A CONDITION UNDER WHICH \( B = A = U^*BU \) 
FOLLOWS FROM \( B \leq A \leq U^*BU \)

TAKATERU OKAYASU AND YASUNORI UETA

(Communicated by Joseph A. Ball)

Abstract. We will give some sufficient conditions for a \( p \)-hyponormal operator, \( p > 0 \), to be normal, and a sufficient condition for a triplet of operators \( A, B, U \) with \( A, B \) self-adjoint and \( U \) unitary such that \( B \leq A \leq U^*BU \) necessarily satisfies \( B = A = U^*BU \).

1. Introduction

Given \( A, B \) self-adjoint operators (by operators we mean bounded linear ones) on a Hilbert space \( \mathcal{H} \), supposed to be separable herein, we denote \( A \leq_u B \) if \( A \leq U^*BU \) for some unitary \( U \) on \( \mathcal{H} \) (see [5]). A noteworthy fact on the binary relation \( \leq_u \) is that if \( A, B \) are self-adjoint and \( A \leq_u B \), then \( e^A \leq_u e^B \) (Kosaki [4]). Our interest in this paper is toward a deeper understanding of the relation \( \leq_u \).

It is obvious that the relation \( \leq_u \) satisfies the reflexive and transitive laws. However, the antisymmetric law, that is, the assertion

\[ A \leq_u B \text{ and } B \leq_u A \Rightarrow A, B \text{ are unitarily equivalent,} \]

is true in some cases, but is not true in general. It would be noteworthy to point out that if \( A, B \) are operators of trace class and satisfy \( A \leq_u B, B \leq_u A \), then they are unitarily equivalent, because, \( U^*BU \leq A \leq V^*BV \) implies that the trace of \( V^*BV - U^*BU \geq 0 \) is zero, and that \( U^*BU = A \). Further, it is known that if either \( A \) or \( B \) is a positive compact operator and they satisfy \( A \leq_u B, B \leq_u A \), then they are unitarily equivalent. This is shown by Takeaki Yamazaki, and the proof is as follows: \( U^*BU \leq A \leq V^*BV \) together with \( B \geq 0 \) implies that \( T = B^{1/2}VU^* \) is compact hyponormal, and is necessarily normal ([1]), so, we have \( B = TT^* = T^*T = UV^*BVU^* = UAU^* \). But, on the other hand, we have examples of triplets of operators \( A, B, U \) with \( A, B \) self-adjoint, not unitarily equivalent, and \( U \) unitary for which \( B \leq A \leq U^*BU \) holds (see Examples 2 and 3 below; the latter is due to Yamazaki).

We will give in Theorem 5 in Section 3 a sufficient condition for a triplet of operators \( A, B, U \) with \( A, B \) self-adjoint and \( U \) unitary, on \( \mathcal{H} \), which satisfy that \( B \leq A \leq U^*BU \) necessarily satisfy \( B = A = U^*BU \), namely, that either \( \sigma(A) \)
or \( \sigma(B) \) is a null set with respect to Lebesgue measure on the real line \( \mathbb{R} \) and \( \sigma(U) \neq T \), the unit circle on the complex plane \( \mathbb{C} \).

In Section 2, we will investigate sufficient conditions for \( p \)-hyponormal operators, \( p > 0 \), to be normal; for one thing for interest in themselves, and for another for the discussion in Section 3. We will show in Theorem 3 that for a \( p \)-hyponormal operator \( T \) to be normal, it suffices that both \( \sigma(|T|) \) and the set \( \{ r \geq 0 : C_r \subseteq \sigma(T) \} \) are null with respect to Lebesgue measure on \( \mathbb{R} \), where \( C_r = \{ \lambda : |\lambda| = r \} \), and in Theorem 4 that \( \sigma(|T|) \) is null with respect to Lebesgue measure on \( \mathbb{R} \) and \( \{ \arg \lambda : \lambda \in \sigma(U) \setminus \{0\} \} \neq (-\pi, \pi] \) holds, where \( T = U|T| \) is the polar decomposition of \( T \).

The authors would like to express their appreciation to T. Yamazaki for sending us his result cited above and Example 3 below, and for giving us a permission to put them into this paper.

2. CONDITIONS FOR \( p \)-HYPONORMAL OPERATORS TO BE NORMAL

An operator \( T \) on \( \mathcal{H} \) is called \( p \)-hyponormal, \( p > 0 \), if it satisfies the inequality
\[
(T^*T)^p \geq (TT^*)^p.
\]

An operator \( T \) is called hyponormal if \( T \) is 1-hyponormal. It is known that any compact hyponormal operator is necessarily normal (T. Andô [1]), and further, that any compact \( p \)-hyponormal operator, \( p > 0 \), is necessarily normal. This fact is due to the following

**Theorem 1** (Chô and Itô [2]). If an operator \( T \) on \( \mathcal{H} \) is \( p \)-hyponormal, \( 1 \geq p > 0 \), then we have
\[
\|(T^*T)^p - (TT^*)^p\| \leq \frac{p}{\pi} \int \int r^{2p-1} \, dr \, d\theta.
\]

Theorem 1 was established for the case \( p = 1 \) by Putnam [6], and improved extensively by D. Xia [7], M. Chô and M. Itô. The inequality above is called the Putnam-type inequality.

The normal approximate point spectrum for an operator \( T \), denoted by \( \sigma_{na}(T) \), is the set of complex numbers \( \lambda \) for which there exists a sequence \( \{\xi_n\} \) of unit vectors of \( \mathcal{H} \) such that
\[
\lim_{n \to \infty} \|(T - \lambda I)\xi_n\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|(T^* - \lambda I)\xi_n\| = 0.
\]

The normal approximate point spectrum is characterized as follows.

**Lemma 1** (cf. [7]). Let \( T \) be an operator on \( \mathcal{H} \), \( T = U|T| \) the polar decomposition of \( T \), and \( \lambda = re^{i\theta} \), \( r \neq 0 \). Then, \( \lambda \in \sigma_{na}(T) \) if and only if there exists a sequence \( \{\xi_n\} \) of unit vectors of \( \mathcal{H} \) such that
\[
\lim_{n \to \infty} \||T| - rI\xi_n\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \||U - e^{i\theta}I\xi_n\| = 0.
\]

It is clear that for any hyponormal operator \( T \) we have the equality \( \sigma_{na}(T) = \sigma_a(T) \), the approximate point spectrum of \( T \). Xia proved that it holds also for any 1/2-hyponormal operator \( T \), while Chô and Huruya [2] further proved the following

**Theorem 2**. For any \( p \)-hyponormal \( T \), \( p > 0 \), we have \( \sigma_{na}(T) = \sigma_a(T) \).

Combining Theorem 2 with Lemma 1, we have the following, where \( |S| \) denotes the set \( \{ |\xi| : \xi \in S \} \) for a set \( S \) on \( \mathbb{C} \).
Lemma 2. For any $p$-hyponormal operator $T$ on $\mathcal{H}$, we have $|\sigma_a(T)| \subseteq \sigma(|T|)$.

Now we show the following theorems.

Theorem 3. Let $T$ be a $p$-hyponormal operator on $\mathcal{H}$, $p > 0$. If both $\sigma(|T|)$ and the set $\{r \geq 0 : C_r \subseteq \sigma(T)\}$ are null sets with respect to Lebesgue measure on $\mathbb{R}$, where $C_r = \{\lambda : |\lambda| = r\}$, then $T$ is normal.

Proof. Since the boundary $\partial \sigma(T)$ of $\sigma(T)$ is contained in $\sigma_a(T)$, we have $|\partial \sigma(T)| \subseteq |\sigma_a(T)| \subseteq |\sigma(|T|)|$. So, we have

$$|\sigma(T)| = |\partial \sigma(T)| \cup \{r \geq 0 : C_r \subseteq \sigma(T)\} \subseteq |\sigma(|T|)| \cup \{r \geq 0 : C_r \subseteq \sigma(T)\}$$

which shows that, if both $\sigma(|T|)$ and the set $\{r \geq 0 : C_r \subseteq \sigma(T)\}$ are null with respect to Lebesgue measure on $\mathbb{R}$, then $\sigma(T)$ is normal. Hence, by Theorem 1, it follows that $T$ is normal. \(\square\)

Theorem 4. Let $T$ be a $p$-hyponormal operator on $\mathcal{H}$, $p > 0$, and let $T = U|T|$ be the polar decomposition of $T$. If $\sigma(|T|)$ is a null set with respect to Lebesgue measure on $\mathbb{R}$, and $\{\text{Arg} \lambda : \lambda \in \sigma(U) \setminus \{0\}\} \neq (-\pi, \pi)$ holds, then $T$ is normal.

Proof. Let $r > 0$ be such that $C_r \subseteq \sigma(T)$. Then, for any $\theta \in (-\pi, \pi)$, $r e^{i\theta}$ is in $\sigma(T)$. Hence, there is an $r_{\theta} > 0$ such that $r_{\theta} e^{i\theta} \in \partial \sigma(T)$. But, $\partial \sigma(T)$ is contained in $\sigma_a(T) = \sigma_a(|T|)$ by Theorem 2, so, $e^{i\theta}$ is in $\sigma(U)$ by Lemma 1. It follows that no $r > 0$ satisfies that $C_r \subset \sigma(T)$. Therefore, by the preceding theorem we conclude that $T$ is normal. \(\square\)

We give an example which shows that Theorem 4 fails to be true if the assumption $\{\text{Arg} \lambda : \lambda \in \sigma(U) \setminus \{0\}\} \neq (-\pi, \pi)$ fails to be fulfilled.

Example 1. Let $T$ be a weighted shift defined by

$$T = \begin{pmatrix} 0 & 0 & \cdots \\ \lambda_1 & 0 & \cdots \\ & \lambda_2 & \cdots \\ & & \ddots \\ & & & \ddots \\ \\ \end{pmatrix}.$$ 

Then, $T$ is $p$-hyponormal if and only if $|\lambda_n| \geq |\lambda_{n+1}|$ ($n = 1, 2, \ldots$). If $T = U|T|$ is the polar decomposition of $T$, then $U$ is the unilateral shift

$$U = \begin{pmatrix} 0 & 0 & \cdots \\ 1 & 0 & \cdots \\ & 1 & \cdots \\ & & \ddots \\ \end{pmatrix},$$

whose spectrum is the closed unit disk, and $|T|$ the diagonal operator with the diagonal sequence $\{|\lambda_n|\}$. So, it suffices to choose $\{\lambda_n\}$ so that $\lambda_n \neq 0$ ($n = 1, 2, \ldots$) and $\sigma(|T|)$ null with respect to Lebesgue measure on $\mathbb{R}$. 
3. A condition under which $B = A = U^*BU$
follows from $B \leq A \leq U^*BU$

Now we are about to show the following

**Theorem 5.** Let $A$, $B$ be self-adjoint operators and let $U$ be a unitary operator. If either $\sigma(A)$ or $\sigma(B)$ is a null set with respect to Lebesgue measure on $\mathbb{R}$, $\sigma(U) \neq T$, and $B \leq A \leq U^*BU$, then we have $B = A = U^*BU$.

**Proof.** Without loss of generality, we may assume that $\sigma(B)$ is a null set on $\mathbb{R}$. Choose $\lambda > 0$ such that $B + \lambda I$ becomes positive and invertible. Put $T = (B + \lambda I)^{1/2}U$; then by assumption, $TT^* \leq A + \lambda I \leq T^*T$. We also have that $|T| = U^*(B + \lambda I)^{1/2}U = U^*T$, and that $\sigma(|T|)$ is a null set on $\mathbb{R}$ because $\sigma((B + \lambda I)^{1/2})$ is also a null set. Let $T = V|T|$ be the polar decomposition of $T$. Then $T = VU^*T$, and so, $I = VU^*$, that is, $U = V$. Now we have found that $T$ satisfies the conditions in Theorem 4. Hence $T$ turns out to be normal, which implies that $B = A = U^*BU$.

The following is immediate.

**Corollary.** Let $A$, $B$ be self-adjoint operators, and let $U$, $V$ be unitaries. If either $\sigma(A)$ or $\sigma(B)$ is a null set with respect to Lebesgue measure on $\mathbb{R}$, $\sigma(UV) \neq T$, and $A \leq U^*BU$, $B \leq V^*AV$, then we have $A = U^*BU$, $B = V^*AV$.

We conclude by giving two examples which show that for Theorem 5 to be true the condition $\sigma(U) \neq T$ cannot be removed.

**Example 2.** Let $A$, $B$ be bilateral diagonal operators with diagonal sequences $\{\alpha_n\}_{-\infty}^{\infty}$, $\{\beta_n\}_{-\infty}^{\infty}$ of real numbers, respectively, and let $U$ be the bilateral shift. Then, $U^*BU$ is the diagonal operator with diagonal sequence $\{\beta_{n+1}\}_{-\infty}^{\infty}$. So, for our purpose, it suffices to take sequences $\{\alpha_n\}$, $\{\beta_n\}$ so that $\beta_n \leq \alpha_n \leq \beta_{n+1}$ and that $\{\alpha_n\} \neq \{\beta_n\}$ set-theoretically. Indeed, the former requirement is equivalent to $B \leq A \leq U^*BU$, and the latter implies that $A$ and $B$ are not unitarily equivalent.

**Example 3.** Let $U$ be the unilateral shift, $V$ the diagonal operator with the diagonal sequence $\{1/2, 1, 1, \cdots\}$, and put

$$
A = \begin{pmatrix} V & O \\ O & O \end{pmatrix}, \quad B = \begin{pmatrix} UU^* & O \\ O & O \end{pmatrix}, \quad W = \begin{pmatrix} U & I - UU^* \\ O & U^* \end{pmatrix}.
$$

Then $A$, $B$ are positive operators and $W$ a unitary operator. It is seen that $B \leq A \leq W^*BU$ because $O \leq UU^* \leq V \leq I = U^*U$. But $\sigma(A) \neq \sigma(B)$, which shows that $A$ and $B$ are not unitarily equivalent.

**References**


CONDITION $B = A = U^*BU$ follows from $B \leq A \leq U^*BU$


Faculty of Science, Yamagata University, Yamagata 990-8560, Japan

Graduate School of Science and Engineering, Yamagata University, Yamagata 990-8560, Japan