TRACE CLASS CRITERIA FOR BILINEAR HANKEL FORMS OF HIGHER WEIGHTS

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Abstract. In this paper we give a complete characterization of higher weight Hankel forms, on the unit ball of $\mathbb{C}^d$, of Schatten-von Neumann class $S_p$, $1 \leq p \leq \infty$. For this purpose we give an atomic decomposition for certain Besov-type spaces. The main result is then obtained by combining the decomposition and our earlier results.

1. Introduction

Hankel operators on the unit disc have been studied extensively; see [Pe1] for a systematic treatment. One of the main topics is to study Schatten-von Neumann properties of Hankel operators; see [Pe1] and [Pe2]. In [JP] Janson and Peetre introduced Hankel forms of higher weights on the unit disc. Their Schatten-von Neumann properties were studied in [Ro] and [Z].

In [P1] Peetre introduced Hankel forms of higher weights on the unit ball in $\mathbb{C}^d$. Their Schatten-von Neumann, $S_p$, properties were studied in [Su] for $2 \leq p \leq \infty$. See also [FR] for a different approach.

The results for $2 \leq p \leq \infty$ in [Su] were proved by using interpolation between $S_2$ and $S_\infty$ (bounded operators) and boundedness of certain matrix-valued Bergman projections, but the case of $1 \leq p < 2$ was left open there.

In this paper we extend the results in [Su] to $1 \leq p \leq \infty$. For this purpose we study the atomic decomposition for some Besov spaces of vector-valued holomorphic functions, see Section 4, which then gives $S_1$ properties. Our results follow by interpolation, and we get a full characterization for $1 \leq p \leq \infty$. Some of the proofs in this paper are based on techniques used in [Su] and will therefore be given briefly. The reader is referred to that article for more details.

The paper is organized as follows. In section 2 we recall briefly some notation and we prove Theorem 2.1 generalizing the result for $p = 2$ in [Su]. Section 3 is devoted to duality relations for the spaces of symbols. In Section 4 we give an atomic decomposition for a certain space of symbols, which will be used in Section 5 to prove the $S_1$ criterion.

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2. Preliminaries

2.1. The Banach space $\mathcal{H}^p_{\nu,s}$ for $1 \leq p \leq \infty$. Let $dm$ denote the Lebesgue measure on the unit ball $B \subset \mathbb{C}^d$ and let $dt(z)$ be the measure $(1 - |z|^2)^{\nu - d - 1} dm(z)$. For $d < \nu < \infty$ let $dt(z)$ be the measure $c_{\nu} (1 - |z|^2)^\nu dt(z)$, where $c_{\nu}$ is chosen such that

$$\int_B dt(z) = 1.$$ 

The closed subspace of all holomorphic functions in $L^2(dt(z))$ is denoted by $L^2_{\nu}(dt(z))$ and is called a weighted Bergman space. Note that the space $L^2_{\nu}(dt(z))$ has a reproducing kernel $K_z(w) = (1 - \langle w, z \rangle)^{-\nu}$, that is,

$$f(z) = \langle f, K_z \rangle = \int_B f(w) K_z(w) dt(w), \quad f \in L^2_{\nu}(dt(z)), \quad z \in B.$$ 

Denote by $B(z, w)$ the Bergman operator on $V = \mathbb{C}^d$ as in [L], namely

$$B(z, w) = (1 - \langle z, w \rangle)(I - z \otimes w^*),$$

where $z \otimes w^*$ stands for the rank one operator given by $(z \otimes w^*)(v) = \langle v, w \rangle z$.

The Bergman metric at $z \in B$, when we identify the tangent space with $V$, is $\langle B(z, z)^{-1}u, v \rangle$ for $u, v \in V$. We note that

$$B(z, w)^{-1} = (1 - \langle z, w \rangle)^{-2}((1 - \langle z, w \rangle)I + z \otimes w^*).$$

Let $B^t(z, w)$ denote the dual of $B(z, w)$ acting on the dual space $V'$ of $V$. When acting on a vector $v' \in V'$ it is

$$B^t(z, w)v' = (1 - \langle z, w \rangle)v'(I - z\bar{w}).$$

For a nonnegative integer $s$, let $\otimes^s V'$ be the tensor product of $s$ copies of $V'$ and let $\otimes^0 V' = \mathbb{C}$. The space $\otimes^s V'$ is equipped with a natural Hermitian inner product induced by that of $V'$. Denote by $\otimes^s V'$ the subspace of symmetric tensors of length $s$ and denote by $\otimes^s B^t(z, z)$ the operator on $\otimes^s V'$ induced by the action of $B^t(z, z)$ on $V'$, where $\otimes^0 B^t(z, z) = I$. Recall, generally, that if $A$ acts on $V'$, $\otimes^s A$ acts on $\otimes^s V'$ by

$$(\otimes^s A)(u_1 \otimes u_2 \otimes \cdots \otimes u_s) = (Au_1) \otimes (Au_2) \otimes \cdots \otimes (Au_s).$$

For example, in the case $s = 2$ the operator $\otimes^2 B^t(z, z)$ becomes

$$(1 - |z|^2)^2 (I \otimes I - I \otimes A_z - A_z \otimes I + A_z \otimes A_z),$$

where $A_z = \bar{z} \otimes z^*$. Let $L^p_{\nu,s} = L^p_{\nu}(\mathbb{B}, \otimes^s V')$ be the space of functions $G : \mathbb{B} \to \otimes^s V'$ such that

$$\|G\|_{\nu,s,p} = \left( \int_B \langle (1 - |z|^2)^{2\nu} \otimes \otimes^s B^t(z, z)G(z), G(z) \rangle^{p/2} dt(z) \right)^{1/p} < \infty,$$

where $1 \leq p \leq \infty$, and let $L^\infty_{\nu,s}$ be the space of functions $G : \mathbb{B} \to \otimes^s V'$ such that

$$\|G\|_{\nu,s,\infty} = \sup_{z \in B} \langle (1 - |z|^2)^{2\nu} \otimes \otimes^s B^t(z, z)G(z), G(z) \rangle^{1/2} < \infty.$$ 

Let $\mathcal{H}^p_{\nu,s}$ be the closed subspace of all holomorphic functions in $L^p_{\nu,s}$, $1 \leq p \leq \infty$.
Also, we need the group $G$ of biholomorphic mappings of $\mathbb{B}$. Let $P_z$ be the orthogonal projection of $\mathbb{C}^d$ onto $\mathbb{C}_z$ and let $Q_z = I - P_z$. Put $s_z = (1 - |z|^2)^{1/2}$ and define a linear fractional mapping $\varphi_z$ on $\mathbb{B}$ by (see [Rul])

$$\varphi_z(w) = \frac{z - P_z w - s_z Q_z w}{1 - \langle w, z \rangle}.$$  

If $g \in G$ and $g(z) = 0$, then there is a unique unitary operator $U : \mathbb{C}^d \to \mathbb{C}^d$ such that

$$g = U \varphi_z.$$  

Define the complex Jacobian $J_g$ by $J_g(w) = \det(g'(w))$. Now, let $z_0 \in \mathbb{B}$. Then by arguments in Remark 3.1 in [Su] it follows that there is a constant $c$ with $|c| = 1$ such that

$$J_{\varphi_{z_0}}(w)^{2\nu/(d+1)} = c \cdot \frac{(1 - |z_0|^2)\nu}{(1 - \langle w, z_0 \rangle)^{2\nu}}.$$  

The next theorem gives the reproducing properties for $\mathcal{H}_{\nu,s}^p$.

**Theorem 2.1.** Let $1 \leq p < \infty$. There is a nonzero constant $c$ such that, for any $G \in \mathcal{H}_{\nu,s}^p$ and any $v \in \odot^s V'$,

$$\langle G(z), v \rangle = c \int_{\mathbb{B}} \langle \odot^s B^t(w, w) G(w), K_{\nu,s}(w, z) v \rangle (1 - |w|^2)^{2\nu} d\mu(w),$$  

where

$$K_{\nu,s}(w, z) = (1 - \langle w, z \rangle)^{-2\nu} \odot^s B^t(w, z)^{-1}.$$  

The proof of this theorem is given at the end of this subsection.

**Remark 2.2.** Consider $\mathcal{H}_{\nu,s}^2 \subset L_{\nu,s}^2$. According to Lemma 3.5 in [Su] the orthogonal projection operator $P_{\nu,s}$ of $L_{\nu,s}^2$ onto $\mathcal{H}_{\nu,s}^2$ is given by

$$P_{\nu,s}G(z) = c \int_{\mathbb{B}} (1 - |w|^2)^{2\nu} K_{\nu,s}(z, w) \odot^s B^t(w, w) G(w) d\mu(w).$$  

Namely, for any $G \in L_{\nu,s}^2$ and any $v \in \odot^s V'$ it follows that

$$\langle P_{\nu,s}G(z), v \rangle = c \int_{\mathbb{B}} \langle \odot^s B^t(w, w) G(w), K_{\nu,s}(w, z) v \rangle (1 - |w|^2)^{2\nu} d\mu(w).$$  

The orthogonal projection operator has the following boundedness property.

**Proposition 2.3.** If $1 \leq p < \infty$, then $P_{\nu,s} : L_{\nu,s}^p \to \mathcal{H}_{\nu,s}^p$ is bounded.

**Proof.** The case $1 < p < \infty$ is just Corollary 7.4 in [Su]. Now, consider the case $p = 1$. Let $F \in L_{\nu,s}^1$. Then it follows from Theorem 2.1 above and Lemma 7.1 in [Su] that

$$\left\| \odot^s B^t(z, z)^{1/2} P_{\nu,s} F(z) \right\| \leq C_s \int_{\mathbb{B}} T(z, w) \left\| \odot^s B^t(w, w)^{1/2} F(w) \right\| (1 - |w|^2)^{2\nu} d\mu(w),$$  

where

$$T(z, w) = \frac{(1 - |z|^2)^{s/2}(1 - |w|^2)^{s/2}}{|1 - \langle z, w \rangle|^{2\nu + s}}.$$
Thus, by Fubini-Tonelli’s theorem and Proposition 1.4.10 in [Ru] it follows that

\[ \|P_{\nu, s}F\|_{\nu, s, 1} \leq C_s \int_{\mathbb{B}} \|\otimes^s B^t(w, w)\|^{1/2} F(w) \| (1 - |w|^2)^{2\nu} \]

\[ \cdot \left( \int_{\mathbb{B}} T(z, w)(1 - |z|^2)^{2\nu} \, dt(z) \right) \, dt(w) \]

\[ \leq C'_s \int_{\mathbb{B}} \|\otimes^s B^t(w, w)\|^{1/2} F(w) \| (1 - |w|^2)^{2\nu} \, dt(w) = C'_s \|F\|_{\nu, s, 1}. \]

\[ \square \]

Note that it is proved in [Su], using the complex interpolation method of Banach spaces, that \( \mathcal{H}^p_{\nu, s} = (\mathcal{H}^2_{\nu, s}, \mathcal{H}^\infty_{\nu, s})_{[1-2/p, p]} \) if \( 2 < p < \infty \); see Theorem 8.2 in [Su]. However, Proposition 2.3 allows us to use the same proof as in [Su] to get the following result.

**Corollary 2.4.** If \( 1 < p < \infty \), then

\[ \mathcal{H}^p_{\nu, s} = (\mathcal{H}^1_{\nu, s}, \mathcal{H}^\infty_{\nu, s})_{[1-1/p, p]} . \]

Now we go back to Theorem 2.1. First we need a proposition.

**Proposition 2.5.** Let \( s \) be a nonnegative integer and let \( \nu > d, 2\nu > \alpha > d \). Then there is a constant \( C_s > 0 \) such that

\[ (1 - |z|^2)^{2\nu - \alpha} \left\| K_{\nu, s}(\cdot, z) \otimes^s B^t(z, z) \right\|_{\alpha, s, 1} \leq C_s \|v\| \]

for all \( z \in \mathbb{B} \) and all \( v \in \mathcal{S} V' \).

**Proof.** Let \( v \in \mathcal{S} V' \). It follows from Lemma 7.1 in [Su] and Proposition 1.4.10 in [Ru] that

\[ \left\| K_{\nu, s}(\cdot, z) \otimes^s B^t(z, z) \right\|_{\alpha, s, 1} \]

\[ = \int_{\mathbb{B}} \left\| \otimes^s B^t(w, w) \|^{1/2} \otimes B^t(w, z)^{-1} \otimes^s B^t(z, z) \|^{1/2} v \| \frac{(1 - |w|^2)^{\alpha}}{|1 - (w, z)|^{2\nu}} \, dt(w) \]

\[ \leq C_s \|v\| \int_{\mathbb{B}} \frac{(1 - |z|^2)^{2\nu - \alpha} (1 - |w|^2)^{\alpha + s/2} |1 - (w, z)|^{2\nu + s}}{|1 - (w, z)|^{2\nu}} \, dt(w) \leq C'_s (1 - |z|^2)^{2\nu - 2\nu} \|v\|. \]

\[ \square \]

**Lemma 2.6.** Let \( z \in \mathbb{B} \). Then there is a constant \( C_s > 0 \) such that, for any \( v \in \mathcal{S} V' \) and any \( 1 \leq p \leq \infty \), it follows that

\[ \left\| (1 - |z|^2)\alpha K_{\nu, s}(\cdot, z) \otimes^s B^t(z, z) \right\|_{\alpha, s, p} \leq C_s \|v\|. \]

**Proof.** Let \( T_z = (1 - |z|^2)^\nu K_{\nu, s}(\cdot, z) \otimes^s B^t(z, z) \). By Proposition 2.3 and by Lemma 7.1 in [Su] it follows that \( \|T_z v\|_{\nu, s, 1} \leq C_s \|v\| \) and \( \|T_z v\|_{\nu, s, \infty} \leq C'_s \|v\| \) respectively, for all \( v \in \mathcal{S} V' \). Thus the result follows from Riesz-Thorin’s interpolation theorem. \[ \square \]

Now we can prove Theorem 2.1.

**Proof of Theorem 2.1.** Let \( G \in \mathcal{H}^p_{\nu, s}, 1 \leq p \leq \infty \). Then it follows from Lemma 2.6 that, for all \( v \in \mathcal{S} V' \),

\[ \int_{\mathbb{B}} \left\| \otimes^s B^t(w, w)G(w), K_{\nu, s}(w, z) \right\| \, dw \leq \|G\|_{\nu, s, p} \|K_{\nu, s}(\cdot, z)\|_{\nu, s, q} < \infty . \]
In particular, if \( z = 0 \), then
\[
\int_{\mathbb{B}} \left| \langle \otimes^s B^t(w, w) G(w), v \rangle \right| (1 - |w|^2)^{2\nu} \, dw(w) < \infty.
\]
By the mean-value property for holomorphic functions and rotation invariance for integration,
\[
\int_{\mathbb{B}} \langle (1 - |w|^2)^{2\nu} \otimes^s B^t(w, w) G(w), v \rangle \, dw(w) = c' \langle G(0), v \rangle,
\]
where \( c' \neq 0 \) only depends on \( d, \nu \) and \( s \). Hence, there exists a nonzero constant \( c \) such that, for all \( G \in \mathcal{H}^p_{\nu,s} \) and all \( v \in \otimes^s V' \),
\[
(2.8) \quad \langle G(0), v \rangle = c \langle G, v \rangle_{\nu,s,2},
\]
where \( \langle \cdot, \cdot \rangle_{\nu,s,2} \) is the \( \mathcal{H}^2_{\nu,s} \)-pairing. Now, define an isometry \( \pi_{\nu,s} \) on \( \mathcal{H}^2_{\nu,s} \) by
\[
\pi_{\nu,s} : g \in G, S(z) \to \left( \otimes^s (dg^{-1}(z))^t \right) S(g^{-1}(z)) (J_g^{-1}(z))^{2\nu/(d+1)},
\]
as in [Su]. Let \( z_0 \in \mathbb{B} \). For notational convenience we prove the reproducing property only for \( s = 1 \); the case for general \( s \) is identically the same. On the one hand,
\[
(2.9) \quad \langle \pi_{\nu,s}(\varphi_{z_0}), G \rangle(0, v) = \langle G(z_0), J_{\varphi_{z_0}}(0)^{2\nu/(d+1)} (\varphi_{z_0}'(0)^t)^* v \rangle.
\]
By equation \( (2.6) \), \( (1 - |z_0|^2)^{(d+1)/2} < |J_{\varphi_{z_0}}(w)| < (1 - |z_0|^2)^{-d-1} \) on \( \mathbb{B} \), so \( \pi_{\nu,1}(\varphi_{z_0}) G \in \mathcal{H}^p_{\nu,1} \). However, using equation \( (2.8) \) above for \( \pi_{\nu,1}(\varphi_{z_0}) G \) and the transformation properties
\[
B(\varphi_{z_0}(w), \varphi_{z_0}(z)) = \varphi_{z_0}'(w) B(w, z) (\varphi_{z_0}'(z))^t
\]
(see equation (9) in [Su]) and
\[
K_{\nu,1}(\varphi_{z_0}(w), \varphi_{z_0}(z)) = J_{\varphi_{z_0}}(w)^{-2\nu/(d+1)} J_{\varphi_{z_0}}(z)^{-2\nu/(d+1)} \cdot (\varphi_{z_0}'(w))^t K_{\nu,1}(w, z) (\varphi_{z_0}'(z))^{-1}
\]
(see equation (9) in [Su] and Theorem 2.2.5 in [Ru]), the left-hand side in equation \( (2.9) \) above is
\[
\langle G(z_0), u \rangle = c \langle G, K_{\nu,s}(\cdot, z_0)u \rangle_{\nu,s,2},
\]
where \( u = J_{\varphi_{z_0}}(0)^{2\nu/(d+1)} (\varphi_{z_0}'(0)^t)^* v \). Since \( v \) is arbitrary, then so is \( u \in \otimes^s V' \), which proves the theorem. \( \square \)

2.2. Hankel forms of higher weights. Let \( H_1 \) and \( H_2 \) be Hilbert spaces and let \( T : H_1 \to H_2 \) be a linear operator. Define the singular numbers \( s_n(T) = \inf \{ ||T - K|| : \text{rank}(K) \leq n \} \), \( n \geq 0 \). If \( T \) is compact, these singular numbers are equal to the eigenvalues of \( |T| = (T^* T)^{1/2} \). We denote by \( S_p \) the ideal of operators for which \( \{s_n(T)\} \in l^p, 0 < p \leq \infty \); see [S].

The transvector \( T_s \) on \( L^2(d\nu) \otimes L^2(d\nu) \) (introduced in [PI]; see also [P2], [PZ] and [Su]) is defined by
\[
(2.10) \quad T_s(f, g)(z) = \sum_{k=0}^s \binom{s}{k} (-1)^k \partial^{s-k} f(z) \otimes \partial^k g(z)/\nu^{s-k}(v)_k.
\]
where
\[ \partial^s f(z) = \sum_{j_1, \ldots, j_s = 1}^d \partial_{j_1} \cdots \partial_{j_s} f(z) \, dz_{j_1} \otimes \cdots \otimes dz_{j_s} \in \otimes^s V' \]
and \((\nu)_k = \nu(\nu + 1) \cdots (\nu + k - 1), (\nu)_0 = 1\), is the Pochammer symbol.

**Lemma 2.7.** There is a constant \( C_s > 0 \) such that
\[ \|T_s(f, g)\|_{\nu, s, 1} \leq C_s \|f\|_{\nu} \|g\|_{\nu} \]
for all \( f, g \in L^2_\alpha(dt_\nu) \).

First we need a lemma, which actually is a consequence of Theorem 4.1 in \([Su]\), but we give an independent and easier proof.

**Lemma 2.8.** There is a constant \( C_{\nu, s} > 0 \) such that
\[ \int_B \langle \otimes^s B^t(z, z) \partial^s f(z), \partial^s f(z) \rangle (1 - |z|^2)^\nu \, dt(z) \leq C_{\nu, s} \|f\|_{\nu} \]
for all \( f \in L^2_\alpha(dt_\nu) \).

**Proof.** First,
\[ \partial^s f(z) = c_\nu(s) \int_B \frac{f(w) \otimes^s w}{(1 - \langle z, w \rangle)^{\nu + s}} \cdot (1 - |w|^2)^\nu \, dt(w), \]
so that
\[ \left\| \otimes^s B^t(z, z)^{1/2} \partial^s f(z) \right\| \leq C_{\nu, s} \int_B \frac{|f(w)| \cdot \left\| B^t(z, z)^{1/2} \right\| \nu}{(1 - \langle z, w \rangle)^{\nu + s}} \cdot (1 - |w|^2)^\nu \, dt(w). \]
We can estimate
\[ \left\| B^t(z, z)^{1/2} \right\| = s_z \left( \|s_z Pz\bar{w}\|^2 + \|Qz\bar{w}\|^2 \right)^{1/2} \]
\[ = s_z \left( |w|^2 - |\langle z, w \rangle|^2 \right)^{1/2} \]
\[ \leq \sqrt{2} \cdot s_z \left| 1 - \langle z, w \rangle \right|^{1/2}. \]
Hence,
\[ \left\| \otimes^s B^t(z, z)^{1/2} \partial^s f(z) \right\| \leq C'_{\nu, s} \int_B T(z, w)|f(w)|(1 - |w|^2)^\nu \, dt(w), \]
where
\[ T(z, w) = \frac{(1 - |z|^2)^{s/2}}{|1 - \langle z, w \rangle|^{\nu + s/2}}. \]
Now, the result follows by exactly the same arguments as in the proof of Theorem 7.2 in \([Su]\) (where we let \( t = -\nu - d/4 \)).

**Proof of Lemma 2.7.** The transvectant is a linear combination of terms \( \partial^k f(z) \otimes \partial^{s-k} g(z) \) so we need only to estimate \( \|\partial^k f(z) \otimes \partial^{s-k} g(z)\|_{\nu, s, 1} \) for \( 0 \leq k \leq s \). First we observe that
\[ \left\| \otimes^s B^t(z, z)^{1/2} \partial^k f(z) \otimes \partial^{s-k} g(z) \right\| = \left\| \otimes^k B^t(z, z)^{1/2} \partial^k f(z) \right\| \cdot \left\| \otimes^{s-k} B^t(z, z)^{1/2} \partial^{s-k} g(z) \right\|. \]
Thus by Hölder’s inequality and Lemma 2.8 it follows that
\[
\int_{\mathbb{B}} \left\| \otimes^s B^t(z, z)^{1/2} \partial^k f(z) \otimes \partial^s - k g(z) \right\| (1 - |z|^2)^\nu \, dt(z) \\
\leq C\|f\|_{\nu,k} \|g\|_{\nu,s-k} \leq C_s\|f\|_{\nu} \|g\|_{\nu}.
\]
\[\square\]

The Hankel bilinear form \( H^p_F \) on \( L^2_\nu(dt_\nu) \otimes L^2_\nu(dt_\nu) \) is defined by
\[
(2.11) \quad H^p_F(f, g) = \int_{\mathbb{B}} \left\langle \otimes^s B^t(z, z), T_\nu(f, g)(z), F(z) \right\rangle \, dt_{2\nu}(z)
\]
where \( F : \mathbb{B} \rightarrow \otimes^s V^* \) is holomorphic. We call \( F \) the symbol of the corresponding Hankel form. We remark that
\[
H^p_F(f, g) = \int_{\mathbb{B}} f(z)g(z) \overline{F(z)} \, dt_{2\nu}(z).
\]
This is the classical Hankel form studied in [JPR].

With the form \( H^p_F \) one can associate the operator \( A^p_F \) defined by
\[
H^p_F(f, g) = \langle f, A^p_F g \rangle_{\nu},
\]
as in [JPR]. Notice that \( A^p_F \) is an anti-linear operator on \( L^2_\nu(dt_\nu) \). To get a linear operator one combines \( A^p_F \) with a conjugation, i.e., one instead considers the operator \( \overline{A^p_F} : g \rightarrow \overline{A^p_F}g \). We say that \( H^p_F \) is of Schatten-von Neumann class \( \mathcal{S}_p \), for \( 0 < p < \infty \), if and only if \( \overline{A^p_F} : L^2_\nu(dt_\nu) \rightarrow L^2_\nu(dt_\nu) \) is of class \( \mathcal{S}_p \).

3. Duality of \( \mathcal{H}^p_{\nu,s} \)

In this section we determine the dual space \((\mathcal{H}^p_{\nu,s})^*\) of \( \mathcal{H}^p_{\nu,s} \), \( 1 \leq p < \infty \).

**Lemma 3.1.** Let \( 1 \leq p < \infty \). If \( \Phi \in (\mathcal{L}^p_{\nu,s})^* \), then there is a function \( G \in \mathcal{L}^q_{\nu,s} \) such that
\[
\Phi(F) = \int_{\mathbb{B}} \left\langle \otimes^s B^t(z, z), F(z), G(z) \right\rangle (1 - |z|^2)^2 \, dt(z)
\]
denoting \( \|\Phi\| = \|G\|_{\nu,s,q} \) where \( 1/q + 1/p = 1 \).

**Proof.** Define \( A(z) = (1 - |z|^2)^\nu \otimes^s B^t(z, z)^{1/2} \) and \( M_A F(z) = A(z)F(z) \). Then \( M_A \) is an isometry from \( \mathcal{L}^p_{\nu,s} \) onto \( \mathcal{L}^p \), where \( \mathcal{L}^p = \{ F : \mathbb{B} \rightarrow V : \|F\|_p < \infty \} \) and
\[
\|F\|_p = \left( \int_{\mathbb{B}} \|F(z)\|^p \, dt(z) \right)^{1/p}.
\]
Consider \( \Theta = \Phi M^{-1}_A \). Then \( \Theta \) is a bounded linear functional on \( \mathcal{L}^p \) and \( \Theta(AF) = \Phi(F) \). Then we can find a function \( H \in \mathcal{L}^q \) such that
\[
\Phi(F) = \int_{\mathbb{B}} \left\langle (AF)(z), H(z) \right\rangle \, dt(z)
\]
with \( \|\Theta\| = \|H\|_q \). Let \( G = M^{-1}_A H \). Then \( G \in \mathcal{L}^q_{\nu,s} \) and
\[
\Phi(F) = \int_{\mathbb{B}} \left\langle \otimes^s B^t(z, z), F(z), G(z) \right\rangle (1 - |z|^2)^2 \, dt(z)
\]
Also \( \|\Phi\| = \|G\|_{\nu,s,q} \). \[\square\]
Theorem 3.2. For $1 \leq p < \infty$ we have $(\mathcal{H}^p_{v,s})^* = \mathcal{H}^q_{v,s}$, under the integral pairing

$$(F, G)_{\nu,s,2} = \int_{\mathbb{B}} \langle \otimes^s B^i(z, z) F(z), G(z) \rangle (1 - |z|^2)^{2p} \, d\nu(z), \quad F \in \mathcal{H}^p_{v,s}, \, G \in \mathcal{H}^q_{v,s},$$

where $1/p + 1/q = 1$. Namely, for any bounded linear functional $\Phi : \mathcal{H}^p_{v,s} \to \mathbb{C}$ there is a function $G \in \mathcal{H}^q_{v,s}$ such that $\Phi(F) = \langle F, G \rangle_{\nu,s,2}$ for all $F \in \mathcal{H}^p_{v,s}$ with

$$C \|G\|_{\nu,s,q} \leq \|\Phi\| \leq \|G\|_{\nu,s,q}.$$

Proof. By H"older's inequality, every function $G \in \mathcal{H}^q_{v,s}$ defines a bounded linear functional $\tilde{\Phi}$ on $\mathcal{H}^p_{v,s}$ under the above integral pairing $\|\Phi\| = \|\tilde{\Phi}\|$. By Lemma 3.3 there is a function $H \in L^p_{v,s}$ such that

$$\Phi(F) = \int_{\mathbb{B}} \langle \otimes^s B^i(z, z) F(z), H(z) \rangle (1 - |z|^2)^{2p} \, d\nu(z)$$

for all $F \in L^p_{v,s}$, with $\|\tilde{\Phi}\| = \|H\|_{\nu,s,q}$. However, Theorem 2.1 implies that, for any $F \in \mathcal{H}^p_{v,s}$,

$$F(z) = (P_{v,s} F)(z) = c \int_{\mathbb{B}} (1 - |w|^2)^{2p} K_{v,s}(w, z)^* \otimes^s B^i(w, w) F(w) \, d\nu(w).$$

Substituting this into formula (3.1) and using Fubini-Tonelli's theorem we get that

$$\Phi(F) = \tilde{\Phi}(F) = \int_{\mathbb{B}} \langle \otimes^s B^i(w, w) F(w), (P_{v,s} H)(w) \rangle (1 - |w|^2)^{2p} \, d\nu(w).$$

Let $G = P_{v,s} H$. By Proposition 2.3 $\|P_{v,s} H\|_{\nu,s,q} \leq C \|H\|_{\nu,s,q}$. Then $G \in \mathcal{H}^q_{v,s}$, $\Phi(F) = \langle F, G \rangle_{\nu,s,2}$ for all $F \in \mathcal{H}^p_{v,s}$ and $C \|G\|_{\nu,s,q} \leq \|\Phi\|$.

4. Atomic decomposition of $\mathcal{H}^1_{v,s}$

Following [JPR], we denote by $l^1(\mathbb{B}, \otimes^s V')$ the space of all functions $a : \mathbb{B} \to \otimes^s V'$, with support in $\{z_j\}_{j=1}^\infty \subset \mathbb{B}$, such that

$$\|a\|_{l^1} = \sum_{j=1}^\infty \|a(z_j)\| < \infty.$$

Also, denote by $l^\infty(\mathbb{B}, \otimes^s V')$ the space of all functions $a : \mathbb{B} \to \otimes^s V'$ such that

$$\|a\|_{l^\infty} = \sup_{z \in \mathbb{B}} \|a(z)\| < \infty.$$

Then it is elementary that

$$l^\infty(\mathbb{B}, \otimes^s V') = (l^1(\mathbb{B}, \otimes^s V'))^*,$$

under the pairing

$$\langle a, b \rangle' = \sum_{j=1}^\infty \langle a(z_j), b(z_j) \rangle$$

where $a \in l^1(\mathbb{B}, \otimes^s V')$ with support $\{z_j\}_{j=1}^\infty \subset \mathbb{B}$ and $b \in l^\infty(\mathbb{B}, \otimes^s V')$. Namely, for any bounded linear functional $\Phi : l^1(\mathbb{B}, \otimes^s V') \to \mathbb{C}$ there is a function $b$ in $l^\infty(\mathbb{B}, \otimes^s V')$ such that $\Phi(a) = \langle a, b \rangle'$ for all $a \in l^1(\mathbb{B}, \otimes^s V')$ with $\|\Phi\| = \|b\|_{l^\infty}$. 
Theorem 4.1. It follows that $F \in H^1_{\nu,s}$ if and only if there is a sequence $\{z_j\}_{j=1}^{\infty} \subset \mathbb{B}$ and a sequence $\{a_j\}_{j=1}^{\infty} \in l^1(\mathbb{B}, \circ^* V')$ such that

$$F(w) = \sum_{j=1}^{\infty} (1 - |z_j|^2)^\nu K_{\nu,s}(w, z_j) \circ^* B^j(z_j, z_j)^{1/2} a_j.$$ 

Proof. By Proposition 2.10 for any $v \in \circ^* V'$ and any $z \in \mathbb{B}$,

$$\left\| K_{\nu,s}(\cdot, z) \circ^* B^j(z, z)^{1/2} v \right\|_{\nu,s,1} \leq C_s (1 - |z|^2)^{-\nu} \|v\|.$$ 

Thus, the operator $T : l^1(\mathbb{B}, \circ^* V') \to H^1_{\nu,s}$ defined by

$$(Ta)(w) = \sum_{j=1}^{\infty} (1 - |z_j|^2)^\nu K_{\nu,s}(w, z_j) \circ^* B^j(z_j, z_j)^{1/2} a_j$$

is bounded, where $a_j = a(z_j)$ and the support of $a$ is $\{z_j\}_{j=1}^{\infty}$. We need to prove that $T$ is onto. Consider $T^* : (H^1_{\nu,s})^* \to (l^1(\mathbb{B}, \circ^* V'))^*$, $T^*(\Phi)(a) = \Phi(Ta)$, which is bounded, where $\Phi \in (H^1_{\nu,s})^*$ and $a \in l^1(\mathbb{B}, \circ^* V')$. By Theorem 3.2, for any $\Phi \in (H^1_{\nu,s})^*$ there is a $G \in H^\infty_{\nu,s}$ such that $\Phi(F) = (F, G)_{\nu,s,2}$ for all $F \in H^1_{\nu,s}$ with $C\|G\|_{\nu,s,\infty} \leq \|\Phi\| \leq \|G\|_{\nu,s,\infty}$. Now, let $a \in l^1(\mathbb{B}, \circ^* V')$ with support $\{z_j\}_{j=1}^{\infty} \subset \mathbb{B}$. By the reproducing property in Theorem 2.1 it follows that

$$T^*(\Phi)(a) = \Phi(Ta) = \langle Ta, G \rangle_{\nu,s,2} = c \sum_{j=1}^{\infty} \langle a_j, (1 - |z_j|^2)^\nu \circ B^j(z_j, z_j)G(z_j) \rangle.$$ 

Hence, by (4.1) and Theorem 3.2 it follows that

$$\frac{1}{c} \cdot \|T^*\|_{(l^1)^*} = \sup_{z \in \mathbb{B}} \left\| (1 - |z|^2)^\nu \circ^* B^j(z, z)G(z) \right\| = \|G\|_{\nu,s,\infty} \geq \|\Phi\|.$$ 

On the one hand, (4.2) yields that $\ker T^* = \{0\}$ and consequently the range of $T$ is dense in $H^1_{\nu,s}$. On the other hand, (4.2) yields that the range of $T^*$ is closed and so is the range of $T$ by the Closed Range Theorem. □

5. Trace class $S_1$

We consider now the trace class property of $H^p_{\nu,s}$ in (2.11).

Theorem 5.1. The Hankel form $H^p_{\nu,s}$ is of trace class $S_1$ if and only if $F \in H^1_{\nu,s}$.

Combining the results in [Su], we have now a complete characterization of the Schatten-von Neumann class Hankel forms.

Theorem 5.2. The Hankel form $H^p_{\nu,s}$ is of Schatten-von Neumann class $S_p$ if and only if $F \in H^p_{\nu,s}$, $1 \leq p \leq \infty$.

Proof of Theorem 5.2. It follows from Lemma 5.5 below and Theorem 1.1(a) in [Su] that the operator $\Gamma : F \to H^p_{\nu,s}$ is bounded from $H^1_{\nu,s}$ into $S_1$ and from $H^\infty_{\nu,s}$ into $S_\infty$, respectively. Since $S_p = (S_1, S_\infty)_{\nu,1/p}$ if $1 < p < \infty$, then it follows by Riesz-Thorin’s interpolation theorem and Corollary 2.1 that $\Gamma$ is bounded from $H^p_{\nu,s}$ into $S_p$ if $1 < p < \infty$.

On the other hand, it follows from Lemma 5.6 below and Theorem 1.1(a) in [Su] that $\tilde{T}_s$, defined in (5.2), is bounded from $S_1$ into $H^1_{\nu,s}$ and from $S_\infty$ into $H^\infty_{\nu,s}$, respectively. Again, by interpolation $\tilde{T}_s$ is bounded from $S_p$ into $H^p_{\nu,s}$ if $1 < p < \infty$. 


Theorem 1.1(a) in [Su] we get that $H$. Also, if $1386 \ M. \ S U N D H \ddot{\text{A}} L L$

First, let $I$ fix $z \in H$. We use the Closed Graph Theorem. Assume that $F_n \to F$ in $H_{\nu,s}$ and that $\Gamma(F_n) \to B$ in $S_1$. We shall prove that $H_{\nu} = B$. On the one hand, by Theorem 1.1(a) in [Su] and Lemma 5.3 it follows that

$$\|H_{F_n,F} \|_{S_1} \leq C\|F_n - F\|_{\nu,s,\infty} \leq C'\|F_n - F\|_{\nu,s,1}$$
so that $H^p_{F_n} \to H^p_\Gamma$ in $S_\infty$. On the other hand,

$$||\Gamma(F_n) - B||_{S_\infty} \leq ||\Gamma(F_n) - B||_{S_1}$$

so that $H^p_{F_n} \to B$ in $S_\infty$. Thus $H^p_\Gamma = B$ so that $\Gamma$ has the closed graph property. Hence, $\Gamma$ is bounded. \qed

We recall the transvectant $\tilde{T}_s : S_\infty(L^2(\sigma_\nu), L^2(\sigma_\nu)) \to A_s(\mathbb{B} \times \mathbb{B})$ defined in [Su] (see also [FR] and [PZ]), where $A_s(\mathbb{B} \times \mathbb{B})$ consists of all holomorphic functions $G : \mathbb{B} \times \mathbb{B} \to \phi^\nu \phi^{V'}$. We recall further that the transvectant $T_s$ in (2.10) can be defined for any holomorphic function $G(z, w)$ on $\mathbb{B} \times \mathbb{B}$, namely

$$(T_sG)(z, w) = \sum_{k=0}^{\infty} \binom{s}{k} (-1)^k \partial^k \bar{z} \partial^{s-k} G(z, w)$$

For bounded bilinear forms $A$ on $L^2(\sigma_\nu)$, we define

$$(5.2) \quad \tilde{T}_s(A) = (T_sG)(z, z),$$

where $G(z, w) = A(K_z, K_w)$.

**Lemma 5.6.** The operator $\tilde{T}_s : S_1 \to H^1_{\nu,s}$ defined in (5.2) is bounded. Also, $\tilde{T}_s(H^p_\Gamma) = F$ if $H^p_\Gamma \in S_1$.

**Proof.** First, let $B \in S_1$ be of rank one. Then there exists $\phi, \varphi \in L^2(\sigma_\nu)$ such that

$$B(f, g) = \langle f, \phi \rangle \langle g, \varphi \rangle$$

for all $f, g \in L^2(\sigma_\nu)$. Then $\|B\|_{S_1} = \|\phi\|_{\sigma_\nu} \|\varphi\|_{\sigma_\nu}$ and $\tilde{T}_s(B)(z) = T_s(\phi, \varphi)(z)$, so by Lemma 2.7 it follows that

$$(5.3) \quad \|\tilde{T}_s(B)\|_{\nu,s,1} \leq C_s \|\phi\|_{\sigma_\nu} \|\varphi\|_{\sigma_\nu} \leq C_s \|B\|_{S_1}.$$

In general, if $B \in S_1$ we can write $B = \sum_{n=1}^{\infty} B_n$, rank $B_n = 1$ such that

$$\|B^N\|_{S_1} = \sum_{n=1}^{N} \|B_n\|_{S_1} \to \|B\|_{S_1} \quad \text{as } N \to \infty,$$

where $B^N = \sum_{n=1}^{N} B_n$. By (5.3) the sequence $\{\tilde{T}_s(B^N)\}_{n=1}^{\infty}$ is Cauchy and hence converges to some $G$ in $H^1_{\nu,s}$. Now, since $B^N \to B$ in $S_\infty$ it follows by Lemma 8.4 in [Su] that $\tilde{T}_s(B^N) \to \tilde{T}_s(B)$ in $H^\infty_{\nu,s}$. Hence $\tilde{T}_s(B) = G$ so that (5.3) holds for any $B \in S_1$.

Also, if $H^p_\Gamma \in S_1$, then $\tilde{T}_s(H^p_\Gamma) = F$. (As in the proof of Theorem 5.2 we refer to the proof of Lemma 8.6 in [Su].) \qed

**References**


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