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COUNTABLE COMPACT HAUSDORFF SPACES NEED NOT BE METRIZABLE IN ZF

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Abstract. We show that the existence of a countable, first countable, zero-dimensional, compact Hausdorff space which is not second countable, hence not metrizable, is consistent with ZF.

1. Notation and terminology

Definition 1.1. Let (X, T) be a topological space and $\mathcal{B}$ a base for T.

(i) X is said to be **compact** if every open cover $\mathcal{U}$ of X has a finite subcover $\mathcal{V}$. X is said to be **compact with respect to the base** $\mathcal{B}$ if every open cover $\mathcal{U} \subset \mathcal{B}$ of X has a finite subcover $\mathcal{V}$.

(ii) A collection $\mathcal{U} \subset \wp(X)$ is said to be **locally finite (point finite)** if every $x \in X$ has an open neighborhood which meets only finitely many members of $\mathcal{U}$ (each $x \in X$ belongs to only finitely many members of $\mathcal{U}$).

(iii) Let $\mathcal{U}$ be an open cover of X. A collection $\mathcal{V} \subset \wp(X)$ is said to be **an open refinement of** $\mathcal{U}$ if $\bigcup \mathcal{V} = X$ and each $V \in \mathcal{V}$ is open and is contained in some $U \in \mathcal{U}$.

(iv) X is said to be **paracompact** if every open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$.

(v) X is said to be **metacompact** if every open cover $\mathcal{U}$ of X has a point finite open refinement $\mathcal{V}$.

(vi) X is said to be **separable** if it has a countable dense subset.

(vii) X is said to be **first countable** if every $x \in X$ has a countable open neighborhood base.

(viii) X is said to be **second countable** if there is a countable base for T.

(ix) X is said to be **zero-dimensional** if each of its points has a neighborhood base consisting of clopen (closed and open) sets.

(x) X is said to be **metrizable** if there is a metric d on X such that the topology T_d induced by d coincides with T.

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Definition 1.2.

AC$_{\triangle\omega}(\mathbb{R})$: *Every family* $\mathcal{A} = \{A_i \subset \wp(\omega) \setminus \{\emptyset\} : i \in I\}$ *such that,* $\forall i \in I, \forall x, y \in A_i, |x \triangle y| < \omega$, *has a choice function* ($\triangle$ denotes the operation of symmetric difference between sets).

CAC$_{\triangle\omega}(\mathbb{R})$: *AC$_{\triangle\omega}(\mathbb{R})$ restricted to countable families.*

CUC$_{\triangle\omega}(\mathbb{R})$: *Every family* $\mathcal{A} = \{A_i \subset \wp(\omega) \setminus \{\emptyset\} : i \in \omega\}$ *such that,* $\forall i \in \omega, \forall x, y \in A_i, |x \triangle y| < \omega$, *has a countable union.*

2. Introduction and some preliminary results

There is no question that without the axiom of choice (AC) modern General Topology would have been shrunk considerably in size and become less interesting. The reason is that many everyday used theorems are equivalent to AC or to some weaker forms of it. As an example, AC and Tychonoff's compactness theorem express the same truth in mathematics (see [10]). For horrors and disasters which we may encounter in topology without AC, we refer the reader to [1], [3], [4], [5], [7], [11], [14], and [16].

An example of a "beautiful" but "suspicious" result which holds true in ZF (see C. Good and I. Tree [[3], Corollary 4.8, p. 86]) is Urysohn's metrization theorem. i.e.,

Theorem 2.1 (UMT)**.** *Every regular, second countable topological space is metrizable.*

Since compact T_2 spaces are regular without employing any choice principle (the standard proof (see [15]) goes through in ZF with some minor changes), the following consequence of UMT is also provable in ZF

2CM: *Every second countable compact T_2 space (X, T) is metrizable.*

However, if in 2CM we require that X be **countable** rather than its base, then the situation for the resulting statement

CCM: *Every countable compact T_2 space is metrizable*,

is **strikingly** different. Even more, the weaker than CCM proposition:

CC$_0$M: *Every countable, zero-dimensional, compact T_2 space is metrizable*,

is not a theorem of ZF (see Theorem 3.4). However, it is part of the folklore that CCM is a theorem of **ZFC** (= ZF+AC). In fact, CCM is a theorem of a weaker system than ZFC, namely ZF+CAC($\mathbb{R}$), where

CAC($\mathbb{R}$): AC *restricted to countable families of nonempty sets of reals.*

Theorem 2.2. *CCM is provable in ZF+CAC($\mathbb{R}$).*

Proof. Let (X, T) be a countable, compact T_2 space. Without loss of generality we may assume that $X = \omega$. Consider the family $\mathcal{A} = \{A_{nm} : n, m \in \omega, n \neq m\}$, where $A_{nm} = \{(U, V) : U, V$ are disjoint open neighborhoods of n and m, respectively$\}$. Since $|\omega \times \omega| = |\omega|$, $|\wp(\omega)| = |\mathbb{R}|$, and $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$ in ZF (see [9]), let, by CAC($\mathbb{R}$), f be a choice function of $\mathcal{A}$. Then $F = \mathrm{dom}(f) \cup \mathrm{ran}(f)$ is countable, and consequently $\mathcal{B} = \{\bigcap \mathcal{Q} : \mathcal{Q} \in [F]^{<\omega}\}$ is countable. We show that $\mathcal{B}$ is a base for (ω, T). Fix an open set O in ω and let $n \in O$. Since O^c is compact, it follows that the open cover $\mathcal{U} = \{\pi_2(f(A_{nm})) : m \in O^c\}$ of O^c, where π_2 is the canonical projection on the second coordinate, has a finite subcover, say $\{\pi_2(f(A_{nm_1})), \ldots, \pi_2(f(A_{nm_k}))\}$. Then $U = \pi_1(f(A_{nm_1})) \cap \ldots \cap \pi_1(f(A_{nm_k}))$ is an open neighborhood of n which

is contained in O. Thus $\mathcal{B}$ is a countable base of (ω, T) and the conclusion follows from UMT. □

In the sequel we shall use the following two results which are valid in ZF and whose proof we leave as an easy exercise.

Theorem 2.3. *Every separable metric space is second countable.*

Theorem 2.4. *Let (X, T) be a topological space and $\mathcal{B}$ a base for T. Then X is compact iff X is compact with respect to the base $\mathcal{B}$.*

3. Main results

Since the statements $\mathrm{AC}_{\triangle\omega}(\mathbb{R})$, $\mathrm{CAC}_{\triangle\omega}(\mathbb{R})$ and $\mathrm{CUC}_{\triangle\omega}(\mathbb{R})$ are all new here, we shall attempt to locate them on the map of choice principles. Consider the following propositions:

$\mathbf{AC}_\omega(\mathbb{R})$: AC *restricted to families of nonempty, countable sets of reals.*
$\mathbf{CAC}_\omega(\mathbb{R})$: $\mathrm{AC}_\omega(\mathbb{R})$ *restricted to countable families.*
$\mathbf{CUC}(\mathbb{R})$: *A countable union of countable sets of reals is countable.*

Theorem 3.1. (i) *$AC_\omega(\mathbb{R})$ implies $AC_{\triangle\omega}(\mathbb{R})$.*
(ii) *$CAC_\omega(\mathbb{R})$ implies $CAC_{\triangle\omega}(\mathbb{R})$.*

Proof. Fix a family $\mathcal{A}$ as in the statement of $\mathrm{AC}_{\triangle\omega}(\mathbb{R})$ ($\mathrm{CAC}_{\triangle\omega}(\mathbb{R})$, respectively). Note that if $A \in \mathcal{A}$ and $z \in A$, then $A \subset \{z \triangle y : y \in [\omega]^{<\omega}\}$, where $[\omega]^{<\omega}$ is the set of all finite subsets of ω. Since $[\omega]^{<\omega}$ is countable in ZF, it follows that A is countable. Thus, $\mathcal{A}$ is a family of countable sets (which without loss of generality (wlog) can be considered as sets of reals since $|\wp(\omega)| = |\mathbb{R}|$ in ZF); thus by $\mathrm{AC}_\omega(\mathbb{R})$ ($\mathrm{CAC}_\omega(\mathbb{R})$, respectively) $\mathcal{A}$ has a choice function. □

Clearly $\mathrm{CUC}(\mathbb{R}) \to \mathrm{CAC}_\omega(\mathbb{R})$ and $\mathrm{CUC}_{\triangle\omega}(\mathbb{R}) \to \mathrm{CAC}_{\triangle\omega}(\mathbb{R})$. It is unknown (see [7]) whether $\mathrm{CAC}_\omega(\mathbb{R}) \to \mathrm{CUC}(\mathbb{R})$. Unlike this situation, the implication $\mathrm{CAC}_{\triangle\omega}(\mathbb{R}) \to \mathrm{CUC}_{\triangle\omega}(\mathbb{R})$ is surprisingly true as the next theorem indicates.

Theorem 3.2. *$CAC_{\triangle\omega}(\mathbb{R})$ iff $CUC_{\triangle\omega}(\mathbb{R})$.*

Proof. It suffices to show $\mathrm{CAC}_{\triangle\omega}(\mathbb{R}) \to \mathrm{CUC}_{\triangle\omega}(\mathbb{R})$ as the other implication is straightforward. Fix a family $\mathcal{A} = \{A_i : i \in \omega\}$ as in the statement of $\mathrm{CUC}_{\triangle\omega}(\mathbb{R})$. Wlog we may assume that $\mathcal{A}$ is pairwise disjoint. Let, by $\mathrm{CAC}_{\triangle\omega}(\mathbb{R})$, f be a choice function of the family $\mathcal{A}$. Define a function $h : \bigcup \mathcal{A} \to [\omega]^{<\omega} \times \omega$ as follows: $h(x) = (f(A_i) \triangle x, i)$, $x \in A_i$. It can be readily verified that h is $1:1$. Thus, $|\bigcup \mathcal{A}| \leq |[\omega]^{<\omega} \times \omega| = \aleph_0$ and the desired result follows. □

In the next theorem, we use a family $\mathcal{A}$ satisfying the hypothesis of $\mathrm{CAC}_{\triangle\omega}(\mathbb{R})$ in order to construct a countable, first countable, zero-dimensional, compact T_2 space which fails to be second countable in case $\mathcal{A}$ has no choice function. This shows the way to go in order to prove the consistency of ZF+$\neg\mathrm{CC}_0$M.

Theorem 3.3. *$CC_0M \to CAC_{\triangle\omega}(\mathbb{R})$.*

Proof. Fix $\mathcal{A} = \{A_i \subset \wp(\omega) \setminus \{\emptyset\} : i \in \omega\}$ a family satisfying the hypothesis of $\mathrm{CAC}_{\triangle\omega}(\mathbb{R})$. Wlog we may assume that $\mathcal{A}$ is pairwise disjoint and that it satisfies the following two conditions:

(a) $\forall i \in \omega, \forall x \in A_i, \forall Q \in [\omega]^{<\omega}, (x \backslash Q) \in A_i$.

(If not, then we can replace A_i with $B_i = \{x \backslash Q : x \in A_i, Q \in [\omega]^{<\omega}\}$ and from a choice function f of $\mathcal{B} = \{B_i : i \in \omega\}$ we can easily pass to a choice function h of $\mathcal{A}$. Indeed, let $\{F_n : n \in \omega\}$ be an enumeration of $[\omega]^{<\omega}$ and define for each $i \in \omega$, $h(i) = f(i) \cup F_{n_i}$, where $n_i = \min\{n \in \omega : f(i) \cup F_n \in A_i\}$.)

(b) $\forall i, j \in \omega, i \neq j, \forall x \in A_i, \forall y \in A_j, x \cap y = \emptyset$.

(If not, then fix $f : \omega \times \omega \to \omega$ a bijection and let for every $n \in \omega, f_n : \omega \to \omega, f_n(m) = f((n, m))$. Then $\mathcal{B} = \{B_i = \{f_i[x] : x \in A_i\} : i \in \omega\}$ satisfies the above-mentioned requirement and we may replace $\mathcal{A}$ by $\mathcal{B}$.)

Let $X = \omega \cup \mathcal{A}$ and consider the collection

$$\mathcal{B} = \{\emptyset\} \cup \{\{x\} : x \in \omega\} \cup \{\{A_i\} \cup F : i \in \omega, F \in A_i\}.$$

Since $\mathcal{B}$ is closed under finite intersections it follows that $\mathcal{B}$ is a base for a topology T on X. (If $i \in \omega$ and $F, G \in A_i$, then $G = F \triangle x$ for some $x \in [\omega]^{<\omega}$. Therefore, $(\{A_i\} \cup F) \cap (\{A_i\} \cup G) = \{A_i\} \cup (F \setminus x) \in \mathcal{B}$ due to (a).) Furthermore, as each A_i is a countable set, it follows that X is a first countable space.

We assert that (X, T) is T_2. Fix $x, y \in X$ such that $x \neq y$. If $x, y \in \omega$, then $\{x\}, \{y\}$ are disjoint open sets including x and y, respectively. If $x \in \omega$ and $y = A_i$, for some $i \in \omega$, then from (a), $\{x\}$ and $\{y\} \cup (F \backslash \{x\})$, where F is any element of A_i, are the required disjoint open sets including x and y, respectively. Finally, if $x = A_i, y = A_j$, where $i, j \in \omega$, $i \neq j$, then $\{x\} \cup F$, $F \in A_i$, and $\{y\} \cup G$, $G \in A_j$, are, due to (b), disjoint open sets containing x and y, respectively.

(X, T) is locally compact. If $x \in \omega$, then $\{x\}$ is certainly a compact neighborhood of x. If $x = A_i$, for some $i \in \omega$, then $V_x = \{x\} \cup F$, $F \in A_i$, is a compact neighborhood of x. (If $\mathcal{U} \subset \mathcal{B}$ is a basic open cover of V_x, then there exists a $U \in \mathcal{U}$ such that $x \in U$. Then $U = \{x\} \cup G$ for some $G \in A_i$. Since $|F \triangle G| < \omega$, it follows that $|V_x \setminus U| < \omega$. Therefore, U together with finitely many other members of $\mathcal{U}$ covers V_x. Thus, V_x is compact with respect to $\mathcal{B}$ and by Theorem 2.4, V_x is compact.)

Let $Y = (X \cup \{\infty\}, P)$, where $\infty \notin X$, be the Alexandroff one-point compactification of X. By $\mathrm{CC_0M}$, Y is metrizable, and being countable, it follows from Theorem 2.3 that Y is second countable; hence its subspace X also has a countable base $\mathcal{W}$. It follows that $Z = [\mathcal{W}]^{<\omega}$ is countable, so let $Z = \{Z_n : n \in \omega\}$. Define a function $f : \bigcup \mathcal{A} \to \omega$ by letting for each $i \in \omega$ and each $U \in A_i$, $f(U) = \min\{n \in \omega : \bigcup Z_n = \{A_i\} \cup U\}$. Note that $f(U)$ is definable since $\mathcal{W}$ is a base and the members of $\mathcal{B}$ are compact. Clearly, f is injective; hence $\bigcup \mathcal{A}$ is countable. This completes the proof of the theorem. □

Theorem 3.4. *The principle $CAC_{\triangle\omega}(\mathbb{R})$ is not provable in ZF. Hence, CC_0M is not provable in ZF.*

Proof. We shall construct a symmetric extension model $\mathcal{N}$ of a countable transitive model $\mathcal{M}$ of ZF + $V = L$, in which $\mathrm{CAC}_{\triangle\omega}(\mathbb{R})$ is false (hence, by Theorem 3.3, $\mathrm{CC_0M}$ will also be false in $\mathcal{N}$). The model $\mathcal{N}$ is a variation of Feferman's model $\mathcal{M}2$ in [7] (see also [2]). Let $\mathcal{M}$ be a countable transitive model of ZF + $V = L$ and $\mathbb{P} = \mathrm{Fn}(\omega \times \omega, 2)$ the set of all finite partial functions p with $\mathrm{dom}(p) \subset \omega \times \omega$ and $\mathrm{ran}(p) \subset 2 = \{0, 1\}$ partially ordered by reverse inclusion, that is, $p \leq q$ iff $p \supset q$. Let G be a $\mathbb{P}$-generic set over $\mathcal{M}$ and $\mathcal{M}[G]$ the corresponding generic extension of $\mathcal{M}$. Now each $X \in [\omega \times \omega]^{<\omega}$ induces an automorphism (order-preserving bijection) π_X on the partially ordered set $(\mathbb{P}, \leq)$ which is defined as follows: For all $p \in \mathbb{P}$,

$\mathrm{dom}(\pi_X p) = \mathrm{dom}(p)$ and $(\pi_X p)(n,m) = p(n,m)$ if $(n,m) \notin X$ and $1 - p(n,m)$ otherwise. Define $\mathcal{G} = \{\pi_X : X \in [\omega\times\omega]^{<\omega}\}$. Then $\mathcal{G}$ endowed with the composition operation $\circ$ is a group of automorphisms on $(\mathbb{P}, \leq)$. Indeed, it is easy to verify that for all $X, Y \in [\omega \times \omega]^{<\omega}$, $\pi_X \circ \pi_Y = \pi_{X\triangle Y}$; hence $\mathcal{G}$ is closed with respect to $\circ$. Furthermore, $\pi_\emptyset$ is the identity element of $\mathcal{G}$, and for all $X \in [\omega \times \omega]^{<\omega}$, $(\pi_X)^{-1} = \pi_X$. Consider the collection $\mathcal{E} = \{\mathrm{fix}(E) : E \in [\omega]^{<\omega}\}$, $\mathrm{fix}(E) = \{\pi_X : (X \in [\omega \times \omega]^{<\omega}) \wedge (X \cap (E \times \omega) = \emptyset)\}$, of subgroups of $\mathcal{G}$. Clearly, $\mathcal{E}$ is a filterbase since for all $E_1, E_2 \in [\omega]^{<\omega}$, $\mathrm{fix}(E_1) \cap \mathrm{fix}(E_2) = \mathrm{fix}(E_1 \cup E_2)$. Let $\mathcal{F}$ be the normal filter (see [8] on the notion of normal filter) which is generated by $\mathcal{E}$ and $\mathcal{N}$ the corresponding symmetric model of ZF.

In $\mathcal{M}[G]$, let for each $n \in \omega$, $a_n = \{m \in \omega : (\exists p \in G)p(n,m) = 1\}$. □

Claim 1. For each $n \in \omega$, the sets a_n and $[a_n]$ belong to $\mathcal{N}$, where $[x] = \{x\triangle y : y \in [\omega]^{<\omega}\}$.

Proof of Claim 1. For each $n \in \omega$, define the following names: $\underline{a_n} = \{(\check{m}, p) : (m \in \omega) \wedge (p \in \mathbb{P}) \wedge (p(n,m) = 1)\}$ and $\underline{[a_n]} = \{(\underline{a_n\triangle x}, \emptyset) : x \in [\omega]^{<\omega}\}$, where $\underline{a_n\triangle x} = \{(\check{m}, p) : (m \in (\omega\setminus x) \wedge p(n,m) = 1) \vee (m \in x \wedge p(n,m) = 0)\}$. Clearly, these sets are names for a_n and $[a_n]$, respectively, and since their elements are hereditarily symmetric, it suffices to show that they are symmetric. It is straightforward to verify that $\mathrm{fix}(\{n\}) \subset \mathrm{sym}(\underline{a_n}) \cap \mathrm{sym}(\underline{[a_n]})$, where for a name τ, $\mathrm{sym}(\tau) = \{\pi_X : (X \in [\omega \times \omega]^{<\omega}) \wedge (\pi_X(\tau) = \tau)\}$. This completes the proof of Claim 1. □

Claim 2. The family $\mathcal{A} = \{[a_n] : n \in \omega\}$ is a countable set of $\mathcal{N}$.

Proof of Claim 2. First we show that for each $n \in \omega$, $\mathcal{G} \subset \mathrm{sym}(\underline{[a_n]})$. Fix an $n \in \omega$ and a set $X \in [\omega \times \omega]^{<\omega}$. Let $x \in [\omega]^{<\omega}$ and put $z_x = \{m \in x : (n,m) \notin X\} \cup \{m \in (\omega \setminus x) : (n,m) \in X\}$. We assert that $\pi_X(\underline{a_n\triangle x}) = \underline{a_n\triangle z_x}$. Indeed, let $(\check{m}, \pi_X p) \in \pi_X(\underline{a_n\triangle x})$. We consider the following two cases:

(i) $m \in x$. Then $p(n,m) = 0$ (see the proof of Claim 1). If $(n,m) \in X$, then $m \notin z_x$ and $(\pi_X p)(n,m) = 1$. If $(n,m) \notin X$, then $m \in z_x$ and $(\pi_X p)(n,m) = 0$. In each case, $(\check{m}, \pi_X p) \in \underline{a_n\triangle z_x}$.

(ii) $m \notin x$. Then $p(n,m) = 1$ and we may continue similarly to case (i).

We deduce that $\pi_X(\underline{a_n\triangle x}) \subset \underline{a_n\triangle z_x}$. Conversely, let $(\check{m}, p) \in \underline{a_n\triangle z_x}$. Suppose that $m \notin z_x$. Then $p(n,m) = 1$. We consider the following two cases:

(a) $m \in x$ and $(n,m) \in X$. Since π_X is an automorphism on $\mathbb{P}$, let $q \in \mathbb{P}$ be such that $\pi_X(q) = p$. Necessarily we must have that $q(n,m) = 0$; hence $(\check{m}, q) \in \underline{a_n\triangle x}$, and consequently $(\check{m}, p) = (\check{m}, \pi_X(q)) \in \pi_X(\underline{a_n\triangle x})$.

(b) $m \notin x$ and $(n,m) \notin X$. This can be treated similarly to case (a).

Now suppose that $m \in z_x$. Then either $m \in x$ and $(n,m) \notin X$ or $m \notin x$ and $(n,m) \in X$. For both cases we may work similarly to cases (a) and (b) in order to verify that $(\check{m}, p) \in \pi_X(\underline{a_n\triangle x})$. Thus, $\underline{a_n\triangle z_x} \subset \pi_X(\underline{a_n\triangle x})$ and consequently $\pi_X(\underline{a_n\triangle x}) = \underline{a_n\triangle z_x}$ as required.

We may now easily conclude that $\pi_X(\underline{[a_n]}) = \underline{[a_n]}$; hence $\mathcal{G} \subset \mathrm{sym}(\underline{[a_n]})$.

Let $F = \{(\mathrm{op}(\check{n}, \underline{[a_n]}), \emptyset) : n \in \omega\}$ where $\mathrm{op}(\sigma, \tau)$ is the name given in Definition 2.16 of [13] on page 191. Clearly, F is a name for the enumeration $f = \{(n, [a_n]) : n \in \omega\}$ and from the above we infer that $\mathcal{G} \subset \mathrm{sym}(F)$; hence F is hereditarily symmetric meaning that $f \in \mathcal{N}$. This completes the proof of Claim 2. □

Claim 3. The family $\mathcal{A} = \{[a_n] : n \in \omega\}$ has no choice function in $\mathcal{N}$.

Proof of Claim 3. Assume the contrary and let $f \in \mathcal{N}$ be a choice function of $\mathcal{A}$. Let F be a hereditarily symmetric name for f and $E \in [\omega]^{<\omega}$ such that $\mathrm{fix}(E) \subset \mathrm{sym}(F)$. Let $n \in (\omega \setminus E)$ and $x \in [\omega]^{<\omega}$ such that $f([a_n]) = a_n \triangle x$. Then there is a condition $p \in G$ such that

$$p \Vdash \mathrm{op}(\underline{[a_n]}, \underline{a_n \triangle x}) \in F. \tag{1}$$

Let $m_0 \in \omega$ be such that for all $m \geq m_0$, $(n, m) \notin \mathrm{dom}(p)$ and $m \notin x$. Put $X = \{(n, m_0)\}$. Then $X \cap (E \times \omega) = \emptyset$; hence $\pi_X \in \mathrm{fix}(E)$ and so $\pi_X(F) = F$. In addition, $\pi_X p = p$ and $z_x = \{m \in x : (n, m) \notin X\} \cup \{m \in (\omega \setminus x) : (n, m) \in X\} = x \cup \{m_0\}$. From (1) and the proof of Claim 2 it follows that

$$p \Vdash \mathrm{op}(\underline{[a_n]}, \underline{a_n \triangle (x \cup \{m_0\})}) \in F. \tag{2}$$

Since $p \in G$, from (2) we deduce that $f([a_n]) = a_n \triangle (x \cup \{m_0\})$, which is a contradiction. This completes the proof of Claim 3 and of the theorem. □

We show next that, besides $\mathrm{CC_0M}$, the model $\mathcal{N}$ of the proof of Theorem 3.4 fails to satisfy certain other propositions which are standard theorems of ZFC topology. These statements concern topological sums and Tychonoff products of spaces sharing particular properties and have been studied in [6], [11], and [12] (without the requirement that the coordinate spaces are countable).

Theorem 3.5. *Each one of the following statements fails in the model $\mathcal{N}$ of the proof of Theorem* 3.4.

(i) *Topological sums of countably many countable compact metrizable spaces are metrizable, hence normal.*
(ii) *Topological sums of countably many countable paracompact spaces are paracompact.*
(iii) *Topological sums of countably many countable metacompact spaces are metacompact.*
(iv) *Tychonoff products of countably many countable compact metrizable spaces are compact metrizable.*
(v) *Tychonoff products of countably many countable second countable spaces are second countable.*

Proof. Consider the family $\mathcal{A} = \{A_n = [a_n] : n \in \omega\}$ defined in the proof of Theorem 3.4. Let (X, T) be the topological space defined in the proof of Theorem 3.3. For each $n \in \omega$, consider the subspace $Y_n = \{A_n\} \cup (\bigcup\{F : F \in A_n\})$ of X and let $Z_n = Y_n \cup \{\infty_n\}$, $\infty_n \notin Y_n$, be the one-point compactification of the locally compact T_2 space Y_n. (Assume that the ∞_n's are pairwise distinct.) Since A_n is a countable set, it follows that $[A_n]^{<\omega}$ is countable, and consequently the family $\mathcal{C}_n = \mathcal{B}_n \cup \{Z_n \setminus (\bigcup Q) : Q \in [\mathcal{B}_n]^{<\omega}\}$, where $\mathcal{B}_n$ is the restriction of the base $\mathcal{B}$ of X to Y_n, is a countable base for Z_n. As Z_n is compact T_2, it is regular; thus by UMT, Z_n is metrizable. Let $W = \sum_{n \in \omega} W_n$, $W_n = Z_n \times \{n\}$, be the topological sum of the spaces W_n, $n \in \omega$. Clearly the sets $A = \{(A_n, n) : n \in \omega\}$ and $B = \{(\infty_n, n) : n \in \omega\}$ are disjoint and closed. Suppose that U_A and U_B are disjoint open sets containing A and B, respectively. Then for each $n \in \omega$, $V_n = \{F \in A_n : (U_B \cap W_n) \cap ((\{A_n\} \cup F) \times \{n\}) = \emptyset\}$ is clearly finite. Furthermore, since $V_n \subset \wp(\omega)$ and $|\wp(\omega)| = |\mathbb{R}|$, we may choose, for each $n \in \omega$, an element $F_n \in V_n$. Then $f = \{(n, F_n) : n \in \omega\}$ is a choice function for $\mathcal{A}$ contradicting Claim 3 of the proof of Theorem 3.4. Thus, W is not normal, hence not metrizable.

Clearly, W_n is paracompact, hence metacompact, for all $n \in \omega$. However, the sum W fails to be metacompact in $\mathcal{N}$. Assume the contrary and let $\mathcal{C} = \{C \times \{n\} : n \in \omega, C \in \mathcal{C}_n\}$ and $\mathcal{D}$ a point finite open refinement of the cover $\mathcal{C}$. For each $n \in \omega$, let $D_n = \{D \in \mathcal{D} : (\infty_n, n) \in D\}$. Then D_n is finite for all $n \in \omega$, and we may continue similarly to the proof of the last paragraph in order to define a choice function for $\mathcal{A}$. This is a contradiction; hence W is not metacompact. Therefore, (ii) and (iii) fail in the model $\mathcal{N}$.

In [11] it is shown that the statements (iv) and (v) (without the requirement that the coordinate spaces are countable) imply the weak choice axiom CAC_ω, where CAC_ω is AC restricted to countable families of nonempty, countable sets. Since CAC_ω fails for the family $\mathcal{A} = \{[a_n] : n \in \omega\}$ it follows that (iv) and (v) fail in $\mathcal{N}$. The proof of the theorem is complete. □

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