\Vdash_P$ “$\exists \dot{x}$ such that $T(\dot{x})$ is well-founded”. As P has the ccc, there is $\xi < \omega_2$ such that $\Vdash_P rank(T(\dot{x})) < \check{\xi}$, and there is a name $\dot{F}$ for a ranking function on $T(\dot{x})$, $|\dot{F}| = \omega_1$. Now let M be an elementary submodel of H_{ω_2} such that $\dot{x}, \dot{F}$ and P are elements of M and $\omega_1 + 1 \subseteq M$. As the forcing relation for Δ_0 sentences is uniformly Δ_0 definable for transitive models of ZF^-, we can take the transitive collapse of M, and we have $\Vdash_{P'}$“$\dot{F}'$ is an order preserving function from $T(\dot{x}')$ into the ordinals”, where $\dot{x}', \dot{F}'$ and P' are the images of $\dot{x}, \dot{F}$ and P under the collapsing map. Thus, since P' preserves ω_1, $\Vdash_{P'} \phi(r)$. This proves we can assume P has size ω_1.

For the moment, we work in $W := V^{Coll(\omega,\{\alpha\})}$. Let $\dot{P}$ be a $Coll(\omega,\kappa)$-name for P. As $Coll(\omega,\kappa) \in H_{\kappa^+}$, we may assume $\dot{P} \in H_{\kappa^+}$, whence

$$H_{\kappa^+} \models \exists Q \Vdash_Q \phi(r), \tag{1}$$

as witnessed by $Q := Coll(\omega,\kappa) * P$. In W^Q,

$$\phi(r) \iff H_{\kappa^+} \models \exists u \forall w \psi(u,w,r)$$

for a suitable Δ_0 formula ψ (e.g. such that $\forall w \psi(u,w,r)$ says that a certain tree $T(u,r)$ on ω is ill-founded, i.e. has no ranking function; use the tree representation for Π^1_1 sets). Using this equivalence and arguing as in Fact 3.3(3), in W (1) is equivalent to

$$\exists Q \in H_{\kappa^+} \quad \exists \dot{u} \in H_{\kappa^+} \quad \forall \dot{w} \in H_{\kappa^+} \quad H_{\kappa^+} \models \psi'(\dot{u},\dot{w},r,Q), \tag{2}$$

where ψ' is Δ_1. By Σ^1_2-indescribability of κ, (2) holds with κ^+ replaced by κ. As (1) and (2) are still equivalent when κ^+ is replaced by κ, there is $Q' \in H_\kappa$, $\Vdash_{Q'} \phi(r)$.

In $V^{Coll(\omega,\kappa)}$, there is H which is generic for Q' over W, as $|\mathcal{P}(Q')|^W$ is collapsed to ω. $\phi(r)$ holds in $W[H]$, and $\phi(r)$ is upward absolute between $W[H] \subseteq V^{Coll(\omega,\kappa)}$ and $V^{Coll(\omega,\kappa)}$. So $\phi(r)$ holds in $V^{Coll(\omega,\kappa)}$, whence Σ^1_3-absoluteness holds between this model and any subsequent ccc extension. $\square$

Open questions

What are other applications of lightface indescribable cardinals? E.g. what is the consistency strength of "two-step" Σ^1_3-absoluteness for ccc forcing plus ω_1 inaccessible to reals? Two-step Σ^1_3-absoluteness for ccc forcing means that for any ccc forcing P and a P-name $\dot{Q}$ such that P forces "$\dot{Q}$ is a ccc partial ordering", ${H_{\omega_1}}^V \prec_{\Sigma_2} {H_{\omega_1}}^{V^P}$ and $H_{\omega_1}^{V^P} \prec_{\Sigma_2} {H_{\omega_1}}^{V^{P*\dot{Q}}}$.

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Kurt Gödel Research Center for Mathematical Logic, Währinger Strasse 25, A-1090 Wien, Austria

E-mail address: `david@logic.univie.ac.at`

URL: `http://www.logic.univie.ac.at/~david`