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EQUIVALENCE OF COMPLETENESS AND CONTRACTION PROPERTY

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ABSTRACT. In this paper, we consider the completeness and the contraction property in metric spaces and show that the contraction property implies Lipschitz-completeness or arcwise-completeness in a metric space. However, in a metric space, the contraction property does not imply the usual completeness. We prove that a locally Lipschitz-connected metric space has the contraction property if and only if it is Lipschitz-complete and that a locally arcwise-connected metric space is arcwise-complete if and only if X has the strong contraction property.

1. INTRODUCTION

Many authors have considered the topological characterization and the equivalence of the contraction property (see [1]–[6], [8]). In [7], Kirk showed that the Caristi fixed point theorem holds only in a complete metric set and Sullivan [9] showed the same for the Ekeland principle. These lead naturally to the question whether a metric space with the contraction property is complete.

Borwein [1] proved the interesting result that a uniformly Lipschitz-connected subset (such as a convex subset of a normed space) has the contraction property if and only if the subset is complete. This implies that a normed space is complete if and only if every contraction on the space has a fixed point. In [1], Borwein also showed that one cannot hope to extend this result much, as the following examples illustrate (see, Examples 3, 4 in [1]).

Example 1.1. There exists an incomplete nonuniformly Lipschitz-connected subset of the Euclidean plane with the contraction property. Let

$$C = \{(x, y)|0 < x \leq 1, y = \sin(1/x)\}.$$

Example 1.2. There exists an incomplete starshaped subset of the Euclidean plane with the contraction property. Let

$$C = \bigcup\{L_k|k \in N\},$$

where

$$L_k = \text{conv}\{(0, 0), (1, 2^{-k})\}.$$

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In fact, Examples 1.1, 1.2 show that the sufficient and necessary condition fails if "uniformly Lipschitz-connected" is replaced by "Lipschitz-connected" or "star-shaped". Thus, in a metric space, we are led to ask what completeness the contraction property can guarantee and under what conditions it implies that the contraction property is equivalent to completeness.

In this paper, based on Lipschitz-completeness and arcwise-completeness, we show that a metric space has the contraction property only if the space is Lipschitz-complete and the converse remains true in Lipschitz-connected spaces. Our proof of Lipschitz-completeness of the space depends on the technique used by Borwein [1] to construct a contraction map. Furthermore, in the sense of equivalent metric, we prove that an arcwise-connected metric space has the strong contraction property if and only if the space is arcwise-complete.

2. Preliminaries

Let (X, d) be a metric space. X is said to be of the contraction property if and only if for any selfmap T on X, T has a fixed point whenever T is a Banach contraction under the metric d. X is said to be of the strong contraction property if and only if for any selfmap T on X, T has a fixed point in X whenever T is a Banach contraction under some metric d^* uniformly equivalent to d.

Let us say that a subset C of the metric space X endowed with a metric d is uniformly Lipschitz-connected if there exists a positive constant L such that given any x^0 and x^1 in C there exists an arc $g : [0,1] \mapsto C$ with $g(0) = x^0$ and $g(1) = x^1$ such that

$$d(g(s), g(t)) \leq L|s-t| d(g(0), g(1))$$

for $0 \leq s, t \leq 1$ (see [1]). An arc $g : [0,1] \to C$ with $g(0) = x^0$ and $g(1) = x^1$ is called a Lipschitz arc if g satisfies the Lipschitz condition, i.e., there exists $L > 0$ such that

$$d(g(s), g(t)) \leq L|s-t| \tag{2.1}$$

for $0 \leq s, t \leq 1$. A subset C of X is said to be Lipschitz-connected if for arbitrary x^0 and x^1 in C there exists a Lipschitz arc $g : [0,1] \to C$ such that $g(0) = x^0$ and $g(1) = x^1$. It is easy to see that a uniformly Lipschitz-connected space is Lipschitz-connected but the converse is not true (see [1]). A subset C of X is said to be locally arcwise-connected (respectively, locally Lipschitz-connected) if there is some $\delta > 0$ such that for any x^0 and x^1 in C there exists an arc (respectively, a Lipschitz arc) g linking x^0 and x^1 whenever $d(x_0, x_1) < \delta$.

Lemma 2.1. *Let $g_1, g_2 : [0,1] \to C$ be Lipschitz arcs with $g_1(1) = g_2(0)$. Then*

$$g(t) = \begin{cases} g_1(2t), & 0 \leq t < \frac{1}{2}, \\ g_2(2t-1), & \frac{1}{2} \leq t \leq 1, \end{cases}$$

is also a Lipschitz arc.

Proof. Let $s, t \in [0,1]$. For $s, t \in [0, \frac{1}{2}]$ there exists a constant $L_1 > 0$ such that

$$d(g(s), g(t)) = d(g_1(2s), g_1(2t)) \leq L_1|s-t|,$$

and for $s, t \in [\frac{1}{2}, 1]$, there exists a constant $L_2 > 0$ such that

$$d(g(s), g(t)) = d(g_2(2s), g_2(2t)) \leq L_2|s-t|.$$

For $s \in [0, \frac{1}{2}]$ and $t \in [\frac{1}{2}, 1]$, we have

$$\begin{aligned} d(g(s), g(t)) &\leq d(g(s), g(\tfrac{1}{2})) + d(g(\tfrac{1}{2}), g(t)) \leq L_1|s - \tfrac{1}{2}| + L_2|\tfrac{1}{2} - t| \\ &\leq L_1(\tfrac{1}{2} - s) + L_2(t - \tfrac{1}{2}) \leq L_1(t - s) + L_2(t - s) \\ &= (L_1 + L_2)|t - s|. \end{aligned}$$

This implies g is a Lipschitz arc. □

Theorem 2.2. *Let $C = \bigcup\{L_\alpha : \alpha \in I\}$, where $\{L_\alpha : \alpha \in I\}$ is a family of Lipschitz-connected subsets of a metric space. For arbitrary $L_\beta, L_\gamma \in \{L_\alpha : \alpha \in I\}$, there exist $L_\beta = L_0, L_1, \cdots, L_k, L_{k+1} = L_\gamma$ such that $L_i \cap L_{i+1} \neq \emptyset$ for all $i = 0, 1, \cdots, k$. Then C is Lipschitz-connected.*

Proof. For $x_0, x_1 \in C$, assume $x_0 \in L_\beta, x_1 \in L_\gamma$. Then there exist $L_\beta = L_0, L_1, \cdots,$ L_k, $L_{k+1} = L_\gamma$ such that $L_i \cap L_{i+1} \neq \emptyset$. Since L_0, L_1 are Lipschitz-connected and $L_0 \cap L_1 \neq \emptyset$, it is clear from Lemma 2.1 that $L_0 \cup L_1$ is Lipschitz-connected. By Lemma 2.1 and induction, it follows that $\bigcup_{i=0}^{k+1} L_i$ is also Lipschitz-connected and there is a Lipschitz arc in C linking x_0 and x_1. Hence C is Lipschitz-connected. □

In particular $C = \bigcup\{L_\alpha : \alpha \in I\}$ is Lipschitz-connected whenever $\{L_\alpha : \alpha \in I\}$ is a family of Lipschitz-connected subsets of a metric space whose intersection is nonempty. Therefore a starshaped subset of a normed space is Lipschitz-connected. It is easy to check that the subsets of Examples 1.1, 1.2 are Lipschitz-connected. The following example is also a Lipschitz-connected subset.

Example 2.3. Let C be a subset of the Euclidean plane defined by

$$C = \{(t, 0) : t \in [0, 1]\} \cup \left(\bigcup_{k=0}^{\infty} \{(2^{-k}, t) : t \in [0, 1]\} \right).$$

By Theorem 2.2, C is Lipschitz-connected.

A continuous mapping $g : (0, 1] \to C$ is called a semi-closed arc if for each $\epsilon > 0$, there exists some $\delta > 0$ such that $d(g(s), g(t)) < \epsilon$ for all $0 \leq s, t < \delta$. A semi-closed arc g is called a Lipschitz semi-closed arc if g satisfies the Lipschitz condition.

Definition 2.4. Let X be a metric space.

(1) X is said to be arcwise-complete if for each semi-closed arc $g : (0, 1] \to X$, $\lim_{s \to 0} g(s)$ exists in X;
(2) X is said to be Lipschitz-complete if for each Lipschitz semi-closed arc $g : (0, 1] \to X$, $\lim_{s \to 0} g(s)$ exists in X.

Remark 2.1. Arcwise-completeness is weaker than usual completeness even if in an arcwise connected space, so is Lipschitz-completeness (see Examples 1.1, 1.2 and 2.3). It is obvious from the definitions that Lipschitz-completeness is weaker than arcwise-completeness.

It is well known that X has usual completeness if and only if $\bigcap F_n \neq \emptyset$ whenever $\{F_n : n \in N\}$ is a sequence of nonempty closed subsets of X with $F_{n+1} \subset F_n$ and $\operatorname{diam}(F_n) \to 0$. Concerning Lipschitz-completeness and arcwise-completeness, we have the following properties.

Theorem 2.5. *Let X be a metric space.*

(1) *X is arcwise-complete if and only if $\bigcap F_n \neq \emptyset$ whenever $\{F_n : n \in N\}$ is a sequence of arcwise-connected and nonempty closed subsets of X with $F_{n+1} \subset F_n$ and $\mathrm{diam}(F_n) \to 0$, where $\mathrm{diam}(F_n) := \sup_{x,y \in F_n} d(x,y)$ denotes the diameter of F_n.*

(2) *X is Lipschitz-complete if and only if $\bigcap F_n \neq \emptyset$ whenever $\{F_n : n \in N\}$ is a sequence of Lipschitz-connected and nonempty closed subsets of X with $F_{n+1} \subset F_n$ and $\mathrm{diam}(F_n) \to 0$.*

Proof. (i)"$\Rightarrow$": Let X be an arcwise-complete space. $\{F_n : n \in N\}$ is a sequence of arcwise-connected and nonempty closed subsets of X with $F_{n+1} \subset F_n$ and $\mathrm{diam}(F_n) \to 0$. For each F_n, choose $x_n \in F_n$ and an arc $g_n : [0,1] \mapsto F_n$ such that $g_n(0) = x_{n+1}, g_n(1) = x_n$. Define $g : (0,1] \mapsto X$ as follows:

$$g(s) = g_n(2^{n+1}s - 1), \quad \forall \frac{1}{2^{n+1}} < s \leq \frac{1}{2^n}, \ \ n = 1, 2, \cdots .$$

This implies from $\mathrm{diam}(F_n) \to 0$ that g is a semi-closed arc in X. The arcwise-completeness of X implies $\lim_{s\to 0} g(s) = \overline{x} \in X$. For each $n \in N$, since F_n is closed and $g(s) \in F_n$ for all $s \in (0, \frac{1}{2^n}]$, $\overline{x} \in F_n$, say $\bigcap F_n \neq \emptyset$.

"$\Leftarrow$": Let $\overline{X}$ be the completion of X. Suppose that X were not arcwise-complete. Then there exists some semi-closed arc g in X such that $\lim_{s\to 0} g(s) = \overline{x} \notin X$. Define F_n by

$$F_n = \{g(s) : s \in (0, \frac{1}{2^n}]\}, \quad n \in N.$$

It is easy to check that $\{F_n : n \in N\}$ is a sequence of arcwise-connected and nonempty closed subsets of X with $F_{n+1} \subset F_n$ and $\mathrm{diam}(F_n) \to 0$. So $\bigcap F_n \neq \emptyset$, say $\overline{x} \in F_n \subset X$, which is a contradiction.

(ii) Similar to the proof of (i), we only need to replace "arcwise-connected" by "Lipschitz-connected". □

3. The contraction property and Lipschitz-completeness

Theorem 3.1. *Let X be a metric space. If X has the contraction property, then X is Lipschitz-complete.*

Proof. Let $\overline{X}$ be the completion of X and $g : (0,1] \to X$ be a Lipschitz semi-closed arc. Then

$$d(g(s), g(t)) \leq L|s - t|, \quad \forall 0 < s, t \leq 1. \tag{3.1}$$

Set $\lim_{s\to 0} g(s) = \overline{x} \in \overline{X}$ and let $h : X \to [0,1]$ be given by

$$h(x) = \frac{d(x, \overline{x})}{2L(1 + d(x, \overline{x}))}.$$

Define $g(0) = \overline{x} \in \overline{X}$ and $X' = X \cup \{\overline{x}\}$. Then g extends to a mapping satisfying (3.1) on $[0,1]$. Define $T : X' \to X'$ by

$$T = g \circ h.$$

T is a metric contraction (with contraction constant $\frac{1}{2}$) since

$$\begin{aligned} d(Tx,Ty) &= d(g(\tfrac{d(x,\overline{x})}{2L(1+d(x,\overline{x}))}), g(\tfrac{d(y,\overline{x})}{2L(1+d(y,\overline{x}))})) \\ &\leq L|\tfrac{d(x,\overline{x})}{2L(1+d(x,\overline{x}))} - \tfrac{d(y,\overline{x})}{2L(1+d(y,\overline{x}))}| \\ &\leq \tfrac{1}{2}|d(x,\overline{x}) - d(y,\overline{x})| \\ &\leq \tfrac{1}{2}d(x,y). \end{aligned}$$

Note that

$$\overline{x} = g(0) = g(h(\overline{x})) = T(\overline{x}).$$

$\overline{x}$ is the unique fixed point of T in X'. Finally observing that $T : X \to X$ and T is also a metric contraction on X, we have that the unique fixed point $\overline{x}$ of T is in X and this completes the proof. □

In [1], Borwein remarked with some examples that the contraction property holds in some incomplete metric spaces (see Examples 1.1, 1.2). Theorem 3.1 shows that the contraction property cannot ensure usual completeness but can ensure Lipschitz-completeness.

Theorem 3.2. *A locally Lipschitz-connected metric space has the contraction property if and only if it is Lipschitz-complete.*

Proof. From Theorem 3.1, it remains to show that if X is Lipschitz-complete, then X has the contraction property. Let $T : X \to X$ be a metric contraction with contraction constant $0 \leq h < 1$. Note that T is contractive. Select x_0 in X with $d(x_0, Tx_0) < \delta$ and arc $g_0 : [0,1] \to X$ which connect x_0 and Tx_0 in Lipschitz fashion as in (1). For each $k \in N$, define $g_k : [0,1] \to X$ by $g_k(s) = T^k g_0(s)$ for all $s \in [0,1]$. Then g_k connects $x_k = T^k(x_0)$ and $x_{k+1} = T^{k+1}(x_0)$ for each $k \in N$. Define $g : (0,1] \to X$ by

$$g(s) = g_k(k(k+1)t - k), \quad \frac{1}{k+1} < t \leq \frac{1}{k}, \quad k = 1, 2, \cdots .$$

On each interval of the form $(\frac{1}{k+1}, \frac{1}{k}]$ we have

$$\begin{aligned} d(g(s), g(t)) &= (g_k(k(k+1)s - k), g_k(k(k+1)t - k)) \\ &= d(T^k g_0(k(k+1)s - k), T^k g_0(k(k+1)t - k)) \\ &\leq h^k d(g_0(k(k+1)s - k), g_0(k(k+1)t - k)) \\ &\leq Lk(k+1)h^k|s - t|. \end{aligned}$$

Since $0 < h < 1$ and $\lim\limits_{k\to\infty} k(k+1)h^k = 0$, there exists some $N_0 > 0$ such that $k(k+1)h^k \leq 1$ for all $k > N_0$. Let $M_1 = \max\{1 \cdot 2h, 2 \cdot 3h^2, \cdots, N_0(N_0+1)h^{N_0}\}$ and $L_1 = \max\{LM_1, L\}$. Then for each $k \in N$, we have

$$d(g(s), g(t)) \leq L_1|s - t|, \qquad \forall s, t \in (\frac{1}{k+1}, \frac{1}{k}). \tag{3.2}$$

Let $s \in (\frac{1}{k+1}, \frac{1}{k})$, $t \in (\frac{1}{k+l+1}, \frac{1}{k+l})$ $(k, l \in N)$. Thus by (3.2) and the continuity of g, we have

(3.3)
$$\begin{aligned} d(g(s), g(t)) &\le d(g(s), g(\tfrac{1}{k+1})) + d(g(\tfrac{1}{k+1}), g(\tfrac{1}{k+2})) + \cdots + d(g(\tfrac{1}{k+l}), g(t)) \\ &\le L_1|s - \tfrac{1}{k+l}| + L_1|\tfrac{1}{k+l} - \tfrac{1}{k+2}| + \cdots + L_1|\tfrac{1}{k+l} - t| \\ &\le L_1|s - t|. \end{aligned}$$

From (3.2) and (3.3), $g : (0, 1] \to X$ is a Lipschitz semi-closed arc in X. Then $\overline{x} = \lim_{s\to 0} g(s) \in X$. Finally observing that T is metric contraction and that g connects $\{x^k = T^k x_0 : k \in N\}$, we have that $\overline{x} \in X$ is a fixed point of T. □

Corollary 3.3. *A starshaped subset of a normed space has the contraction property if and only if it is Lipschitz-complete.*

By Theorem 3.2 and Corollary 3.3, we know that the subsets of Examples 1.1, 1.2 and 2.3 have Lipschitz-completeness, so that they have the contraction property. Theorem 3.2 can be considered as a generalization of the Banach contraction principle under weaker completeness.

4. The contraction property and arcwise-completeness

In this section, we consider the equivalence relation between completeness and the contraction principle in the sense of the equivalence metric.

Lemma 4.1 ([8])**.** *Let T operate on a metric space (X, ρ). There exists a bounded metric σ uniformly equivalent to ρ on X such that T is a Banach contraction under σ if, and only if, T is uniformly continuous and*
$$\mathrm{diam}(T^n X) \to 0.$$

Theorem 4.2. *Let (X, d) be a metric space. If X has the strong contraction property, then X is arcwise-complete.*

Proof. Let $\overline{X}$ be the completion of X, $g : (0, 1] \mapsto X$ be a semi-closed arc and $\overline{x} = \lim_{s\to 0} g(s)$. Set $g(0) = \overline{x}$ and $X' = X \cup \{\overline{x}\}$. Then g continuously extends to a mapping on $[0, 1]$.

Observe that $g(s) \to \overline{x}$ as $s \to 0$. For each number sequence $\{\epsilon_k\}_{k=1}^{\infty}$ with $1 = \epsilon_0 > \epsilon_1 > \epsilon_2 > \cdots$ and $\epsilon_k \to 0$ as $k \to \infty$, there exists a sequence $\{\delta_k\}_{k=1}^{\infty}$ such that $0 \le \delta_{k+1} < \delta_k \le \epsilon_k$ and

$$d(g(t'), g(t'')) < \epsilon_k, \qquad \forall 0 \le t', t'' < \delta_k. \tag{4.1}$$

Let $\overline{d}(x, y) = \frac{d(x,y)}{1+d(x,y)}$ and $h : X' \to [0, 1]$ be given by

$$h(x) = \begin{cases} \frac{\delta_{k+1}}{2^{k+1}} \cdot \frac{\bar{d}(x,\overline{x}) - \epsilon_{k-1}}{\epsilon_k - \epsilon_{k-1}} + \frac{\delta_k}{2^k} \cdot \frac{\epsilon_k - \bar{d}(x,\overline{x})}{\epsilon_k - \epsilon_{k-1}}, & \epsilon_k < \bar{d}(x, \overline{x}) \le \epsilon_{k-1},\ k = 1, 2, \cdots; \\ 0, & x = \overline{x}. \end{cases}$$

Define $T : X' \to X'$ by
$$T = g \circ h.$$

It is easy to check that T is uniformly continuous and $T\bar{x} = \bar{x}$. Furthermore, for each $x \in X$, we have

$$|h(x) - 0| = |\frac{\delta_{k+1}}{2^{k+1}} \cdot \frac{\bar{d}(x, \overline{x}) - \epsilon_{k-1}}{\epsilon_k - \epsilon_{k-1}} + \frac{\delta_k}{2^k} \cdot \frac{\epsilon_k - \bar{d}(x, \overline{x})}{\epsilon_k - \epsilon_{k-1}}| \le \frac{\delta_{k+1}}{2^{k+1}} + \frac{\delta_k}{2^k} < \delta_k.$$

By (4.1),

$$d(Tx, \bar{x}) = d(g(h(x)), g(0)) < \epsilon_k.$$

Since $\overline{d}(x, y) \le d(x, y)$, without loss of generality, assume $\epsilon_{k'+1} < \bar{d}(Tx, \bar{x}) \le \epsilon_{k'}$ where $k' \ge k$. Then

$$|h(Tx) - 0| = |\frac{\delta_{k'+2}}{2^{k'+2}} \cdot \frac{\bar{d}(x, \overline{x}) - \epsilon_{k'}}{\epsilon_{k'+1} - \epsilon_{k'}} + \frac{\delta_{k'+1}}{2^{k'+1}} \cdot \frac{\epsilon_{k'+1} - \bar{d}(x, \overline{x})}{\epsilon_{k'+1} - \epsilon_{k'}}| \le \frac{\delta_{k'+2}}{2^{k'+2}} + \frac{\delta_{k'+1}}{2^{k'+1}} < \delta_{k+1}.$$

By (4.1) again, this implies $d(T^2x, \bar{x}) = d(g(h(Tx), g(0))) < \epsilon_{k+1}$. Continuing in this vein we get

$$d(T^n x, \bar{x}) < \epsilon_{k+(n-1)} \le \epsilon_n, \forall x \in X'.$$

Hence $\mathrm{diam}(T^n(X')) \to 0$ as $n \to \infty$ and the conditions of Lemma 4.1 are satisfied.

By Lemma 4.1, there is a metric d^* uniformly equivalent to d such that $T : X' \mapsto X'$ is a Banach contraction mapping with respect to d^*. Thus $\overline{x}$ is the unique fixed point of T in X'. Finally observe that $T : X \to X$ and T is also a metric contraction on X. The unique fixed point $\overline{x}$ of T is in X since X has the strong contraction property. □

Theorem 4.3. *Let (X, d) be a locally arcwise-connected metric space. If X is arcwise-complete, then X has the strong contraction property.*

Proof. Let d^* be a metric uniformly equivalent to d, $T : X \to X$ a metric contraction under d^* with contraction constant $0 \le h < 1$. Note that T is contractive and d^* is uniformly equivalent to d. Select, as we may, x_0 in X with $d(x_0, Tx_0) < \delta$ and arc $g_0 : [0, 1] \to X$ which connect x_0 and Tx_0. For each $k \in N$, define $g_k : [0, 1] \to X$ by $g_k(s) = T^k g_0(s)$ for all $s \in [0, 1]$. Then g_k connects $x_k = T^k(x_0)$ and $x_{k+1} = T^{k+1}(x_0)$ for each $k \in N$. Now $\{T^n x_0\}$ is a Cauchy sequence under d^*, so that it is also a Cauchy sequence under d.

Define $g : (0, 1] \mapsto X$ by

$$g(s) = g_k(2^{k+1}s - 1), \quad \forall \frac{1}{2^{k+1}} < s \le \frac{1}{2^k}, \quad k = 1, 2, \cdots .$$

Since T is a metric contraction, it follows that g is a semi-closed arc, $g(s) \to x^* \in X$ as $s \to 0$ and $\{T^n x_0\} \to x^*$. Finally, observe that $\{T^n x_0\}$ is a Cauchy sequence under d. It is easy to check that x^* is the fixed point of T. □

Theorem 4.3 can also be considered as a generalization of the Banach contraction principle under weaker completeness.

Combining Theorems 4.2 and 4.3, we have the following theorems.

Theorem 4.4. *Let X be a locally arcwise-connected metric space. Then X is arcwise-complete if and only if X has the strong contraction property.*

Corollary 4.5. *Let X be an arcwise-connected metric space. Then X is arcwise-complete if and only if X has the strong contraction property.*

By Corollary 4.5, the subsets in Examples 1.1, 1.2 and 2.3 have the strong contraction property because, as we see, they have arcwise-completeness.

References

1. J. M. Borwein, *Completeness and contraction principle*, Proc. Amer. Math. Soc. **87** (1983), 246–250. MR0681829 (84a:54080)
2. J. R. Jachymski, *Equivalence of some contractivity properties over metrical structures*, Proc. Amer. Math. Soc. **125**(1997),2327–2335. MR1389524 (97j:54047)
3. J. R. Jachymski, *Short proof of the converse to the contraction principle and some related results*, Topol. Methods Nonlinear Anal. **15**(2000),179–186. MR1786260 (2001h:54071)
4. L. Janos, *A converse of Banach's contraction theorem*, Proc. Amer. Math. Soc. **18**(1968),287–289. MR0208589 (34:8398)
5. L. Janos, *On pseudo-complete spaces*, Notices Amer. Math. Soc. **18**(1971), 97–163.
6. L. Janos, *The Banach contraction mapping principle and cohomology*, Comment. Math. Univ. Carolinae, 43:3(2000),605-610. MR1795089 (2001m:54039)
7. W. A. Kirk, *Caristi's fixed point theorem and metric convexity*, Colloq. Math. **36**(1976), 81–86. MR0436111 (55:9061)
8. S. Leader, *A topological characterization of Banach contractions*, Pacific J. Math. **69:2**(1977),461-466. MR0436093 (55:9044)
9. F. Sullivan, *A characterization of complete metric spaces*, Proc. Amer. Math. Soc. **83**(1981),345–346. MR0624927 (83b:54036)

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