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THE GLOBAL ATTRACTIVITY OF THE RATIONAL DIFFERENCE EQUATION $y_n = 1 + \frac{y_{n-k}}{y_{n-m}}$

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ABSTRACT. This paper studies the behavior of positive solutions of the recursive equation

$$y_n = 1 + \frac{y_{n-k}}{y_{n-m}},\ n = 0, 1, 2, \ldots,$$

with $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$ and $k, m \in \{1, 2, 3, 4, \ldots\}$, where $s = \max\{k, m\}$. We prove that if $\gcd(k, m) = 1$, with k odd, then y_n tends to 2, exponentially. When combined with a recent result of E. A. Grove and G. Ladas (*Periodicities in Nonlinear Difference Equations*, Chapman & Hall/CRC Press, Boca Raton (2004)), this answers the question when $y = 2$ is a global attractor.

1. INTRODUCTION

This paper studies the behavior of positive solutions of the recursive equation

$$y_n = 1 + \frac{y_{n-k}}{y_{n-m}},\ n = 0, 1, \ldots, \tag{1}$$

with $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$ and $k, m \in \{1, 2, 3, 4, \ldots\}$, where $s = \max\{k, m\}$.

The study of properties of rational difference equations has been an area of intense interest in recent years; cf. [7], [8] and the references therein.

In [9], the authors proved that if $(k, m) = (2, 3)$, then every positive solution of (1) converges to a period two solution. More generally, it follows from Theorem 5.3 in [7] that if k is even and m is odd, then every positive solution of (1) converges to a nonnegative periodic solution with period $2\gcd(m, k)$. For a discussion of related equations, see also [1], [2], [3], [4], [6] and [11]. Here we prove the following complimentary result which answers the question when $y = 2$ is a global attractor.

Theorem 1. *Suppose that* $\gcd(m, k) = 1$ *and that* $\{y_i\}$ *satisfies* (1) *with* $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$ *where* $s = \max\{m, k\}$. *Then, if k is odd, the sequence* $\{y_i\}$ *converges to the unique equilibrium* 2.

The paper proceeds as follows. In Section 2, we introduce some preliminary lemmas and notation. Section 3 contains a proof of Theorem 1, while in Section

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4, exponential convergence of solutions to (1) is examined. Section 5 then combines Theorem 1 with existing results to fully determine the periodic character for solutions to equation (1).

2. Preliminaries and notation

In this section, we introduce some preliminary lemmas and notation. Since the case $k = m$ is trivial, we will assume, throughout, that $k \neq m$.

First, consider the transformed sequence $\{y_i^*\}$ defined by

$$y_i^* = \begin{cases} y_i, & \text{if } y_i \geq 2, \\ 3 - \frac{1}{y_i - 1}, & \text{otherwise,} \end{cases} \tag{2}$$

for $i \geq 0$. As well, define the sequence $\{\delta_i\}$ via $\delta_i = |2 - y_i^*|$ for $i \geq 0$. The following inequality will be crucial to our arguments.

Lemma 1. *We have*

$$\delta_n \leq \max\{\delta_{n-k}, \delta_{n-m}\}, \tag{3}$$

for all $n \geq s$.

Proof. Suppose that

$$\max\{\delta_{n-k}, \delta_{n-m}\} < \delta_n. \tag{4}$$

If $y_n > 2$, then (4) implies that $y_{n-k} < y_n$ and $y_{n-m} > \frac{1}{y_n - 1} + 1$. To see the second inequality note that the result is trivial when $y_{n-m} \geq 2$, and if $y_{n-m} \leq 2$, then it follows from (2) and the fact that $y_n - 2 > 2 - y_{n-m}^* = \frac{1}{y_{n-m} - 1} - 1$. Hence,

$$y_n = 1 + \frac{y_{n-k}}{y_{n-m}} < 1 + \frac{y_n}{\frac{1}{y_n - 1} + 1} = y_n. \tag{5}$$

Similarly, if $y_n < 2$, we have $y_{n-k} > y_n$ and $y_{n-m} < \frac{1}{y_n - 1} + 1$, and hence,

$$y_n = 1 + \frac{y_{n-k}}{y_{n-m}} > 1 + \frac{y_n}{\frac{1}{y_n - 1} + 1} = y_n. \tag{6}$$

In either case, we have a contradiction, and the lemma follows. □

Now, set

$$D_n = \max_{n-s \leq i \leq n-1} \{\delta_i\}, \tag{7}$$

for $n \geq s$.

The following lemma is a simple consequence of Lemma 1 and (7).

Lemma 2. *The sequence* $\{D_i\}$ *is monotonically nonincreasing in* i, *for* $i \geq s$.

Since $D_i \geq 0$ for $i \geq s$, Lemma 2 implies that, as i tends to infinity, the sequence $\{D_i\}$ converges to some limit, say D, where $D \geq 0$.

We now turn to a proof of Theorem 1.

3. Convergence of solutions to equation (1)

In this section, we prove Theorem 1.

Proof of Theorem 1. Note that it suffices to show that the transformed sequence $\{y_i^*\}$ converges to 2.

By the definition in (7), the values of D_i are taken on by entries in the sequence $\{\delta_j\}$, and as well, by Lemma 1, $y_i^* \in [2-D_i, 2+D_i]$ for $i \geq s$. Suppose $D > 0$. Then, for any $\epsilon \in (0, D)$, we can find an N such that $y_N^* \in [2-D-\epsilon, 2-D+\epsilon]$ or $y_N^* \in [2+D-\epsilon, 2+D+\epsilon]$ and for $i \geq N - mk - s$,

$$y_i^* \in [2-D-\epsilon, 2+D+\epsilon]. \tag{8}$$

Suppose that

$$y_N^* \in [2+D-\epsilon, 2+D+\epsilon]. \tag{9}$$

Note that for ϵ sufficiently small, the hypotheses above guarantee that $y_{N-k} \geq 2$ and $y_{N-m} \leq 2$, where at least one of the inequalities is strict. To see this, suppose that for instance $y_{N-k} \geq 2$ and $y_{N-m} \geq 2$. Then

$$y_N^* = 1 + \frac{y_{N-k}^*}{y_{N-m}^*} \leq 1 + \frac{2+D+\epsilon}{2} = 2 + D - \epsilon - \left(\frac{D}{2} - \frac{3}{2}\epsilon\right) < 2 + D - \epsilon \tag{10}$$

for ϵ sufficiently small, since $D > 0$. Equation (10) then contradicts the assumption in (9). In the case $y_{N-k} \leq 2$ and $y_{N-m} \leq 2$, we have

$$\begin{aligned} y_N^* &= 1 + \frac{1 + \frac{1}{3-y_{N-k}^*}}{1 + \frac{1}{3-y_{N-m}^*}} \leq 1 + \frac{2}{1 + \frac{1}{3-(2-D-\epsilon)}} = 1 + \frac{2}{1 + \frac{1}{1+D+\epsilon}} \\ &= 2 + D - \epsilon - \frac{D^2 + D - (3\epsilon + \epsilon^2)}{2 + D + \epsilon} < 2 + D - \epsilon \end{aligned} \tag{11}$$

for ϵ sufficiently small, since $D > 0$. Again we obtain a contradiction to the assumption in (9).

If $y_{N-k} \leq 2$ and $y_{N-m} \geq 2$, then $y_N = 1 + y_{n-k}/y_{n-m} \leq 2$, which again contradicts (9).

Thus, assume that $y_{N-k} \geq 2$ and $y_{N-m} \leq 2$. Solving for y_{N-k}^* and y_{N-m}^* in

$$y_N^* = y_N = 1 + \frac{y_{N-k}}{y_{N-m}} = 1 + \frac{y_{N-k}^*}{1 + \frac{1}{3-y_{N-m}^*}}, \tag{12}$$

we have

$$y_{N-k}^* = (y_N^* - 1)\left(1 + \frac{1}{3 - y_{N-m}^*}\right) \tag{13}$$

and

$$y_{N-m}^* = 3 - \frac{1}{\frac{y_{N-k}^*}{y_N^*-1} - 1}. \tag{14}$$

Employing the inequalities in (8) and (9) in (13) and (14) gives

$$\begin{aligned} 2+D+\epsilon \geq y^*_{N-k} &\geq (1+D-\epsilon)\left(1+\frac{1}{3-(2-D-\epsilon)}\right) \\ &= (1+D-\epsilon)\left(\frac{2+D+\epsilon}{1+D+\epsilon}\right) = 2+D-\epsilon\left(\frac{3+D+\epsilon}{1+D+\epsilon}\right) \\ &\geq 2+D-\epsilon\left(\frac{3+D}{1+D}\right) \qquad (15) \end{aligned}$$

and

$$\begin{aligned} 2-D-\epsilon \leq y^*_{N-m} &\leq 3-\frac{1}{\frac{2+D+\epsilon}{1+D-\epsilon}-1} \\ &= 3-\frac{1+D-\epsilon}{1+2\epsilon} = 2-D+\epsilon\left(\frac{3+2D}{1+2\epsilon}\right) \\ &\leq 2-D+\epsilon(3+2D). \qquad (16) \end{aligned}$$

Thus

$$2+D+\epsilon\left(\frac{3+D}{1+D}\right) \geq y^*_{N-k} \geq 2+D-\epsilon\left(\frac{3+D}{1+D}\right) \qquad (17)$$

and

$$2-D-\epsilon(3+2D) \leq y^*_{N-m} \leq 2-D+\epsilon(3+2D). \qquad (18)$$

Similarly when $y^*_N \in [2-D-\epsilon, 2-D+\epsilon]$, $y_{N-k} \leq 2$ and $y_{N-m} \geq 2$, we have

$$2+D+\epsilon\left(\frac{3+D}{1+D}\right) \;\geq\; y^*_{N-m} \;\geq\; 2+D-\epsilon\left(\frac{3+D}{1+D}\right) \qquad (19)$$

and

$$2-D-\epsilon(3+2D) \;\leq\; y^*_{N-k} \;\leq\; 2-D+\epsilon(3+2D). \qquad (20)$$

Let $B = 3+2D > \frac{3+D}{1+D}$. Then, when $y^*_N \in [2+D-\epsilon, 2+D+\epsilon]$, iterating the above arguments gives

$$\begin{aligned} 2+D+\epsilon B &\geq y^*_{N-k} \geq 2+D-\epsilon B, \\ 2+D+\epsilon B^2 &\geq y^*_{N-2k} \geq 2+D-\epsilon B^2, \\ &\vdots \\ 2+D+\epsilon B^m &\geq y^*_{N-mk} \geq 2+D-\epsilon B^m \qquad (21) \end{aligned}$$

and

$$\begin{aligned} 2-D-\epsilon B &\leq y^*_{N-m} \leq 2-D+\epsilon B, \\ 2+D-\epsilon B^2 &\leq y^*_{N-2m} \leq 2+D+\epsilon B^2, \\ &\vdots \\ 2+(-1)^k D-\epsilon B^k &\leq y^*_{N-km} \leq 2+(-1)^k D+\epsilon B^k. \qquad (22) \end{aligned}$$

Since k is odd, (21) and (22) give that $y^*_{N-mk} \leq 2-D+\epsilon B^k$ and $y^*_{N-mk} \geq 2+D-\epsilon B^m$. Thus, for sufficiently small ϵ, we obtain a contradiction to the hypothesis that $D > 0$. A similar argument works when $y^*_N \in [2-D-\epsilon, 2-D+\epsilon]$, and the result is proven. $\square$

Remark 1. Note that in (22), the parity of k was crucial to obtaining the contradiction to $D > 0$.

In the next section, we show that (for k odd), the convergence of solutions to (1) is actually exponential.

4. Exponential convergence of solutions to (1)

In the previous section it was shown that for k odd, and $\gcd(k, m) = 1$, all solutions to (1) converge to the unique equilibrium. Here, we employ the following lemma from [10] to prove that the convergence is in fact exponential.

Lemma 3. *Suppose that* $\{a_n\}$ *is a sequence of positive numbers which satisfies the inequality*

$$a_{n+k} \leq A \max\{a_{n+k-1}, a_{n+k-2}, \ldots, a_n\} \tag{23}$$

for $n \in \mathbb{N}$, *where* $A \in (0, 1)$ *and* $k \in \mathbb{N}$ *are fixed. Then, there exists an* $L \in \mathbb{R}_+$ *such that*

$$a_{km+r} \leq LA^m, \tag{24}$$

for all $m \in \mathbb{N} \cup \{0\}$ *and* $1 \leq r \leq k$.

Proof. See [10], Lemma 1. □

We now prove the following.

Theorem 2. *Suppose that* $\gcd(m, k) = 1$, *k is odd and* $\{y_i\}$ *satisfies* (1) *with positive initial conditions. Then* y_i *converges to* 2 *exponentially.*

Proof. Suppose $N > mk$, and set

$$z_i = 2 - y_{N-i} \tag{25}$$

for $0 \leq i \leq mk$. Now, note that

$$\begin{aligned} z_0 = 2 - y_N &= \frac{y_{N-m} - y_{N-k}}{y_{N-m}} \\ &= \frac{z_k - z_m}{2 - z_m}. \end{aligned} \tag{26}$$

Applying (26) successively for $i = k, 2k, 3k, \ldots, (m-1)k$ gives

$$z_0 = \frac{z_{mk}}{\prod_{v=0}^{m-1}(2 - z_{vk+m})} - \sum_{j=0}^{m-1} \frac{z_{jk+m}}{\prod_{v=0}^{j}(2 - z_{vk+m})}. \tag{27}$$

Similarly, applying (26) successively for $i = m, 2m, 3m, \ldots, (k-1)m$ gives

$$-z_m = (-1)^k \frac{z_{km}}{\prod_{v=2}^{k}(2 - z_{vm})} + \sum_{j=1}^{k-1} (-1)^j \frac{z_{jm+k}}{\prod_{v=2}^{j+1}(2 - z_{vm})}. \tag{28}$$

Employing (28) in (27) gives

$$\begin{aligned} z_0 &= \frac{z_{mk}}{\prod_{v=0}^{m-1}(2 - z_{vk+m})} - \sum_{j=1}^{m-1} \frac{z_{jk+m}}{\prod_{v=0}^{j}(2 - z_{vk+m})} \\ &\quad + (-1)^k \frac{z_{km}}{\prod_{v=1}^{k}(2 - z_{vm})} + \sum_{j=1}^{k-1} (-1)^j \frac{z_{jm+k}}{\prod_{v=1}^{j+1}(2 - z_{vm})}. \end{aligned} \tag{29}$$

Hence, since k is odd, we have

$$\begin{aligned} |z_0| &\leq |z_{mk}| \left| \frac{1}{\prod_{v=0}^{m-1}(2-z_{vk+m})} - \frac{1}{\prod_{v=1}^{k}(2-z_{vm})} \right| \\ &\quad + \left| \sum_{j=1}^{m-1} \frac{z_{jk+m}}{\prod_{v=0}^{j}(2-z_{vk+m})} \right| + \left| \sum_{j=1}^{k-1} (-1)^j \frac{z_{jm+k}}{\prod_{v=1}^{j+1}(2-z_{vm})} \right| \\ &\leq C_N \max\{|z_1|, |z_2|, \ldots, |z_{mk+m+k}|\}, \end{aligned} \tag{30}$$

where

$$\begin{aligned} C_N &\stackrel{\text{def}}{=} \left| \frac{1}{\prod_{v=0}^{m-1}(2-z_{vk+m})} - \frac{1}{\prod_{v=1}^{k}(2-z_{vm})} \right| + \sum_{j=1}^{m-1} \left| \frac{1}{\prod_{v=0}^{j}(2-z_{vk+m})} \right| \\ &\quad + \sum_{j=1}^{k-1} \left| \frac{1}{\prod_{v=1}^{j+1}(2-z_{vm})} \right|. \end{aligned} \tag{31}$$

Now, note that as $N \to \infty$, $z_i \to 0$ for $i \in \{1, 2, \ldots, mk+m+k\}$ and hence setting $r = 1/2$, we have

$$\begin{aligned} \lim_{N\to\infty} C_N &= |r^m - r^k| + \sum_{j=1}^{m-1} r^{j+1} + \sum_{j=1}^{k-1} r^{j+1} \\ &= |r^m - r^k| + 2r^2(1-r^{m-1}) + 2r^2(1-r^{k-1}) \\ &= |r^m - r^k| + 1 - r^m - r^k \leq 1 - 2r^{\max\{m,k\}} < 1. \end{aligned} \tag{32}$$

The result then follows upon applying Lemma 3, above. □

In the next section, we combine Theorem 1 with existing results to fully determine the periodic character for solutions to Equation (1).

5. The periodic character of equation (1)

In this section we combine a recent theorem of Grove and Ladas with the result in Theorem 1 to determine the periodic character of equation (1).

First, note that (as in [5]), if $g = \gcd(m,k) > 1$, then $\{y_i\}$ can be separated into g different equations of the form

$$y_n^{(j)} = 1 + \frac{y_{n-\frac{k}{g}}^{(j)}}{y_{n-\frac{m}{g}}^{(j)}}, \tag{33}$$

where $j \in \{1, 2, \ldots, g\}$. Hence, we may assume that $\gcd(m,k) = 1$.

In [7] the authors proved the following.

Theorem 3. *Suppose that* $\gcd(m,k) = 1$ *with* $k \geq 2$ *even and* $m \geq 1$ *odd. Then every positive solution of* (1) *converges to a nonnegative solution of* (1) *with period* 2.

Proof. See [7], Theorem 5.3. □

The next theorem follows upon application of Theorems 1, 2 and 3.

Theorem 4. *Suppose that* $2^i \| m$ *(i.e.* 2^i *is the largest power of* 2 *which divides* m*). Then, every solution of* (1) *converges to a period* t *solution, where* t *is given by*

$$t = \begin{cases} 1, & \text{if } 2^{i+1} \not| k, \\ 2\gcd(m,k), & \text{otherwise.} \end{cases} \tag{34}$$

Additionally, if $t = 1$*, then all solutions converge exponentially to the value* 2.

Remark 2. Note that the argument used to prove Theorem 1 can be modified to show that in the case that $\gcd(m,k) = 1$ with k even, the period two solution for $\{y_n^*\}$ is in fact of the form

$$\ldots, 2-D, 2+D, 2-D, 2+D, \ldots, \tag{35}$$

where D is defined as in Section 2.

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