

SPREADING OF QUASIMODES IN THE BUNIMOVICH STADIUM

NICOLAS BURQ, ANDREW HASSELL, AND JARED WUNSCH

(Communicated by Mikhail Shubin)

ABSTRACT. We consider Dirichlet eigenfunctions u_λ of the Bunimovich stadium S , satisfying $(\Delta - \lambda^2)u_\lambda = 0$. Write $S = R \cup W$ where R is the central rectangle and W denotes the “wings,” i.e., the two semicircular regions. It is a topic of current interest in quantum theory to know whether eigenfunctions can concentrate in R as $\lambda \rightarrow \infty$. We obtain a lower bound $C\lambda^{-2}$ on the L^2 mass of u_λ in W , assuming that u_λ itself is L^2 -normalized; in other words, the L^2 norm of u_λ is controlled by λ^2 times the L^2 norm in W . Moreover, if u_λ is an $o(\lambda^{-2})$ quasimode, the same result holds, while for an $o(1)$ quasimode we prove that the L^2 norm of u_λ is controlled by λ^4 times the L^2 norm in W . We also show that the L^2 norm of u_λ may be controlled by the integral of $w|\partial_N u|^2$ along $\partial S \cap W$, where w is a smooth factor on W vanishing at $R \cap W$. These results complement recent work of Burq-Zworski which shows that the L^2 norm of u_λ is controlled by the L^2 norm in any pair of strips contained in R , but adjacent to W .

1. INTRODUCTION

The Bunimovich stadium S is a planar domain given by the union of a rectangle $R = \{(x, y) \mid x \in [-\alpha, \alpha], y \in [-\beta, \beta]\}$ with two “wings,” i.e., the two semicircular regions centered at $(\pm\alpha, 0)$ with radius β which lie outside R . Geodesic flow in S (obeying the law of reflection at the boundary) was proved to be ergodic by Bunimovich [1]. Work of Gérard-Leichtnam [5], later generalized by Zelditch-Zworski [11], shows that as a consequence the stadium is quantum ergodic. This means that there is a density one sequence of Dirichlet eigenfunctions which becomes uniformly distributed; in particular, along this density one sequence the weak limit of the L^2 mass distribution becomes uniform. (Quantum ergodicity on boundaryless manifolds whose geodesic flow is ergodic was known earlier, by results of Schnirelman [9], Zelditch [10] and Colin de Verdière [3].) One can ask whether the entire sequence of eigenfunctions becomes uniformly distributed; if so, the domain is called quantum unique ergodic (QUE). It has been conjectured by Rudnick and Sarnak that

Received by the editors July 18, 2005 and, in revised form, October 21, 2005.

2000 *Mathematics Subject Classification*. Primary 35Pxx, 58Jxx.

Key words and phrases. Eigenfunctions, quasimodes, stadium, concentration, quantum chaos.

This research was partially supported by a Discovery Grant from the Australian Research Council for the second author, and by National Science Foundation grants DMS-0323021 and DMS-0401323 for the third author. The first and third authors gratefully acknowledge the hospitality of the Mathematical Sciences Institute of the Australian National University. The authors thank an anonymous referee for helpful comments.

©2006 American Mathematical Society
Reverts to public domain 28 years from publication

complete surfaces with negative curvature which are classically ergodic are QUE; this has been proved recently by Lindenstrauss [7] for arithmetic surfaces, subject to the additional hypothesis that the eigenfunctions are also Hecke eigenfunctions. The Bunimovich stadium, by contrast, is generally believed to be non-QUE; it is thought that there is a sequence of eigenfunctions that concentrates in the rectangle R . Little is currently understood about the way in which such eigenfunctions would concentrate, however. For example, their (hypothetical) rate of decay outside R is unclear. The result in the present paper is intended to shed some light on this question: we show that any sequence of eigenfunctions (or quasimodes) cannot concentrate very rapidly inside R , by obtaining lower bounds (tending to zero as $\lambda \rightarrow \infty$, but only polynomially) on the L^2 mass inside the wings W_{\pm} .

Let $\Delta = -\partial_x^2 - \partial_y^2$ denote the (nonnegative) Laplacian on S with Dirichlet boundary conditions. We denote by $\|\cdot\|$ the norm in $L^2(S)$, and by $\partial_N g$ the outward pointing normal derivative of g at ∂S . We consider an $o(1)$ Dirichlet quasimode u_λ for Δ , by which we mean that we have a sequence $\lambda = \lambda_k \rightarrow \infty$ of real numbers and a corresponding sequence $u_\lambda \in H^2(S)$ satisfying

$$(1) \quad \begin{aligned} (\Delta - \lambda^2)u_\lambda &= f_\lambda, \\ u_\lambda|_{\partial S} &= 0, \\ \|u_\lambda\| &= 1, \end{aligned}$$

where

$$(2) \quad \|f_\lambda\| = o(1) \text{ as } \lambda \rightarrow \infty.$$

We more generally define an $O(\lambda^{-j})$ or $o(\lambda^{-j})$ quasimode by modifying the right-hand side of (2) accordingly. Of course a sequence of eigenfunctions is an $o(\lambda^{-j})$ quasimode for any j .

It is easy to see that an $O(1)$ quasimode can be localized to a small rectangle of the form $[\gamma, \delta] \times [-\beta, \beta]$, where $[\gamma, \delta]$ is an arbitrary subinterval of $[-\alpha, \alpha]$; indeed the family $u_{(n+1/2)\pi/\beta} = \phi(x) \cos((n+1/2)\pi y/\beta)$ is (after normalization) such a quasimode, where ϕ is any nonzero smooth function supported in $[\gamma, \delta]$. By contrast, an $o(1)$ quasimode cannot be so localized: Burq and Zworski [2] have shown that the L^2 norm of u_λ is controlled by (that is, bounded above by a constant times) its L^2 norm in the union of any two rectangles of the form $([-\alpha, \gamma_1] \times [-\beta, \beta]) \cup ([\gamma_2, \alpha] \times [-\beta, \beta])$. In particular, for an $o(1)$ quasimode, the L^2 mass cannot shrink to a closed region disjoint from the wings of the stadium as $\lambda \rightarrow \infty$.

Although the stadium is classically ergodic, there is a codimension one invariant set for the classical flow, consisting of vertical “bouncing-ball” orbits parallel to the y -axis and within the rectangle R , and the union of these orbits is the most likely place where localization of eigenfunctions, or more generally $o(1)$ quasimodes, can occur.¹ There is a rather convincing plausibility argument in the physics literature due to Heller and O’Connor [6] which indicates that a density-zero sequence of eigenfunctions, with eigenvalues $((n+1/2)\pi/\beta)^2 + O(1)$, does concentrate to some extent at these bouncing-ball orbits. The rigorous essence of this argument has been developed by Donnelly [4] who showed that there are sequences of functions lying in the range of spectral projectors $E_{I_n}(\Delta)$, where I_n are intervals of the form $[(n+1/2)\pi/\beta)^2 - C, ((n+1/2)\pi/\beta)^2 + C]$ which concentrate at the bouncing-ball

¹The explicit quasimode in the paragraph above concentrates along a subset of these orbits.

orbits.² On the other hand, the result of Burq-Zworski [2] (recently generalized to other domains by Marzuola [8]) shows that such localization cannot be too extreme: the control region must extend to the boundary of the rectangle.

Our main result here is that we may in fact push our control region outside the rectangle altogether and into the wings, in return for a loss, either from restriction to the boundary, or of powers of λ . To state this concisely, it is convenient to introduce an auxiliary coordinate in the wings given by $w = |x| - \alpha$; thus w is nonnegative on the wings and vanishes exactly on the vertical lines $R \cap W$.

Theorem 1.1. *There is a $C > 0$, depending only on α/β , such that any family u_λ satisfying (1) obeys the estimates*

$$(3) \quad \|f_\lambda\|^2 + \int_{\partial S \cap W} w |\partial_N u_\lambda|^2 dl \geq C,$$

$$(4) \quad \|f_\lambda\|^2 + \lambda^8 \|u_\lambda\|_{L^2(W)}^2 \geq C$$

and

$$(5) \quad \lambda^2 \|f_\lambda\|^2 + \lambda^4 \|u_\lambda\|_{L^2(W)}^2 \geq C.$$

Therefore, if u_λ is an $o(1)$ quasimode, we have for sufficiently large λ

$$(6) \quad \begin{aligned} \|w^{\frac{1}{2}} \partial_N u\|_{L^2(\partial S \cap W)} &\geq C, \\ \|u_\lambda\|_{L^2(W)} &\geq C\lambda^{-4}, \end{aligned}$$

while if u_λ is an $o(\lambda^{-2})$ quasimode (e.g. an eigenfunction), then

$$(7) \quad \|u_\lambda\|_{L^2(W)} \geq C\lambda^{-2}.$$

Note that the results of Theorem 1.1 still leave open the possibility of quasimodes concentrated along bouncing-ball orbits in the rectangle with $o(1)$ mass in the wings. They also do not rule out the possibility that all the energy in the wings may asymptotically concentrate in a boundary layer near R .

2. PRELIMINARIES TO L^2 ESTIMATES

Our main tool is positive commutator estimates, which we use in the following form.

Lemma 2.1. *Let u be real, equal to zero at ∂S , and satisfy $(\Delta - \lambda^2)u = f$, where f is smooth. Then for any real vector field B ,*

$$(8) \quad \langle u, [\Delta - \lambda^2, B]u \rangle = \langle 2Bu + (\operatorname{div} B)u, f \rangle + \int_{\partial S} (\partial_N u)Bu dl.$$

Proof. We integrate twice by parts, using the Dirichlet boundary conditions in the first instance, to write

$$\langle u, [\Delta - \lambda^2, B]u \rangle = \langle f, Bu \rangle + \int_{\partial S} \partial_N u Bu dl - \langle u, Bf \rangle.$$

Applying Green's Theorem to the last term now gives two terms: $\langle Bu, f \rangle + \langle (\operatorname{div} B)u, f \rangle$. Since u and f are real this yields the desired identity. \square

²This was shown for surfaces without boundary containing a flat cylinder, but the arguments go through for the stadium.

We also record here an inequality that will be of use in estimating derivative terms.

Lemma 2.2. *Let u, f be as in Lemma 2.1. Then for all $s > 0$, for λ sufficiently large,*

$$\|\nabla u\|^2 \leq C_s(\lambda^{\max(2,s)}\|u\|^2 + \lambda^{-s}\|f\|^2).$$

Proof. We compute

$$\begin{aligned} \|\nabla u\|^2 &= \int_S u_x^2 + u_y^2 dA \\ &= \int_S (\Delta u)u dA \\ &= \lambda^2 \int_S u^2 dA + \langle f, u \rangle. \end{aligned}$$

Applying Cauchy-Schwarz to $\langle f, u \rangle$ gives the estimate. □

3. PROOF OF (3)

It suffices to prove (3) under the assumption that u_λ , and hence f_λ , are real, since we can treat the real and imaginary parts separately. We make this assumption from now on.

We begin with the standard commutator $[\Delta, x\partial_x] = -2\partial_x^2$. Applying (8) with $B = x\partial_x$, and dropping the subscript on u_λ , we have

$$\begin{aligned} \langle u_x, u_x \rangle &= -\langle \partial_x^2 u, u \rangle = \langle [\Delta - \lambda^2, x\partial_x]u, u \rangle \\ (9) \qquad &= \int_{\partial S} x\partial_x u \partial_N u dl + \int (2x\partial_x u + u)f dA; \end{aligned}$$

in the last equation we integrated twice by parts, using for a second time the fact that $(\Delta - \lambda^2)u = f$ as well as the fact that u satisfies Dirichlet boundary conditions; hence integration by parts produces boundary terms only where derivatives land on both factors of u .

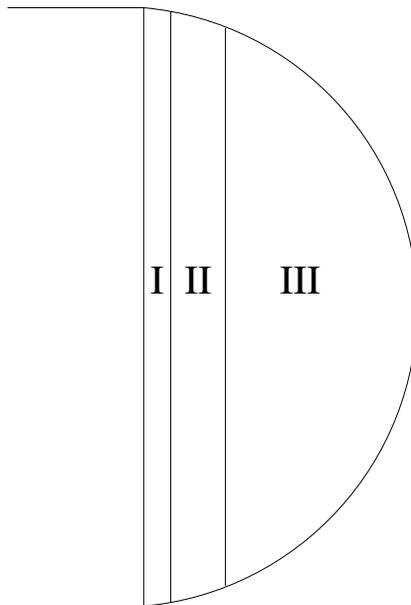
Now at every boundary point we may decompose $x\partial_x$ into $p\partial_l + q\partial_N$ where ∂_l is differentiation tangent to the boundary. Of course ∂_l annihilates u . Now since ∂_x is tangent to the upper and bottom sides of the rectangle we find that the boundary integral in (9) is only over $\partial S \cap W$. Moreover, as ∂_x is tangent to the top and bottom of the circles forming the boundaries of the wings, we have $q = O(w)$ on $\partial S \cap W$. (Recall that $w = |x| - \alpha$.) Hence we have shown that

$$(10) \qquad \|u_x\|^2 \leq \int_{\partial S \cap W} O(w)|\partial_N u|^2 dl + \epsilon \int (u^2 + u_x^2) dA + C \int f^2 dA.$$

We may absorb the $\epsilon\|u_x\|^2$ term, then apply the Poincaré inequality, and absorb the $\epsilon\|u\|^2$ term to obtain

$$(11) \qquad \|u\|_{L^2(S)}^2 \leq C \int_{\partial S \cap W} w|\partial_N u|^2 dl + C\|f\|^2,$$

which is the first part of our theorem, as we took u to be L^2 -normalized.

FIGURE 1. The three regions of interest in W

4. PROOF OF (4)

To prove this estimate we start from

$$(12) \quad \|u_x\|^2 \leq C \int_{\partial S} w_+ |\partial_N u|^2 dl + C \|f\|^2,$$

which follows directly from the considerations of the previous section, and estimate the boundary integral term. We shall obtain upper bounds of the form

$$\lambda^8 \int_W u^2 dA + \|f\|^2$$

and

$$\lambda^4 \int_W u^2 dA + \lambda^2 \|f\|^2,$$

thus proving (4) and (5).

We shall perform this estimate in three separate regions in the wing. Region I is the near-rectangular region, in a boundary layer where $w \leq \delta\lambda^{-2}$. Region II will be outside the boundary layer, where $\delta\lambda^{-2} \leq w \leq \beta/2$. Region III will be the far outer region $w \geq \beta/2$.

We begin with Region III, far away from the rectangle. In this case we employ Lemma 2.1 where B is the operator $\phi(x)\partial_x$, where ϕ is supported where $w > \beta/4$, say, equal to 1 where $w > \beta/2$, and with $\partial_x\phi \geq 0$. Then (8) gives us, with $P = [\Delta, B]$,

$$\int_{\partial S} \phi \partial_x u \partial_N u dl \leq |\langle Pu, u \rangle| + |\langle \phi_x f, u \rangle| + 2\langle \phi u_x, f \rangle.$$

Note that $P = -2\phi_x \partial_{xx}^2 - \phi_{xx} \partial_x$. Thus the LHS is bounded ($\forall \epsilon > 0$) by

$$| \langle -\phi_x u_{xx}, u \rangle | + \epsilon \|u_x\|^2 + C(\|u\|_{L^2(W)}^2 + \|f\|^2)$$

(where of course C depends on ϵ). We can add the positive term $\int_S \phi_x (\partial_y u)^2 dA$ to this estimate. Integrating by parts in y gives us

$$| \langle \phi_x (-u_{xx} - u_{yy}), u \rangle | + \epsilon \|u_x\|^2 + C(\|u\|_{L^2(W)}^2 + \|f\|^2).$$

Using the positivity of the integrand, we thus obtain an estimate

$$(13) \quad \left| \int_{\partial S \cap \text{III}} \phi \partial_x u \partial_N u dl \right| \leq C\lambda^2 \|u\|_{L^2(W)}^2 + C\|f\|^2 + \epsilon \|u_x\|^2$$

with C depending on $\epsilon > 0$.

Now we work on Region I, within an $O(\lambda^{-2})$ boundary layer along the rectangle. We again apply Lemma 2.1, this time with $B = x\partial_x + y\partial_y$. Since B is a tangential vector plus a positive multiple of ∂_N all along ∂S we obtain

$$\int_{\partial S} (\partial_N u)^2 dl \leq | \langle u, [\Delta - \lambda^2, B]u \rangle | + 2| \langle u, f \rangle | + 2| \langle Bu, f \rangle |.$$

Using $[\Delta - \lambda^2, B] = 2\Delta$ and Cauchy-Schwarz this becomes

$$\int_{\partial S} |\partial_N u|^2 dl \leq C\lambda^2 \|u\|^2 + \|Bu\|^2 + C\|f\|^2.$$

Restricting to $\partial S \cap \text{I}$ in the integrand, we can estimate w in L^∞ by $\delta\lambda^{-2}$, and this gives

$$(14) \quad \int_{\partial S \cap \text{I}} w_+ |\partial_N u|^2 dl \leq \delta C(\|u\|^2 + \lambda^{-2}(\|f\|^2 + \|Bu\|^2)).$$

Using Lemma 2.2, we may estimate $\|Bu\|^2$ by $C(\lambda^2 \|u\|^2 + \|f\|^2)$. Hence we may finally write

$$(15) \quad \int_{\partial S \cap \text{I}} w_+ |\partial_N u|^2 dl \leq \delta C_0 \|u\|^2 + \delta C \lambda^{-2} \|f\|^2;$$

note that in the above construction, C_0 can in fact be chosen independently of δ .

Finally, we estimate in Region II. To begin with, we note that for $w \geq \delta\lambda^{-2}$, we can estimate w_+ by $\delta^{-1}\lambda^2 w_+^2$, so we have

$$\int_{\partial S \cap \text{II}} w_+ |\partial_N u|^2 dl \leq C\delta^{-1} \int_{\partial S \cap \text{II}} \lambda^2 w_+^2 \chi(y) \partial_y u \partial_N u dl;$$

here we take χ supported in $|y| > \beta/20$ and equal to -1 for $y < -\beta/10$ and $+1$ for $y > \beta/10$, so that $\chi(y)\partial_y$ is a positive multiple of ∂_N plus a tangential component on $\partial S \cap \text{II}$. To estimate further, we employ Lemma 2.1 with the commutant

$$B = \lambda^2 w_+^2 \chi(y) \partial_y.$$

The point of this commutant is that we have given ourselves two powers of w which will “absorb” two integrations by parts in x *without any boundary terms at $w = 0$* ; this is crucial since we know of no way to deal with such boundary terms. (On the other hand, we pay the price of additional powers of λ with this method.) Thus we obtain, setting $Q = [\Delta, B]$,

$$(16) \quad \int_{\partial S} \lambda^2 w_+^2 \chi(y) \partial_y u \partial_N u dl \leq | \langle Qu, u \rangle | + \lambda^2 \left| \int_S w_+^2 \chi_y(y) u f dA \right| + 2| \langle Bu, f \rangle |.$$

We can estimate the second term on the RHS by $\lambda^4 \|u\|_{L^2(W)}^2 + C\|f\|^2$.

Now consider the terms involving the operator Q . This is given by

$$Q = \lambda^2 \left(-4w_+ \chi \partial_x \partial_y - 2w_+^2 \chi_y \partial_{yy}^2 - 2H(w) \chi \partial_y - w_+^2 \chi_{yy} \partial_y \right)$$

where $H(\cdot)$ is the Heaviside step-function, $H(w) = 0$ for $w < 0$ and 1 for $w \geq 0$. To treat the terms involving one derivative, e.g. the third term above, we integrate by parts:

$$-2\lambda^2 \int_S H(w) \chi(y) \partial_y u u \, dA = \lambda^2 \int_S H(w) \chi_y(y) u^2 \, dA,$$

which is therefore estimated by $\lambda^2 \|u\|_{L^2(W)}^2$. The fourth term is estimated in exactly the same way.

Thus we are left to estimate

$$(17) \quad \lambda^2 \left(-4 \langle w_+ \chi(y) \partial_{xy}^2 u, u \rangle - 2 \langle w_+^2 \chi_y(y) \partial_{yy}^2 u, u \rangle \right).$$

Integrating the first term by parts in x and the second term by parts in y gives us two principal terms

$$(18) \quad 4\lambda^2 \left(\langle w_+^{1/2} \partial_x u, w_+^{1/2} \partial_y u \rangle + \langle w_+^2 \chi_y \partial_y u, \partial_y u \rangle \right)$$

together with two other terms

$$4\lambda^2 \left(\langle H(w) \partial_y u, u \rangle + \langle w_+^2 \chi_{yy}(y) \partial_y u, u \rangle \right),$$

which are estimated in the same way as the first-order terms above. We apply Cauchy-Schwarz to the first term in (18), while in the second term, which is positive, we replace w_+^2 by w_+ (which is larger, up to a constant multiple) and then integrate by parts again, getting (up to another first-order term estimated as above) an upper bound for (18) of the form

$$(19) \quad C\lambda^2 \int_S w_+ |\partial_x u|^2 + w_+ |\partial_y u|^2 \, dA.$$

Now we integrate by parts again, getting

$$(20) \quad C\lambda^2 \int_S \left(w_+ (-u_{xx} - u_{yy}) u + H(w) u u_x \right) \, dA.$$

Writing $-u_{xx} - u_{yy} = \lambda^2 u + f$, we can estimate the integrand of (20) by

$$\begin{aligned} & \lambda^4 w_+ |u|^2 + \lambda^2 w_+ f u + \lambda^2 H(w) u u_x \\ & \leq \lambda^4 w_+ |u|^2 + \frac{1}{2} (\lambda^4 w_+ u^2 + w_+ f^2) + C\lambda^4 H(w) u^2 + \epsilon u_x^2. \end{aligned}$$

Hence we may estimate (18) by

$$C\lambda^4 \|u\|_{L^2(W)}^2 + \epsilon \|u_x\|^2 + C\|f\|^2.$$

The term $2|\langle Bu, f \rangle|$ is bounded in a similar manner: By Cauchy-Schwarz we may estimate it by

$$C\|f\|^2 + C\lambda^4 \|w_+ u_y\|^2$$

or by

$$C\lambda^2 \|f\|^2 + C\lambda^2 \|w_+ u_y\|^2,$$

as needed. Our estimate for (19) turns the latter estimate into

$$C\lambda^2 \|f\|^2 + C\lambda^4 \|u\|_{L^2(W)}^2 + \epsilon \|u_x\|^2.$$

On the other hand, treating $\lambda^4 \|w_+ u_y\|^2$ in the same manner gives the bound

$$C \|f\|^2 + C \lambda^8 \|u\|_{L^2(W)}^2 + \epsilon \|u_x\|^2.$$

On the support of χ and at the boundary, $\chi(y)u_y$ is a positive multiple of $\partial_N u$. So the upshot is that (16) now gives

$$(21) \quad \int_{\partial S} \lambda^2 w_+^2 \chi(y) (\partial_N u)^2 dl \leq C \lambda^8 \|u\|_{L^2(W)}^2 + \epsilon \|u_x\|^2 + C \|f\|^2$$

and

$$(22) \quad \int_{\partial S} \lambda^2 w_+^2 \chi(y) (\partial_N u)^2 dl \leq C \lambda^4 \|u\|_{L^2(W)}^2 + \epsilon \|u_x\|^2 + C \lambda^2 \|f\|^2.$$

At last we can estimate the boundary integral term in (12) by a combination of (13), (15), and (21)/(22), obtaining

$$\int_{\partial S} w_+ |\partial_N u|^2 dl \leq \delta C_0 \|u\|^2 + C \lambda^8 \|u\|_{L^2(W)}^2 + 2\epsilon \|u_x\|^2 + C \|f\|^2$$

and

$$\int_{\partial S} w_+ |\partial_N u|^2 dl \leq \delta C_0 \|u\|^2 + C \lambda^4 \|u\|_{L^2(W)}^2 + 2\epsilon \|u_x\|^2 + C \lambda^2 \|f\|^2.$$

Here C depends on ϵ, δ but C_0 is independent of δ . We now combine this with (12). Absorbing the $\|u_x\|^2$ and $\|u\|^2$ terms on the LHS by taking δ and ϵ sufficiently small, we obtain (4), (5).

5. CONCLUDING REMARKS

The estimates presented here are certainly not optimal. The powers of λ appearing in Theorem 1.1 can probably be improved using refinements of the methods used here, but it seems unlikely that one could achieve an optimal result with them. We have not, therefore, attempted to obtain the best possible powers of λ , but have rather tried to present a polynomial lower bound on $\|u_\lambda\|_{L^2(W)}$ with a simple proof.

It would be of great interest to obtain a polynomial lower bound on the L^2 mass of u_λ in a subregion of W which is a positive distance from R (i.e. region III of the previous section). We do not know whether such a bound holds, but it does not seem to be obtainable using the methods of this paper; possibly it might yield to the use of more sophisticated tools from microlocal analysis.

REFERENCES

- [1] L. Bunimovich, *On the ergodic properties of nowhere dispersing billiards*, Commun. Math. Phys. **65** (1979), 295–312. MR0530154 (80h:58037)
- [2] N. Burq, M. Zworski, *Bouncing Ball Modes and Quantum Chaos*, SIAM Review **47** (2005), no. 1, 43–49. MR2149100 (2006d:81111)
- [3] Y. Colin de Verdière, *Ergodicité et fonctions propres du laplacien*, Commun. Math. Phys. **102** (1985), 187–214. MR0818831 (87d:58145)
- [4] H. Donnelly, *Quantum unique ergodicity*, Proc. Amer. Math. Soc. **131** (2003), no. 9, 2945–2951. MR1974353 (2005a:58048)
- [5] P. Gérard, E. Leichtnam, *Ergodic properties of eigenfunctions for the Dirichlet problem*, Duke Math. J. **71** (1993), no. 2, 559–607. MR1233448 (94i:35146)
- [6] E. Heller and P. O’Connor, *Quantum localization for a strongly classically chaotic system*, Phys. Rev. Lett. **61** (1988), no. 20, 2288–2291. MR0966831 (89j:81069)
- [7] E. Lindenstrauss, *Invariant measures and arithmetic quantum ergodicity*, Annals of Math., to appear.

- [8] J. Marzuola, *Nonconcentration of eigenfunctions for partially rectangular billiards*, Comm. PDE, to appear.
- [9] A. Schnirelman, *Ergodic properties of eigenfunctions*, Uspekhi Mat. Nauk. **29** (1974), 181–182. MR0402834 (53:6648)
- [10] S. Zelditch, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke Math. Jour. **55** (1987) 919–941. MR0916129 (89d:58129)
- [11] S. Zelditch, M. Zworski, *Ergodicity of eigenfunctions for ergodic billiards*, Comm. Math. Phys. **175** (1996), no. 3, 673–682. MR1372814 (97a:58193)

UNIVERSITÉ PARIS SUD, MATHÉMATIQUES, BÂT 425, 91405 ORSAY CEDEX FRANCE AND INSTITUT UNIVERSITAIRE DE FRANCE

E-mail address: `nicolas.burq@math.u-psud.fr`

DEPARTMENT OF MATHEMATICS, AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA 0200 ACT, AUSTRALIA

E-mail address: `hassell@maths.anu.edu.au`

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60208

E-mail address: `jwunsch@math.northwestern.edu`