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AN AFFINE RESTRICTION ESTIMATE IN $\mathbb{R}^3$

BASSAM SHAYYA

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Abstract. We prove that the Fourier transform of an $L^{4/3}$ function can be restricted to any compact convex C^2 surface of revolution in $\mathbb{R}^3$.

1. Affine restriction estimates in $\mathbb{R}^n$

Let $n \geq 2$. A convex body in $\mathbb{R}^n$ is a compact convex subset of $\mathbb{R}^n$ with nonempty interior. A convex body K is symmetric if $K = -K$. Let $\mathcal{K}_n$ be the set of all symmetric convex bodies in $\mathbb{R}^n$.

For $K \in \mathcal{K}_n$, the polar body K^* of K is defined as

$$K^* = \{\xi \in \mathbb{R}^n : |\xi \cdot x| \leq 1 \text{ for all } x \in K\}.$$

The Lebesgue measure of K is related to the Lebesgue measure of its polar body by the inequalities

$$c_1 \leq |K||K^*| \leq c_2, \tag{1}$$

where c_1 and c_2 are constants[1] that depend only on the dimension n.

For $K \in \mathcal{K}_n$, the affine surface area $\Omega(K)$ is defined as

$$\Omega(K) = \int_{\operatorname{bd} K} \kappa(x)^{1/(n+1)} d\sigma(x),$$

where $\operatorname{bd} K$ is the boundary of K, κ is the generalized Gaussian curvature of $\operatorname{bd} K$ (see [2] or [6]), and $d\sigma$ is the surface measure on $\operatorname{bd} K$ (i.e., the restriction of the $(n-1)$-dimensional Hausdorff measure to $\operatorname{bd} K$). We shall denote the measure $\kappa(x)^{1/(n+1)} d\sigma(x)$ by $d\mu(x)$ and call it the affine surface measure on $\operatorname{bd} K$. Two important properties of affine surface area are that it is invariant under volume-preserving affine transformations of $\mathbb{R}^n$, and that it satisfies the affine isoperimetric inequality

$$\Omega(K)^{n+1} \leq c\,|K|^{n-1} \tag{2}$$

for all $K \in \mathcal{K}_n$, where c is a constant[2] that depends only on the dimension n. We refer the reader to [4], [5], [6], [7], [11], and [12] for more information on affine surface area.

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[1] Whereas determining the best value for c_1 is an open problem, the best possible value for c_2 is known to be $|B(0,1)|^2$. The inequality $|K||K^*| \leq |B(0,1)|^2$ is known as Santaló's inequality.

[2] The best value for c is known to be $n^{n+1}|B(0,1)|^2$.

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The most apparent link between the Fourier transform and affine surface area comes from the fact that Plancherel's theorem can be used to prove the second inequality in (1), which can in turn be used to prove (2) (see [7]). To see how Plancherel's theorem gives an upper bound for the quantity $|K||K^*|$, $K \in \mathcal{K}_n$, notice that

$$|\widehat{\chi_K}(\xi)| \geq \frac{1}{2}\,|K| \quad \text{if} \quad \xi \in (1/6)K^*,$$

so that

$$|K|^{1/2} = \|\chi_K\|_{L^2(\mathbb{R}^n)} = \|\widehat{\chi_K}\|_{L^2(\mathbb{R}^n)} \geq \frac{1}{2}\,|K|\,|(1/6)K^*|^{1/2},$$

which gives

$$|K||K^*| \leq 4(6^n).$$

As was observed by Carbery and Ziesler in [3], there is a connection between restriction theory and the affine isoperimetric inequality. Suppose $K \in \mathcal{K}_n$, $1 < r < 2n/(n+1)$, $(n+1)/r + (n-1)/s = n+1$, and consider the (L^r, L^s) restriction estimate[3]

$$\|\widehat{f}\|_{L^s(d\mu)} \leq C\,\|f\|_{L^r(\mathbb{R}^n)} \tag{3}$$

for $f \in L^r(\mathbb{R}^n)$. Carbery and Ziesler noticed that if one establishes (3) with a constant C independent of K, then (2) follows as a corollary. In fact, (3) is equivalent to the adjoint restriction (or extension) estimate

$$\|\widehat{f d\mu}\|_{L^q(\mathbb{R}^n)} \leq C\,\|f\|_{L^p(d\mu)} \tag{4}$$

for $f \in L^p(d\mu)$, where $p = s'$ and $q = r'$ are the exponents conjugate to s and r. When $f \equiv 1$, the right-hand side of (4) is $\Omega(K)^{1/p}$, and since $|\widehat{d\mu}(\xi)| \geq C\,\Omega(K)$ for $\xi \in (1/6)K^*$, the left-hand side is greater than $C\,\Omega(K)|K^*|^{1/q}$. So if (4) holds, then

$$\Omega(K)^{1-1/p} \leq C\,|K^*|^{-1/q},$$

and since $(n-1)/p + (n+1)/q = n-1$, this is the same as

$$\Omega(K)^{n+1} \leq C\,|K^*|^{1-n}.$$

Now applying the first inequality in (1), we get (2).

Given $K \in \mathcal{K}_n$, we are going to refer to the assertion that (4) holds whenever $1 \leq p \leq \infty$, $2n/(n-1) < q \leq \infty$, and $(n-1)/p + (n+1)/q \leq n-1$ as the affine restriction conjecture for K. What we know so far is that in dimension $n = 2$, the affine restriction conjecture is true for $K \in \mathcal{K}_2$ with C^2 boundaries. This is a theorem of Sjölin [13] (see also [8] and [10]). In dimension $n = 3$, we do not know if (4) holds even for members of $\mathcal{K}_3$ with C^2 boundaries and even for the Tomas-Stein indices $(p, q) = (2, 4)$. Some progress, however, has been achieved in the radial case, which we shall focus on for the rest of the paper.

[3]Throughout this paper, the letter C denotes a positive constant whose value may vary from line to line.

2. Restriction on convex radial surfaces in $\mathbb{R}^3$

Let $0 \le a < b$ and consider the surface

$$\Gamma : \{x \in \mathbb{R}^2 : a \le |x| \le b\} \longrightarrow \mathbb{R}^3$$

given by $\Gamma(x) = (x, \gamma(|x|))$, where $\gamma : [a,b] \to [0,\infty)$ is a convex function.

In a recent paper [9], Oberlin proved that if $a = 0$, $\gamma \in C^3([0,b])$, $\gamma(0) = \gamma'(0) = 0$, $\gamma''(t) > 0$ for $0 < t < b$, $\gamma^{(3)}(t) \ge 0$ for $0 \le t < b$, and

$$\sup_{0<t<b} \frac{t\gamma''(t)}{\gamma'(t)} \le C, \tag{5}$$

then

$$\|\widehat{fd\mu}\|_{L^4(\mathbb{R}^3)} \le C\,\|f\|_{L^2(d\mu)} \tag{6}$$

for all $f \in L^2(d\mu)$. An immediate consequence of this estimate is that the Fourier transform of an $L^{4/3}$ function on $\mathbb{R}^3$ can be restricted to Γ if γ satisfies the above conditions. Convex but exponentially flat surfaces such as $\Gamma(x) = (x, e^{-1/|x|^m})$, $m = 1, 2, \ldots$, were excluded from this result because the functions $\gamma(t) = e^{-1/t^m}$ do not satisfy (5). The situation was improved in [1], where (5) was replaced by the weaker condition

$$\sup_{0<t<b} \frac{\gamma(t)\gamma''(t)}{\gamma'(t)^2} \le C, \tag{7}$$

which is satisfied by $\gamma(t) = e^{-1/t^m}$.

The purpose of this paper is to prove that the Fourier transform of an $L^{4/3}$ function on $\mathbb{R}^3$ can be restricted to Γ if $\gamma \in C^2([a,b])$, $\gamma'(a) \ge 0$, and $\gamma''(t) > 0$ for $a < t < b$. So we are able to relax the smoothness requirement from C^3 to C^2 and remove any growth conditions such as (5) or (7). The drawback is that the L^2-norm on the right-hand side of (6) will be replaced by the L^∞-norm.

Theorem. *Suppose $0 \le a < b$, $\gamma : [a,b] \to [0,\infty)$ is C^2, $\gamma'(a) \ge 0$, and $\gamma''(t) > 0$ for $a < t < b$. Let Γ be the surface in $\mathbb{R}^3$ given by $\Gamma(x) = (x, \gamma(|x|))$, $a \le |x| \le b$, and let $d\mu$ be the affine surface measure on Γ. Also, let $d\nu$ be the pushforward under Γ of the 2-dimensional measure $(\gamma'(|x|)/|x|)^{1/4}dx$. Then*

$$\|\widehat{fd\mu}\|_{L^4(\mathbb{R}^3)} \le C_1\,\|f\|_{L^2(d\nu)} \tag{8}$$

for all $f \in L^2(d\nu)$, and

$$\|\widehat{fd\mu}\|_{L^4(\mathbb{R}^3)} \le C_2\,\|f\|_{L^\infty(d\mu)} \tag{9}$$

for all $f \in L^\infty(d\mu)$, with $C_1 = 4\pi^{1/4}(8\sup\gamma'')^{1/8}$ and $C_2 = \|d\nu\|^{1/2}C_1$.

Notice that by (9) and duality, we get

$$\|\widehat{f}\|_{L^1(d\mu)} \le C\,\|f\|_{L^{4/3}(\mathbb{R}^3)}$$

for all Schwartz functions f on $\mathbb{R}^3$, and hence for all $f \in L^{4/3}(\mathbb{R}^3)$. By interpolation, we obtain the same result for all $f \in L^r(\mathbb{R}^3)$, $1 \le r \le 4/3$.

Proof of the Theorem. The surface measure $d\sigma$ on Γ is here the pushforward under Γ of the 2-dimensional measure $\sqrt{1+\gamma'(|x|)^2}dx$, and the Gaussian curvature of Γ is $\kappa(\Gamma(x)) = (\gamma'(|x|)\gamma''(|x|)/|x|)(1+\gamma'(|x|)^2)^{-2}$, and so the affine surface measure $d\mu$

is the pushforward under Γ of the 2-dimensional measure $(\gamma'(|x|)\gamma''(|x|)/|x|)^{1/4}dx$. Hence

$$\|f\|_{L^2(d\mu)} \leq (\sup \gamma'')^{1/8}\|f\|_{L^2(d\nu)} \leq (\sup \gamma'')^{1/8}\|d\nu\|^{1/2}\|f\|_{L^\infty(d\mu)}$$

for all $f \in L^\infty(d\mu)$. This shows that (8) implies (9). So our task for the remainder of this paper is to prove (8). This will be achieved by pushing the methods of [9] a little further.

Let $d\mu_1$ be the measure $d\mu$ restricted to the first octant in $\mathbb{R}^3$. It is enough to prove (8) with $d\mu$ replaced by $d\mu_1$ and C_1 replaced by $C_1/4$. By Plancherel's theorem and duality, it is then enough to prove that

$$\int |\widehat{g}|\, |f|d\mu_1 * |f|d\mu_1 \leq C\, \|\widehat{g}\|_{L^2(\mathbb{R}^3)}\|f\|^2_{L^2(d\nu)} \tag{10}$$

for all $f \in L^2(d\nu)$ and $g \in L^2(\mathbb{R}^3)$, with $C = (C_1/4)^2$. By the Cauchy-Schwarz inequality,

$$\int |\widehat{g}|\, |f|d\mu_1 * |f|d\mu_1 \leq \|\widehat{g}\|_{L^2(d\lambda * d\lambda)}\|f\|^2_{L^2(d\nu)},$$

where $d\lambda = (\gamma'' \circ \gamma^{-1} \circ \zeta)^{1/4}d\mu_1$ and $\zeta : \mathbb{R}^3 \to \mathbb{R}$ is given by $\zeta(z_1, z_2, z_3) = z_3$. So to prove (10), it is enough to show that

$$\int h\, d\lambda * d\lambda \leq C^2 \int_{\mathbb{R}^3} h(z)dz \tag{11}$$

for all measurable $h : \mathbb{R}^3 \to [0, \infty]$.

We can write $\int h\, d\lambda * d\lambda$ in polar coordinates as

$$\int_X \int_Y h(\rho_1 e^{i\theta_1} + \rho_2 e^{i\theta_2}, \gamma(\rho_1) + \gamma(\rho_2))d\theta(\gamma'(\rho_1)\gamma'(\rho_2)\gamma''(\rho_1)^2\gamma''(\rho_2)^2\rho_1^3\rho_2^3)^{1/4}d\rho,$$

where $X = \{\rho = (\rho_1, \rho_2) : a < \rho_1, \rho_2 < b\}$ and $Y = \{\theta = (\theta_1, \theta_2) : 0 < \theta_1, \theta_2 < \pi/2\}$. In the inner integral, we would like to apply the change of variable $x = G(\theta)$, where $G(\theta) = \rho_1 e^{i\theta_1} + \rho_2 e^{i\theta_2}$. But G is not 1-1 on Y and $J_G(\theta) = \rho_1\rho_2 \sin(\theta_2 - \theta_1)$ vanishes on the diagonal $\{\theta \in Y : \theta_1 = \theta_2\}$, so instead, we apply the change of variable to

$$\int_0^{\pi/2} \int_0^{\theta_1} h(\rho_1 e^{i\theta_1} + \rho_2 e^{i\theta_2}, \gamma(\rho_1) + \gamma(\rho_2))d\theta_2 d\theta_1$$

to get

$$\int_{G(\{\theta \in Y : \theta_2 < \theta_1\})} \frac{2\, h(x, \gamma(\rho_1) + \gamma(\rho_2))}{\sqrt{(|x|^2 - (\rho_1 - \rho_2)^2)((\rho_1 + \rho_2)^2 - |x|^2)}}\, dx.$$

If $x = G(\theta)$, then $|x|^2 = |\rho|^2 + 2\rho_1\rho_2 \cos(\theta_1 - \theta_2)$. So

$$G(\{\theta \in Y : \theta_2 < \theta_1\}) \subset \{x \in \mathbb{R}^2 : |\rho| < |x| < \rho_1 + \rho_2\}.$$

Also, if $|\rho| < |x| < \rho_1 + \rho_2$, then

$$\begin{aligned}(|x|^2 - (\rho_1 - \rho_2)^2)((\rho_1 + \rho_2)^2 - |x|^2) &\geq (2\rho_1\rho_2)(\rho_1 + \rho_2 + |x|)(\rho_1 + \rho_2 - |x|) \\ &\geq 4(\rho_1\rho_2)^{3/2}(\rho_1 + \rho_2 - |x|).\end{aligned}$$

Hence the last integral above is less than or equal to

$$\int_{|\rho| < |x| < \rho_1 + \rho_2} \frac{h(x, \gamma(\rho_1) + \gamma(\rho_2))}{\sqrt{\rho_1 + \rho_2 - |x|}}\, (\rho_1\rho_2)^{-3/4}dx.$$

Letting $A = \{x \in \mathbb{R}^2 : x_1, x_2 > 0, \sqrt{2}\,a < |x| < 2b\}$, $E = \{(\rho, x) \in X \times A : |x| < \rho_1 + \rho_2\}$, and $E_x = \{\rho \in X : (\rho, x) \in E \text{ and } \rho_2 < \rho_1\}$, we obtain

$$\begin{aligned}
&\int h\, d\lambda * d\lambda \\
&\quad\le\ 2\int_X \int_A \frac{\chi_E(\rho,x)h(x,\gamma(\rho_1)+\gamma(\rho_2))}{\sqrt{\rho_1+\rho_2-|x|}}(\gamma'(\rho_1)\gamma'(\rho_2))^{1/4}(\gamma''(\rho_1)\gamma''(\rho_2))^{1/2}dxd\rho \\
&\quad=\ 2\int_A \int_X \frac{\chi_E(\rho,x)h(x,\gamma(\rho_1)+\gamma(\rho_2))}{\sqrt{\rho_1+\rho_2-|x|}}(\gamma'(\rho_1)\gamma'(\rho_2))^{1/4}(\gamma''(\rho_1)\gamma''(\rho_2))^{1/2}d\rho dx \\
&\quad=\ 4\int_A \int_{E_x} \frac{h(x,\gamma(\rho_1)+\gamma(\rho_2))}{\sqrt{\rho_1+\rho_2-|x|}}(\gamma'(\rho_1)\gamma'(\rho_2))^{1/4}(\gamma''(\rho_1)\gamma''(\rho_2))^{1/2}d\rho dx.
\end{aligned}$$

Fix $x \in A$, put $m = \max[|x|/2, a]$, and let

$$\mathrm{I} = \int_{E_x} \frac{h(x,\gamma(\rho_1)+\gamma(\rho_2))}{\sqrt{\rho_1+\rho_2-|x|}}(\gamma'(\rho_1)\gamma'(\rho_2))^{1/4}(\gamma''(\rho_1)\gamma''(\rho_2))^{1/2}d\rho.$$

To estimate I, we shall first apply the change of variable $\rho = (R(t,y), S(t,y))$, where

$$(R,S) : \{(t,y) \in (1/2,1) \times (0,\infty) : \gamma(m) < (1-t)y < ty < \gamma(b)\} \longrightarrow E_x$$

is given by $R(t,y) = \gamma^{-1}(ty)$ and $S(t,y) = \gamma^{-1}((1-t)y)$ (since γ is strictly increasing, $S < R$). The Jacobian of the map (R,S) is

$$J_{(R,S)}(t,y) = \frac{y}{\gamma'(R(t,y))\gamma'(S(t,y))},$$

and

$$2\gamma(m) < y = \gamma(R(t,y)) + \gamma(S(t,y)) < 2\gamma(b)$$

for all (t,y) in the domain of (R,S), so

$$\mathrm{I} = \int_{2\gamma(m)}^{2\gamma(b)} h(x,y) \int_{1/2}^{\tau(y)} \frac{\sqrt{\gamma''(R(t,y))\gamma''(S(t,y))}}{\sqrt{R(t,y)+S(t,y)-|x|}} \frac{y}{(\gamma'(R(t,y))\gamma'(S(t,y)))^{3/4}} dtdy,$$

where $\tau : (2\gamma(m), 2\gamma(b)) \to (1/2, 1]$ is a continuous function.[4]

Fix $y \in (2\gamma(m), 2\gamma(b))$, write $r(t)$ and $s(t)$ for $R(t,y)$ and $S(t,y)$, and let

$$\mathrm{II} = \int_{1/2}^{\tau(y)} \frac{\sqrt{\gamma''(r(t))\gamma''(s(t))}}{\sqrt{r(t)+s(t)-|x|}} \frac{y}{(\gamma'(r(t))\gamma'(s(t)))^{3/4}} dt.$$

Our next goal is to estimate $r(t) + s(t) - |x|$ from below. For $1/2 < t < 1$, define $\eta(t) = \sqrt{\gamma'(r(t))/\gamma'(s(t))}$. Then, since $r'(t) = y/\gamma'(r(t))$ and $s'(t) = -y/\gamma'(s(t))$, we have $\eta(t) = \sqrt{|s'(t)|/r'(t)}$ and

$$\eta'(t) = \frac{\gamma''(r(t))\gamma'(s(t))r'(t) - \gamma''(s(t))\gamma'(r(t))s'(t)}{2\eta(t)\gamma'(s)^2} > 0$$

[4]The change of variable $(t,y) \mapsto (R,S)(t,y)$ can be visualized as follows. For each $y \in (2\gamma(m), 2\gamma(b))$, $(R,S)(t,y)$ is a curve in the ρ-plane which "enters" E_x when $t = 1/2$ (i.e., when it intersects the line $\rho_2 = \rho_1$) and "leaves" E_x when $t = \tau(y)$.

for $1/2 < t < 1$. By the definition of $\tau(y)$ and the above observations, we now have

$$\begin{aligned} r(t)+s(t)-|x| &\geq r(t)+s(t)-r(\tau(y))-s(\tau(y)) \\ &= -\int_t^{\tau(y)} (r'(\alpha)+s'(\alpha))d\alpha \\ &= \int_t^{\tau(y)} (|s'(\alpha)|-r'(\alpha))d\alpha \\ &= \int_t^{\tau(y)} \left(\eta(\alpha)-\frac{1}{\eta(\alpha)}\right)\sqrt{r'(\alpha)|s'(\alpha)|}\,d\alpha \\ &\geq \left(\eta(t)-\frac{1}{\eta(t)}\right)\int_t^{\tau(y)} \sqrt{r'(\alpha)|s'(\alpha)|}\,d\alpha \\ &= \frac{\gamma'(r(t))-\gamma'(s(t))}{\sqrt{\gamma'(r(t))\gamma'(s(t))}}\int_t^{\tau(y)} \sqrt{r'(\alpha)|s'(\alpha)|}\,d\alpha \end{aligned}$$

for $1/2 < t < 1$. Since $r(1/2) = s(1/2)$, we also have

$$\begin{aligned} \gamma'(r(t))-\gamma'(s(t)) &= \gamma'(r(t))-\gamma'(s(t))-(\gamma'(r(1/2))-\gamma'(s(1/2))) \\ &= \int_{1/2}^t (\gamma''(r(\alpha))r'(\alpha)-\gamma''(s(\alpha))s'(\alpha))d\alpha \\ &= \int_{1/2}^t (\gamma''(r(\alpha))r'(\alpha)+\gamma''(s(\alpha))|s'(\alpha)|)d\alpha \\ &\geq 2\int_{1/2}^t \sqrt{\gamma''(r(\alpha))\gamma''(s(\alpha))r'(\alpha)|s'(\alpha)|}\,d\alpha \end{aligned}$$

for $1/2 < t < 1$. Thus

$$r(t)+s(t)-|x| \geq \frac{2v(t)(L-v(t))}{(\sup\gamma'')\sqrt{\gamma'(r(t))\gamma'(s(t))}}$$

for $1/2 < t < 1$, where

$$v(t) = \int_{1/2}^t \sqrt{\gamma''(r(\alpha))\gamma''(s(\alpha))r'(\alpha)|s'(\alpha)|}\,d\alpha$$

and $L = v(\tau(y))$. Thus

$$\begin{aligned} \mathrm{II} &\leq \sqrt{(\sup\gamma'')/2}\int_{1/2}^{\tau(y)} \frac{\sqrt{\gamma''(r(t))\gamma''(s(t))}}{\sqrt{v(t)(L-v(t))}}\left(\frac{y^2}{\gamma'(r(t))\gamma'(s(t))}\right)^{1/2} dt \\ &= \sqrt{(\sup\gamma'')/2}\int_{1/2}^{\tau(y)} \frac{\sqrt{\gamma''(r(t))\gamma''(s(t))}}{\sqrt{v(t)(L-v(t))}}\sqrt{r'(t)|s'(t)|}\,dt \\ &= \sqrt{(\sup\gamma'')/2}\int_{1/2}^{\tau(y)} \frac{v'(t)}{\sqrt{v(t)(L-v(t))}}dt. \end{aligned}$$

Applying the change of variable $l = v(t)$ to the last integral, we get

$$\mathrm{II} \leq \sqrt{(\sup\gamma'')/2}\int_0^L \frac{dl}{\sqrt{l(L-l)}} = \sqrt{(\sup\gamma'')/2}\,\pi.$$

Going back to the integral I, we now have

$$\mathrm{I} \leq \pi \sqrt{(\sup \gamma'')/2} \int_{2\gamma(m)}^{2\gamma(b)} h(x,y)dy,$$

and hence

$$\int h \, d\lambda * d\lambda \leq 2\pi \sqrt{2 \sup \gamma''} \int_A \int_{2\gamma(m)}^{2\gamma(b)} h(x,y)dydx \leq 2\pi \sqrt{2 \sup \gamma''} \, \|h\|_{L^1(\mathbb{R}^3)}.$$

This establishes (11) with $C^2 = 2\pi\sqrt{2 \sup \gamma''}$. Thus (8) holds with

$$C_1 = 4\pi^{1/4}(8 \sup \gamma'')^{1/8}.$$

□

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Department of Mathematics, American University of Beirut, Beirut, Lebanon
E-mail address: bshayya@aub.edu.lb