COMPACTNESS PROPERTIES OF OPERATORS DOMINATED BY AM-COMPACT OPERATORS

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(Communicated by Joseph A. Ball)

Abstract. We study several properties about the problem of domination in the class of positive AM-compact operators, and we obtain some interesting consequences on positive compact operators. Also, we give a sufficient condition under which a Banach lattice is discrete.

1. Introduction

An interesting problem in the operator theory on Banach lattices is that of finding conditions under which properties of a positive operator $T$ will be inherited by any positive operator smaller than (or dominated by) $T$. In other words, if $E$ and $F$ are two Banach lattices and $S, T$ are two operators from $E$ into $F$ such that $0 \leq S \leq T$, we have to study conditions on $E$ and $F$ under which a nice property of $T$ will be inherited by $S$.

For compact operators, this problem was studied by Dodds-Fremlin [8], Aliprantis-Burkinshaw [11], Wickstead [16, 17] and Aqzzouz-Nouira [7]. The domination problem for weakly compact operators was studied simultaneously by Aliprantis-Burkinshaw [2] and Wickstead [15]. For Dunford-Pettis operators, this problem was studied by Aliprantis-Burkinshaw [3], Kalton-Saab [13] and Wickstead [16]. Also, Flores-Hernandez studied the domination problem for disjointly strictly singular operators [9] and strictly singular operators [10]. Finally, Flores-Ruiz [11] obtained some interesting results for the class of narrow operators.

The problem of domination in the class of positive AM-compact operators was originally studied by Fremlin in [12]. He showed that if the norm of $F$ is order continuous, then the subspace of all AM-compact operators from $E$ into $F$ is a band (i.e. an order ideal which is order closed). Recall from Zaanen [18] that a regular operator $T$ from a vector lattice $E$ into a Banach lattice $F$ is said to be AM-compact if it carries each order bounded subset of $E$ onto a relatively compact subset of $F$.

Our objective in this paper is to continue the investigation of the domination problem for the class of AM-compact operators. First, we will prove that the second power of an AM-compactly dominated operator is always AM-compact. Also, we shall give a necessary and sufficient condition for when the AM-compactness of a positive operator which is AM-compactly dominated is inherited. As a consequence,
we will obtain some interesting and well-known properties on the domination problem for positive compact operators (Theorem 2.2 of [1] and Theorem 2.1 of [7]). Finally, we will give a sufficient condition under which the topological dual of a Banach lattice is discrete.

2. Major results

A subset $D$ of a Banach lattice $E$ is said to be almost order bounded if for each $\varepsilon > 0$, there exists some $x \in E^+$ such that $D \subset [−x, x] + \varepsilon B_E$, where $B_E$ is the unit ball of $E$. An operator $T$ from a Banach space $E$ into a Banach lattice $F$ is said to be semi-compact if it maps bounded subsets of $E$ onto almost order bounded subsets of $F$. The class of semi-compact operators fails to satisfy the duality problem, but it follows from Theorem 18.20 of [4], that the class of semi-compact operators satisfies the domination property. For unexplained terminology on Banach lattice theory, we refer to Zaanen [18].

It was well known that if $S$ and $T$ are operators from $E$ into $E$ such that $0 \leq S \leq T$ and $T$ is AM-compact and L-weakly compact (resp. M-weakly compact), then $S^2$ is compact ([4], Exercice 12 (resp. Exercice 13), p. 331). The following evident proposition gives a similar result.

**Proposition 2.1.** Let $E$ be a Banach lattice, and let $S$ and $T$ be operators from $E$ into $E$ such that $0 \leq S \leq T$ with $T$ semi-compact and $S$ AM-compact. Then $S^2$ is compact.

**Proof.** In fact, if $T$ is semi-compact, then $S$ is too (Theorem 18.20 of [4]), and since $S$ is AM-compact, then the second power operator $S^2$ is compact.

If $E'$ is the topological dual of $E$, the absolute weak topology $|\sigma|(E, E')$ is the locally convex solid topology on $E$ generated by the family of lattice seminorms $\{P_f : f \in E'\}$, where $P_f(x) = |f|(|x|)$ for each $x \in E$. Similarly, $|\sigma|(E', E)$ is the locally convex solid topology on $E'$ generated by the family of lattice seminorms $\{P_x : x \in E\}$, where $P_x(f) = |f|(|x|)$ for each $f \in E'$. For more information, we refer the reader to [5].

We will need the following lemma which is a consequence of a Grothendieck’s Theorem ([14], Theorem 3, p. 51).

**Lemma 2.2.** Let $E$ and $F$ be two Banach lattices and let $T : E \rightarrow F$ be an operator. Then for each $x \in E^+$, $T ([0, x])$ is norm precompact in $F$ if and only if $T' (B_{E'})$ is precompact for $|\sigma|(E', E)$ in $E'$.

Without any conditions on the Banach lattice, we have the following result.

**Theorem 2.3.** Let $E$ be a Banach lattice and let $S$ and $T$ be two operators from $E$ into $E$ such that $0 \leq S \leq T$ and $T$ is AM-compact; then $S^2$ is AM-compact.

**Proof.** Let $x \in E^+$; then $T ([0, x])$ is norm precompact in $E$. Since the topology defined by the norm of $E$ is finer than $|\sigma|(E, E')$, then $T ([0, x])$ is precompact for $|\sigma|(E, E')$. An application of Lemma 1.1 of [3] implies that $S ([0, x])$ is precompact for $|\sigma|(E, E')$. It now follows from Theorem 1.2 of [11] that $S' ([0, f])$ is precompact for $|\sigma|(E', E)$ for each $f \in (E')^+$. Hence $S'$ maps almost order bounded subsets of $E'$ onto precompact subsets for $|\sigma|(E', E)$.

On the other hand, if $B_{E'}^+$ denotes $B_{E'} \cap (E')^+$, then $S' (B_{E'}^+)$ is almost order bounded. It follows that $(S')^2 (B_{E'}^+)$ is precompact for $|\sigma|(E', E)$. Now, Lemma
2.2 implies that $S^2([0,x])$ is norm relatively compact for each $x \in E^+$. Therefore $S^2$ is AM-compact.

By using the same arguments as in Example 3.2 of [1], we obtain the following consequence.

Corollary 2.4. Let $E$, $F$ and $G$ be Banach lattices. Let $S_1, T_1 : E \to F$ and $S_2, T_2 : F \to G$ be operators such that $0 \leq S_i \leq T_i$ and each $T_i$ is AM-compact, $i = 1, 2$. Then $S_2 S_1$ is an AM-compact operator.

Let us recall that a vector lattice $E$ is said to be order complete if each nonempty subset that is bounded from above has a supremum. Another consequence is the following result.

Corollary 2.5. Let $E$ be an order complete Banach lattice. If $T$ is a regular operator such that $|T|$ is AM-compact, then the second power operator $T^2$ is AM-compact.

Proof. In fact, $(T)^2 = (T^+ - T^-)^2 = (T^+)^2 - T^+ T^- - T^- T^+ + (T^-)^2$ with $0 \leq T^+ \leq |T|$, $0 \leq T^- \leq |T|$ and $|T|$ is an AM-compact operator. The assertion follows immediately from Theorem 2.3 and Corollary 2.4.

Remark 2.6. There exist Banach lattices $E$ and $F$ and there exist positive operators $S$ and $T$ from $E$ into $F$ such that $0 \leq S \leq T$, with $T$ being AM-compact but with $S$ not being AM-compact. In fact

Examples 2.7. 1. Let $S_1$, $T_1$, $S_2$ and $T_2$ be the positive operators defined in Example 3.1 of [1]. We have $0 \leq S_i \leq T_i$ for $i = 1, 2$, and each $T_i$ is compact. In [1], it was proved that $S_2 S_1$ is not compact. We have to show that $S_2$ is not AM-compact. If not, since $0 \leq S_1 \leq T_1$ and the operator $T_1$ is semi-compact, then $S_1$ is semi-compact (Theorem 18.20 of [1]). Finally, $S_2 S_1$ is a compact operator as a product of a semi-compact operator with an AM-compact operator. But this is impossible.

2. Let $E$ be the Banach lattice $l^1 \oplus L^2 \oplus l^\infty$ and let $S, T : E \to E$ be the operators defined in [1], Example 3.2, where $T_1, T_2, S_1$ and $S_2$ are the operators of the above example. It is clear that $0 \leq S \leq T$ and $S$ is AM-compact but $S^2$ is not compact, and hence $S$ is not AM-compact.

Remarks 2.8. 1. If $R$, $S$ and $T$ are operators from $E$ into $E$ such that $R \leq S \leq T$ and $R$, $T$ are AM-compact, then $S^2$ is a AM-compact operator. In fact, since $0 \leq S - R \leq T - R$ and $T - R$ is an AM-compact operator, the second power operator $(S - R)^2 = S^2 - RS - SR + R^2$ is AM-compact. The result follows.

2. If $E$ is an infinite-dimensional AM-space with unit, there is no positive AM-compact operator $T$ on $E$ such that $0 \leq Id_E \leq T$, where $Id_E$ is the identity operator of $E$. In fact, whenever $E$ is an AM-space with unit, the class of AM-compact operators on $E$ coincides with the class of regular compact operators on $E$.

Recall that a nonzero element $x$ of a vector lattice $E$ is discrete if the order ideal generated by $x$ equals the subspace generated by $x$. The vector lattice $E$ is discrete if it admits a complete disjoint system of discrete elements.

A norm $|||\cdot|||$ of a Banach lattice $E$ is order continuous if for each net $(x_\alpha)$ such that $x_\alpha \downarrow 0$ in $E$, the sequence $(x_\alpha)$ converges to 0 for the norm $|||\cdot|||$, where the
from AM-compact. It is clear that $S$ is AM-compact (Exercice 12 of [4], p. 331). The following result gives a sufficient condition on the Banach lattice, under which the AM-compactness of a positive operator $T$ will be inherited by any positive operator smaller than $T$.

**Proposition 2.9.** Let $E$ and $F$ be Banach lattices and let $S$ and $T$ be operators from $E$ into $F$ such that $0 \leq S \leq T$ and $T$ is AM-compact. If for each $x \in E^+$ the vector lattice $(E_x)'$ is discrete, then the operator $S$ is AM-compact.

**Proof.** Let $S$ and $T$ be operators from $E$ into $F$ such that $0 \leq S \leq T$ and $T$ is AM-compact. It is clear that $S$ is AM-compact if and only if for each $x \in E^+$, the restriction $S|_{E_x}$ from $E_x$ into $F$ is compact, where $E_x$ is the order ideal generated by $x$. Since $T|_{E_x}$ from $E_x$ into $F$ is compact, $0 \leq S|_{E_x} \leq T|_{E_x}$ and $(E_x)'$ is discrete with an order continuous norm, it follows from Theorem 1 of [10] that $S|_{E_x}$ is compact. This proves the result.

The following theorem gives a necessary and sufficient condition for which the domination problem admits a positive solution for the class of positive AM-compact operators.

**Theorem 2.10.** Let $E$ and $F$ be Banach lattices. Then the following statements are equivalent:

1) For all operators $S, T : E \rightarrow F$ such that $0 \leq S \leq T$ and $T$ is AM-compact, the operator $S$ is AM-compact.

2) One of the following conditions holds:

i. The norm of $F$ is order continuous.

ii. The topological dual $E'$ is discrete.

**Proof.** The implication $i \implies 1$ is just a theorem of Fremlin [12].

For the implication $ii \implies 1$ let $S$ and $T$ be operators from $E$ into $F$ such that $0 \leq S \leq T$ and $T$ is AM-compact. Then for each $x \in E^+$, $T([0,x])$ is norm precompact in $F$, and hence $T'(B_{F'})$ is precompact for $|\sigma|(E',E)$ (Lemma 2.2). Since $0 \leq S' \leq T'$, it results from Theorem 3.1.b of [6] that $S'(B_{F'})$ is also precompact for $|\sigma|(E',E)$. A second application of Lemma 2.2 gives the result.

$1 \implies 2$. Assume that either of the conditions $i$ and $ii$ is true. Since the norm of $F$ is not order continuous, there exist some $z \in F^+$ and a disjoint sequence $(z_n)$ in $[0,z]$, which does not admit any subsequence converging to 0 for the norm (Theorem 3.22 of [5]). Also, there exist some $\Phi \in (E')^+$ and a sequence $(\Phi_n)$ in $[0,\Phi]$, which converges to 0 for the weak topology $\sigma(E',E)$ but does not converge to 0 for the absolute weak topology $|\sigma|(E',E)$ (Corollary 6.57 of [5]). This implies that there exists some $y \in E^+$ and a sequence $(y_n)$ in $[0,y]$ such that $\Phi_n(y_n) = 1$ for each $n \in \mathbb{N}$.

Let $\hat{E}$ be the completion of $E$ for the absolute weak topology $|\sigma|(E,E')$, and let $P_n$ be the principal projection on the band $B_n$ generated by $y_n$ in $\hat{E}$. We can assume that $\Phi_n(y_n) = 0$ if $n \neq m$ (if not, we replace $\Phi_n$ by $\Phi_n \circ P_n$).

Let $S$ be the positive operator defined by $S(x) = (\sum_{n=1}^{+\infty} \Phi_n(x) z_n) + \Phi(x) z$ for each $x \in E^+$. Since $(z_n)$ is a disjoint sequence and $(\Phi_n)$ converges to 0 weakly, the operator $S$ is well defined.
The operator $S$ is not AM-compact. If not, the sequence $(S(y_n)) = (\Phi(y_n)z + zn)$ admits a convergent subsequence that we also denote by $(\Phi(y_n)z + zn)$. But since the sequence $(\Phi(y_n))$ admits a convergent subsequence, it follows that $(zn)$ admits a convergent subsequence. This presents a contradiction, and hence $S$ is not AM-compact. However the operator $T$ defined by $T(x) = 2\Phi(x)z$ is AM-compact and $0 \leq S \leq T$. This completes the proof.

Now, as a consequence, we obtain Theorem 2.2 of [1] and Theorem 2.1 of [7].

**Corollary 2.11.** Let $E$ be a Banach lattice. Then for each pair of operators $S$ and $T$ from $E$ into $E$ such that $0 \leq S \leq T$ with $T$ compact, the operator $S^2$ is compact if one of the following assertions is valid:
1. The norm of $E$ is order continuous.
2. For each $x \in E^+$, $(E_x)'$ is discrete.
3. The topological dual $E'$ is discrete.
4. The norm of $E'$ is order continuous.

**Proof.** For assertions 1 and 3 (resp. 2) it follows from Theorem 2.10 (resp. Proposition 2.9) that $S$ is AM-compact and an application of Proposition 2.1 implies the assertion.

For assertion 4, since $0 \leq S' \leq T'$ and the norm of $E'$ is order continuous, the result follows from assertion 1.

**Remark 2.12.** If $T : E \to F$ is a bounded operator between two Banach lattices, Aliprantis and Burkinshaw [4] defined the ring ideal $Ring(T)$ generated by $T$ as the norm closure in $L(E,F)$ of the vector subspace consisting of all operators of the form $\sum_{i=1}^n R_iTS_i$, where $S_i \in L(E,E)$ and $R_i \in L(F,F)$, and where $L(E,F)$ is the Banach space of all norm bounded operators from $E$ into $F$. They proved that if $E = F$ and if $S : E \to E$ is another operator that satisfies $0 \leq S \leq T$ such that $T$ is compact, then we have:
- a) $S^3 \in Ring(T)$ (in particular $S^3$ is compact).
- b) $S^2$ belongs to $Ring(T)$ (in particular $S^2$ is compact) whenever $E$ has an order continuous norm.

It is natural to ask if we can obtain similar results for the class of AM-compact operators. Unfortunately, this is not true. In fact,

1) if $S^2 \in Ring(T)$ whenever $0 \leq S \leq T$ with $T$ AM-compact, then in particular $S^2$ is AM-compact. Hence, the operator $S^2$ will be compact whenever $T$ is compact. But this is not true in general.

2) Also, if $S \in Ring(T)$ whenever $0 \leq S \leq T$ with $T$ AM-compact and the norm of $E$ order continuous or the topological dual $E'$ discrete, then in particular $S$ is AM-compact. Hence, under these conditions (i.e. the norm of $E$ is order continuous or the topological dual $E'$ is discrete) the operator $S$ will be compact whenever $T$ is compact. But this is false in general.

Finally, the following result gives a sufficient condition under which a Banach lattice is discrete.

**Theorem 2.13.** Let $E$ be a Banach lattice. If for each $x \in E^+$ the vector lattice $(E_x)'$ is discrete, then the topological dual $E'$ is discrete.

**Proof.** Assume that for each $x \in E^+$, $(E_x)'$ is discrete. Take $G = E \oplus l^1 \oplus c$, where $c$ is the Banach lattice of all convergent sequences. Let $S,T : G \to G$ be operators
such that $0 \leq S \leq T$ and $T$ is compact. Each one of our operators is of the form

$$S = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} a'_1 & b'_1 & c'_1 \\ a'_2 & b'_2 & c'_2 \\ a'_3 & b'_3 & c'_3 \end{pmatrix}. $$

Hence

$$S^2 = \begin{pmatrix} a_1 a_1 + b_1 a_2 + c_1 a_3 & a_1 b_1 + b_1 b_2 + c_1 b_3 & a_1 c_1 + b_1 c_2 + c_1 c_3 \\ a_2 a_1 + b_2 a_2 + c_2 a_3 & a_2 b_1 + b_2 b_2 + c_2 b_3 & a_2 c_1 + b_2 c_2 + c_2 c_3 \\ a_3 a_1 + b_3 a_2 + c_3 a_3 & a_3 b_1 + b_3 b_2 + c_3 b_3 & a_3 c_1 + b_3 c_2 + c_3 c_3 \end{pmatrix}. $$

Since for each $x \in E^+$, $(E_x)'$ is discrete, $a_1 : E \to E$, $a_2 : E \to l^1$ and $a_3 : E \to c$ are AM-compact. On the other hand, if $l^1$ is discrete and its norm is order continuous, then it follows from Theorem 1 of [10] that $c_2 : c \to l^1$, $b_2 : l^1 \to l^1$ and $a_2 : E \to l^1$ are compact. Now, since $c'$ is discrete and its norm is order continuous, Theorem 1 of [10] implies that $c_1 : c \to E$ and $c_3 : c \to c$ are compact. This shows that $S^2$ is a compact operator. Finally, Theorem 1.1 of [7] implies that the norm of $G$ is order continuous or the norm of $G'$ is order continuous or the topological dual $G'$ is discrete. But the two first conditions are false for our space $G$, hence $G'$ is discrete. This proves that $E'$ is discrete.

**Acknowledgements**

The authors thank the referee for his valuable suggestions and remarks concerning the content of this paper.

**References**


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