

SHARP MARCHAUD AND CONVERSE INEQUALITIES IN ORLICZ SPACES

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ABSTRACT. For L_p spaces on \mathbb{T}^d , \mathbb{R}^d and S^{d-1} , sharp versions of the classical Marchaud inequality are known. These results are extended here to Orlicz spaces (on \mathbb{T}^d , \mathbb{R}^d and S^{d-1}) for which $M(u^{1/q})$ is convex for some q , $1 < q \leq 2$, where $M(u)$ is the Orlicz function. Sharp converse inequalities for such spaces are deduced.

1. INTRODUCTION

The sharp Marchaud inequality

$$(1.1) \quad \omega^r(f, t)_p \leq Ct^r \left\{ \int_t^1 \frac{\omega^{r+1}(f, u)_p^q}{u^{rq+1}} du + \|f\|_p^q \right\}, \quad q = \begin{cases} p, & 0 < p \leq 2, \\ 2, & 2 \leq p < \infty, \\ 1, & p = \infty, \end{cases}$$

was proved for some measures of smoothness $\omega^r(f, t)_{L_p(\mathcal{D})} \equiv \omega^r(f, t)_p$ and some domains \mathcal{D} (see [Zy], [Ti], [Di, 88], [To], [Di, 99] and [Da-Di, 05]). Such results are an improvement over the classical Marchaud inequality, that is, (1.1) for $1 \leq p < \infty$ and $q = 1$, and it was shown in [Zy] that q of (1.1) is best possible for $1 \leq p < \infty$. Using a result of the type (1.1), one usually obtains

$$(1.2) \quad \omega^r(f, t)_p \leq Ct^r \left\{ \sum_{n \leq \frac{1}{t}} n^{rq-1} E_n(f)_p^q \right\}^{1/q}$$

with the appropriate best n -th degree approximation $E_n(f)_p$. We will show here that $L_p(\mathcal{D})$, $1 < p \leq 2$ and $2 \leq p < \infty$, can be replaced by Orlicz spaces with the Orlicz function $M(u)$ for which $M(u^{1/p})$ and $M(u^{1/2})$ are convex respectively.

There are two methods of proof for inequalities like (1.1). The first is by using a Littlewood-Paley type inequality (used by [Zy], [Ti] and [Da-Di, 05]) and the second one uses the concept of power-type q from the geometry of Banach spaces (see [Di, 88] and [Di, 99]). Each method has advantages, that is, enables one to

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obtain a more general result than the other in some direction. Here we use a result from the geometry of Banach spaces ([Ma-Tr]) to obtain an inequality of type (1.1) for Orlicz spaces for which $M(u^{1/q})$ is convex for some $q, 1 < q \leq 2$, where $M(u)$ is the Orlicz function. As we do not know a Littlewood-Paley type inequality for these spaces, it is natural to opt for the alternative method.

In Section 2 we derive the ingredients needed, and as corollaries obtain the analogues of (1.1) for $\mathbb{T}^d, \mathbb{R}^d$ and S^{d-1} with the appropriate measure of smoothness $\omega^r(f, t)_B$. In Section 3 we give the analogues of (1.2) with the corresponding concepts of best approximation and the required additional proofs either quoted or given directly.

2. THE SHARP MARCHAUD INEQUALITY

Traditionally, a Banach space B is said to be of power-type q if

$$(2.1) \quad \eta_B(\tau) \equiv \sup_{\|h\|_B=1, \|g\|_B=\tau} \left(\frac{\|h+g\|_B + \|h-g\|_B}{2} - 1 \right) \text{ satisfies } \eta_B(\tau) \leq K\tau^q.$$

In [Di, 88] the following alternative equivalent definition was introduced and used for proving the sharp Marchaud inequality:

$$(2.2) \quad \frac{\|h+g\|_B + \|h-g\|_B}{2} \leq (\|h\|_B^q + L\|g\|_B^q)^{1/q},$$

where L is independent of h and g .

The Orlicz function $M(t)$ on $[0, \infty)$ is an increasing convex function satisfying $M(0) = 0$. We assume here (as is commonly done) the Δ_2 condition, that is, $M(2t) \leq bM(t)$ for $t \in [0, t_0]$ and for $t \in [t_1, \infty)$. The Orlicz class of functions on \mathcal{D} with a positive measure $d\mu(x)$ is the class of all functions f for which

$$\int_{\mathcal{D}} M(|f(x)|) d\mu(x) < \infty,$$

and the norm (in the Luxemburg form) is

$$(2.3) \quad \|f\|_X = \inf\{\sigma > 0 : \int_{\mathcal{D}} M(|f(x)|/\sigma) d\mu(x) \leq 1\}.$$

Maleev and Troyanski [Ma-Tr, Theorem 1] proved the following theorem.

Theorem 2.1 (Maleev-Troyanski). *Suppose that $M(t)$ is an Orlicz function satisfying the Δ_2 condition and that $M(lt) \leq \frac{1}{2}lM(t)$ for some $l < 1$. Then there exists an Orlicz function $N(t)$ equivalent to $M(t)$ such that*

$$(2.4) \quad \eta_B(\tau) \leq \tau^2 \sup \left\{ \frac{M(uv)}{u^2M(v)}; u \in [\tau, 1], v \in (0, \infty) \right\},$$

where B is the Orlicz space generated by the Orlicz function $M(t)$ but endowed with the equivalent norm

$$(2.3)' \quad \|f\|_B = \inf\{\sigma > 0 : \int_{\mathcal{D}} N(|f(x)|/\sigma) d\mu(x) \leq 1\}.$$

Lemma 2.2. *Suppose $M(u^{1/q})$ is convex for some $q, 1 < q \leq 2$, where $M(u)$ is an Orlicz function satisfying the Δ_2 condition. Then there exists an Orlicz function $N(u)$, $A^{-1}N(u) \leq M(u) \leq AN(u)$ and $\eta_B(\tau) \leq c\tau^q$ for $\eta_B(\tau)$ given by (2.1) with the norm of B given by (2.3)'.*

Proof. As $M(u^{1/q})$ is convex and $M(0) = 0$, we have

$$\frac{M(u_1)}{u_1^q} \leq \frac{M(u_2)}{u_2^q} \quad \text{for } 0 < u_1 < u_2.$$

Hence, for $l < 1$

$$\frac{M(lt)}{l^q t^q} \leq \frac{M(t)}{t^q} \quad \text{and} \quad M(lt) \leq l^q M(t),$$

and we may choose l such that $l^q < \frac{1}{2}l$. We now use (2.4) to obtain

$$\begin{aligned} \eta_B(\tau) &\leq \tau^2 \sup \left\{ \frac{M(uv)}{u^2 M(v)}; u \in [\tau, 1], v \in (0, \infty) \right\} \\ &\leq \tau^2 \sup \left\{ \frac{(uv)^q}{u^2 v^q}; u \in [\tau, 1], v \in (0, \infty) \right\} \\ &= \tau^2 \sup \{u^{q-2}; u \in [\tau, 1]\} = \tau^q. \end{aligned}$$

□

Using Lemma 2.2, we obtain the sharp Marchaud inequality for \mathbb{T}^d and \mathbb{R}^d .

Theorem 2.3. *Suppose B is an Orlicz space of functions on \mathbb{T}^d or \mathbb{R}^d with M of Lemma 2.2. Then*

$$(2.5) \quad \omega^r(f, t)_B \leq Ct^r \left\{ \int_t^1 \frac{\omega^{r+1}(f, u)_B^q}{u^{rq+1}} du + \inf_{c \in \mathbb{R}} \|f - c\|_B^q \right\}^{1/q},$$

where

$$(2.6) \quad \omega^r(f, t)_B \equiv \sup_{|h| \leq t} \|\Delta_h^r f\|_B, \quad \Delta_h f(x) \equiv f(x+h) - f(x), \quad \Delta_h^r f(x) \equiv \Delta_h(\Delta_h^{r-1} f(x)),$$

and $\|f\|_B$ is any norm equivalent to (2.3).

The case $q = 1$ in Theorems 2.3 and 2.4 was already covered by the classical Marchaud-type inequality (see also [Di, 88] and [Di, 99]).

Proof of Theorem 2.3. We use the norm given in (2.3)' by $N(u)$ and note that for this norm, translation is an isometry and $\eta_B(\tau) \leq c\tau^q$. Therefore, the conditions of Theorem 1 of [Di, 88] are satisfied and (2.5) is valid with $\omega^r(f, t)_B$ defined by (2.6) using the norm B given in (2.3)'. Once (2.5) is satisfied for the norm given by (2.3)', it is valid for all equivalent norms (including that defined by (2.3)). We may write $f - c$ instead of f as $\omega^r(f - c, t)_B = \omega^r(f, t)_B$ for $r \geq 1$. □

Theorem 2.4. *Suppose B is an Orlicz space of functions on S^{d-1} (the unit sphere in \mathbb{R}^d) with M of Lemma 2.2. Then (2.5) is valid, where*

$$(2.7) \quad \omega^r(f, t)_B \equiv \sup_{\rho \in O_t} \|\Delta_\rho^r f\|_B, \quad \Delta_\rho f(x) \equiv f(\rho x) - f(x), \quad \Delta_\rho^r f(x) \equiv \Delta_\rho(\Delta_\rho^{r-1} f(x)),$$

$$O_t \equiv \{\rho \in \text{SO}(d) : \rho x \cdot x \geq \text{cost for all } x \in S^{d-1}\},$$

where $\text{SO}(d)$ is the group of $d \times d$ matrices of determinant 1, and B is given by any norm, equivalent to (2.3) with $\mathcal{D} \equiv S^{d-1}$ and the measure on the sphere induced by the Lebesgue measure.

Proof. We note that (2.7) is the modulus of smoothness introduced in [Di, 99] and follow the proof of Theorem 2.3 to deduce Theorem 2.4 from [Di, 99, Theorem 3.1]. \square

3. CONVERSE THEOREMS

In this section we will give estimates of $\omega^r(f, t)_B$ in terms of best approximation for the Orlicz spaces described in Lemma 2.2. We note that the concepts of best approximation on \mathbb{T}^d , \mathbb{R}^d and S^{d-1} differ and will be defined separately in each case. In this section, n is a positive integer.

Theorem 3.1. *Suppose B is an Orlicz space of functions on \mathbb{T}^d given by $M(u)$ of Lemma 2.2 for some q , $1 < q \leq 2$. Then*

$$(3.1) \quad \omega^r(f, t)_B \leq Ct^r \left\{ \sum_{n \leq \frac{1}{t}} n^{rq-1} E_n(f)_B^q \right\}^{1/q},$$

where

$$(3.2) \quad E_n(f)_B = \inf \{ \|f - T_n\|_B : T_n \in \text{span } e^{ix \cdot l}, |l| \equiv |(l_1, \dots, l_d)| \equiv \max_{1 \leq i \leq d} |l_i| < n \}.$$

Proof. We first observe that for a trigonometric polynomial of degree n (as in (3.2)) we have

$$\left\| \left(\frac{\partial}{\partial \xi} \right)^r T_n \right\|_{L_\infty(\mathbb{T}^d)} \leq d^{r/2} n^r \|T_n\|_{L_\infty(\mathbb{T}^d)},$$

and hence

$$(3.3) \quad \|\Delta_h^r T_n\|_{L_\infty(\mathbb{T}^d)} \leq d^{r/2} h^r n^r \|T_n\|_{L_\infty(\mathbb{T}^d)}.$$

Therefore, for any Banach space B of functions on \mathbb{T}^d for which translation is an isometry

$$(3.4) \quad \|\Delta_h^r T_n\|_B \leq d^{r/2} h^r n^r \|T_n\|_B.$$

For the proof of (3.4) (which follows from the technique in [Di, 80]), we observe that for $g \in B^*$ (where B^* is the dual to B), $T_n * g(u) \equiv \langle T_n(\cdot + u), g(\cdot) \rangle$ is a trigonometric polynomial that satisfies $\|T_n * g\|_{L_\infty(\mathbb{T}^d)} \leq \|T_n\|_B \cdot \|g\|_{B^*}$. We choose $g_\varepsilon \in B^*$, $\|g_\varepsilon\|_{B^*} = 1$ and set $Q_{n,\varepsilon} = T_n * g_\varepsilon \in L_\infty$ such that

$$\begin{aligned} \|\Delta_h^r T_n\|_B - \varepsilon &\leq |\Delta_h^r(T_n * g_\varepsilon)(0)| = |\Delta_h^r(Q_{n,\varepsilon})(0)| \\ &\leq \|\Delta_h^r Q_{n,\varepsilon}\|_{L_\infty(\mathbb{T}^d)} \leq d^{r/2} h^r n^r \|Q_{n,\varepsilon}\|_{L_\infty(\mathbb{T}^d)} \\ &\leq d^{r/2} h^r n^r \|T_n\|_B, \end{aligned}$$

and as $\varepsilon > 0$ is arbitrary, (3.4) is established.

We observe that it is sufficient to prove (3.1) for some equivalent norm, and we use the Luxemburg form of the Orlicz space with $M(u)$ (or $N(u)$ of the Maleev-Troyanski Theorem) for which translation is an isometry. We choose c in Theorem 2.3 so that $E_1(f)_B = \|f - c\|_B$, and use the following (equivalent) form of (2.5):

$$(3.5) \quad \omega^r(f, 2^{-n})_B \leq C2^{-nr} \left(\sum_{k=1}^n 2^{krq} \omega^{r+1}(f, 2^{-k})_B^q + E_1(f)_B^q \right)^{1/q}.$$

We set for the estimate of $\omega^{r+1}(f, 2^{-k})_B$,

$$f = f - P_{2^k} + \sum_{l=1}^k (P_{2^l} - P_{2^{l-1}}) + P_1,$$

where P_j , which is of degree $j - 1$, satisfies $\|f - P_j\|_B = E_j(f)_B$. Using the above, we have

$$\begin{aligned} \omega^{r+1}(f, 2^{-k})_B &\leq 2^{r+1} E_{2^k}(f)_B + 2d^{r/2} \sum_{l=1}^k 2^{-k(r+1)} 2^{l(r+1)} E_{2^{l-1}}(f)_B \\ &\leq C_1 \sum_{l=0}^k 2^{(l-k)(r+1)} E_{2^l}(f)_B. \end{aligned}$$

Using Jensen’s inequality, we derive

$$(3.6) \quad \omega^{r+1}(f, 2^{-k})_B^q \leq C_2 \sum_{l=0}^k [2^{(l-k)r} E_{2^l}(f)_B]^q 2^{l-k}$$

with C_2 independent of f and k . Substituting (3.6) in (3.5), we conclude

$$\begin{aligned} \omega^r(f, 2^{-n})_B &\leq C \cdot C_2 2^{-nr} \left(\sum_{k=0}^n 2^{krq} \sum_{l=0}^k 2^{(l-k)rq} 2^{l-k} E_{2^l}(f)_B^q \right)^{1/q} \\ &\leq C \cdot C_2 2^{-nr} \left(\sum_{l=0}^n 2^{lrq} E_{2^l}(f)_B^q \sum_{k=l}^n 2^{l-k} \right)^{1/q} \\ &\leq C_3 2^{-nr} \left(\sum_{l=0}^n 2^{lrq} E_{2^l}(f)_B^q \right)^{1/q}, \end{aligned}$$

which is equivalent to (3.1). □

As a corollary of the technique used in the proof of Theorem 3.1, we also obtain the following result.

Theorem 3.2. *Suppose B is a Banach space of functions on \mathbb{T}^d satisfying $\|f(\cdot)\|_B = \|f(\cdot + h)\|_B$, $\|f(\cdot + h) - f(\cdot)\|_B = o(1)$ for $h \rightarrow 0+$, whenever $f \in B$, and $T_n \in B$ for all trigonometric polynomials T_n . Then*

$$(3.7) \quad \omega^r(f, t)_B \leq Ct^r \left\{ \sum_{n \leq \frac{1}{t}} n^{r-1} E_n(f)_B \right\},$$

where $\omega^r(f, t)_B$ and $E_n(f)_B$ are given by (2.6) and (3.2) respectively.

Proof. We use (3.4), which was proved for B of the present theorem, together with (2.5) with $q = 1$ (which is well established for B of the present theorem) to complete the proof of (3.7). □

For \mathbb{R}^d we have the following result.

Theorem 3.3. *Suppose B is an Orlicz space of functions on \mathbb{R}^d given by $M(u)$ of Lemma 2.2 for some $q, 1 < q \leq 2$. Then*

$$(3.8) \quad \omega^r(f, t)_B \leq Ct^r \left\{ \int_1^{1/t} u^{rq-1} E_u(f)_B^q du + \|f\|_B^q \right\}^{1/q},$$

where

$$(3.9) \quad E_u(f)_B = \inf\{\|f - \varphi_u\|_B : \text{supp } \hat{\varphi}_u(y) \subset [y : |y| \leq u]\},$$

and where

$$\hat{f}(y) = \int_{\mathbb{R}^d} f(\xi) e^{-2\pi i y \cdot \xi} d\xi.$$

Proof. By monotonicity of $\omega^r(f, t)_B, u^{r(q-1)}$ and of $E_u(f)_B^q$, we have to prove only

$$(3.8)' \quad \omega^r(f, 2^{-n})_B \leq C2^{-nr} \left\{ \sum_{l=0}^n 2^{lr} E_{2^l}(f)_B^q + \|f\|_B^q \right\},$$

which is equivalent to (3.8). Using (3.5) (and observing that here $E_1(f)_B^q$ is replaced by $\|f\|_B^q$ since $c \neq 0$ does not belong to an Orlicz space on \mathbb{R}^d for which M is finite), we have only to establish

$$(3.10) \quad \|\Delta_h^r \varphi_u\|_B \leq Ah^r u^r \|\varphi_u\|_B$$

with A independent of h and u , and follow the proof of Theorem 3.1. We first show (3.10) for $L_p(\mathbb{R}^d), 1 \leq p \leq \infty$ (needed here only for $p = 1$ and $p = \infty$). Using [Da-Di, 04, Theorem 2.1], we have

$$\|\Delta^r \varphi_u\|_{L_p(\mathbb{R}^d)} \leq Au^{2r} \|\varphi_u\|_{L_p(\mathbb{R}^d)},$$

where Δ is the Laplacian. We now apply [Di, 89, Theorem 6.2] to obtain

$$\left\| \left(\frac{\partial}{\partial \xi} \right)^r \varphi_u \right\|_{L_p(\mathbb{R}^d)} \leq C \|\Delta^r \varphi_u\|_{L_p(\mathbb{R}^d)}^{1/2} \|\varphi_u\|_{L_p(\mathbb{R}^d)}^{1/2} \leq C_1 u^r \|\varphi_u\|_{L_p(\mathbb{R}^d)}.$$

This implies for $1 \leq p \leq \infty$

$$\|\Delta_h^r \varphi_u\|_{L_p(\mathbb{R}^d)} \leq C_2 h^r u^r \|\varphi_u\|_{L_p(\mathbb{R}^d)}.$$

We use the linear operator $O = \Delta_h^r \eta_u$ with the delayed means operator η_u (satisfying $\eta_u \varphi_u = \varphi_u$) given by

$$(\eta_u(f))^\wedge(x) = \eta \left(\frac{|x|}{u} \right) \hat{f}(x), \quad u > 0,$$

where $\eta(y) \in C^\infty(\mathbb{R}^d), \eta(y) = 0$ for $y \leq 1$ and $\eta(y) = 0$ for $y \geq 2$ (see [Da-Di, 04, Section 2]). We now follow [Ka-Ma, Theorem 1.1] to obtain (3.10) for our Orlicz space. Actually (3.10) can be proved directly and for a wider class of spaces, but as it is just a step in our proof, it will lead too far away from the topic. We now follow the proof of Theorem 3.1 to obtain (3.8)'. \square

For the sphere, the analogue of Theorems 3.1 and 3.3 is given in the following result.

Theorem 3.4. *For an Orlicz space B on S^{d-1} and $\omega^r(f, t)_B$ of Theorem 2.4, we have (3.1), where*

$$(3.11) \quad E_n(f)_B = \inf\{\|f - \varphi_n\|_B : \varphi_n \in \text{span} \bigcup_{k < n} H_k\},$$

$$H_k = \{\varphi : \tilde{\Delta}\varphi = -k(k+d-2)\varphi\} \quad \text{and} \quad \tilde{\Delta}f(x) = \Delta f\left(\frac{x}{|x|}\right) \quad \text{for } x \in S^{d-1}.$$

Proof. Following the proof of Theorem 3.1 and of [Di, 99, Theorem 4.2], we observe that it is sufficient to prove for our space B

$$(3.12) \quad \|\Delta_\rho^r \varphi_n\|_B \leq Ct^r n^r \|\varphi_n\|_B \quad \text{for } \rho \in O_t \quad \text{and} \quad \varphi_n \in \text{span} \bigcup_{k < n} H_k.$$

The inequality (3.12) was proved in [Di, 99, (4.5)] for $1 \leq p \leq \infty$, and using [Ka-Ma, Theorem 1.1] on the linear operator $O = \Delta_\rho^r \eta_{1/t}$ with the delayed mean $\eta_{1/t}$, to (3.12) for $B = L_1(S^{d-1})$ and $B = L_\infty(S^{d-1})$, we obtain (3.12) for our Orlicz space and complete the proof. \square

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