A MINIMUM FIXED POINT THEOREM FOR SMOOTH FIBER PRESERVING MAPS

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(Communicated by Paul Goerss)

This paper is dedicated to my advisor, Robert F. Brown

ABSTRACT. Let $p : E \to B$ be a smooth fiber bundle. Given a smooth fiber preserving map $f : E \to E$, we will show that $f$ can be deformed by a smooth, fiber preserving homotopy to a smooth map $g : E \to E$ such that the number of fixed points of $g$ is equal to the fiberwise Nielsen number of $f$. For a given map $f : X \to X$, where $X$ is a compact ANR, the Nielsen number of $f$, denoted $N(f)$, is a lower bound for the number of fixed points of maps homotopic to $f$. Wecken proved that if $X$ is a triangulated manifold of dimension greater than or equal to 3, there is a map $g$ homotopic to $f$ that has $N(f)$ fixed points [6]. A space is said to be Wecken if every self map of it has this property. Brown later proved that topological manifolds of dimension at least 3 are Wecken [1]. The corresponding theorem in the smooth category was proved by Jiang in [4]. He showed that for a smooth manifold $M$ of dimension $\geq 3$, if $f : M \to M$ is a smooth map, then $f$ can always be smoothly deformed to a map $g$ with exactly $N(f)$ fixed points.

The goal of this note is to apply Jiang’s smooth Wecken theorem to prove a smooth version of a Wecken-type theorem for fiber preserving maps of Heath, Keppelmann and Wong [3]. In the setting of this theorem, $p : E \to B$ is a fibration of compact connected ANR’s. Then the pair $(f, \bar{f})$ is called a fiber preserving map of $p$ if $f : E \to E$, $\bar{f} : B \to B$ and the condition $fp = pf$ is satisfied. The fiberwise Nielsen number $N_X(f, p)$ of $(f, \bar{f})$, also known as the naïve addition formula, is then defined to be $N_X(f, p) = \sum_{x \in \xi} N(f_x)$, where $\xi$ is a set consisting of one point from each essential fixed point class of $\bar{f}$. If $g : E \to E$ is homotopic to $f$ by a fiber preserving homotopy, then $g$ has at least $N_X(f, p)$ fixed points.

Theorem 1 (Heath, Keppelmann, Wong). Let $(f, \bar{f})$ be a fiber preserving map from $p$ to itself with the property that $\bar{f}$ is homotopic to a map $\bar{g}$ that has exactly $N(\bar{f})$ fixed points. Suppose further that every fiber over the unique set of essential representatives for $\bar{g}$ is a Wecken space. Then there is a fiber preserving map $(g, \bar{g})$ that is fiber homotopic to $(f, \bar{f})$ with the property that $g$ has exactly $N_X(f, p)$ fixed points.
For the fiber Wecken theorem in the smooth category, we must assume that we have a smooth fiber bundle. This consists of a smooth surjective map \( p : E \to B \), where \( E \) and \( B \) are smooth compact manifolds with or without boundary. Furthermore, \( B \) can be covered by a system of local coordinate neighborhoods \( \{ U_\alpha \} \) such that there are diffeomorphisms \( \phi_{U_\alpha} : U_\alpha \times F \to p^{-1}(U_\alpha) \) satisfying \( p\phi_{U_\alpha}(x, y) = x \).

We will prove the following.

**Theorem 2.** If \( p : E \to B \) is a smooth fiber bundle with \( \dim B, F \geq 3 \) and \( (f, \tilde{f}) \) is a smooth fiber preserving map of \( p \), then there exists a smooth fiber preserving map \( (g, \tilde{g}) \) smoothly fiber preserving homotopic to \( (f, \tilde{f}) \) such that \( g \) has \( N_X(f, p) \) fixed points.

**Proof.** First, we can directly apply Jiang’s smooth Wecken theorem to \( \tilde{f} \). This gives us a smooth map \( \tilde{h} : B \to B \) that is smoothly homotopic to \( f \), by a smooth homotopy \( \alpha_t \), where \( \alpha_0 = \tilde{f} \) and \( \alpha_1 = p \), such that \( h \) has \( N(f) \) fixed points. By the smooth covering homotopy theorem \([2]\), there exists a smooth lift \( \overline{\alpha_t} \circ p : E \to E \) of \( \alpha_t \circ p \) since \((f, \tilde{f})\) is a smooth fiber preserving map of \( p \). Let \( h = \overline{\alpha_1} \circ p ; \) then \((h, \tilde{h})\) is a fiber preserving map of \( p \) that is smoothly fiber homotopic to \((f, \tilde{f})\).

Suppose \( b \) is a fixed point of \( \tilde{h} \) and that \( b \) is contained in some local coordinate chart \( V \) that has the local trivialization property. Since \( \tilde{h} \) has isolated fixed points, we can find an open neighborhood \( U \) of \( b \) where \( U \subseteq \tilde{h}^{-1}(V) \cap V \) and \( U \) has no additional fixed points of \( \tilde{h} \). We may assume that \( U \) is the interior of a geodesic ball with center \( b \) and radius 1 (we can always rescale). This implies that any two points in \( U \) can be joined by a unique arc length geodesic that is contained in \( U \).

Let \( h_b = \tilde{h}|_{p^{-1}(b)} : p^{-1}(b) \to p^{-1}(b) \). Applying the smooth Wecken theorem to \( h_b \), there exists a smooth map \( g_b \) smoothly homotopic to \( h_b \), by a homotopy we will call \( h_t \), where \( h_0 = g_b \) and \( h_1 = h_b \), such that \( g_b \) has \( N(h_b) \) fixed points. Since \( h_b \) is homotopic to \( f_b \) by the homotopy \( h_t \) above, it follows that \( g_b \) has \( N(f_b) \) fixed points. The local triviality conditions on \( U \) and \( V \) give us a homotopy

\[
\phi_U^{-1} \circ h \circ \phi_{U} : \{b\} \times F \to \{b\} \times F
\]
such that

\[
\phi_U^{-1} \circ h \circ \phi_{U}(y) = (\tilde{h}(b, y)),
\]

where \( \tilde{h}_t \) is a smooth homotopy on \( \{b\} \times F \). Let \( \tilde{h}_0(b, y) = \tilde{g}(b, y) \) and \( \tilde{h}_1(b, y) = \tilde{h}(b, y) \).

Since \((h, \tilde{h})\) is a fiber preserving map of \( p \), we have that \( \phi_U^{-1} \circ h \circ \phi_U : U \times F \to V \times F \) is of the form

\[
\phi_U^{-1} \circ h \circ \phi_U(x, y) = (\tilde{h}(x, y)),
\]

where \( \tilde{h} \) is a smooth map on \( U \times F \). For each \( x \in U \), there exists a unique arc length parameter geodesic \( \gamma_x : I \to U \), where \( \gamma_x(0) = b \), \( \gamma_x(1) = x \), \( \gamma_x \) depends smoothly on its endpoints and varies continuously with \( x \). Define \( k_t : U \times F \to F \) by \( k_t(x, y) = \tilde{h}(\gamma_x(t), y) \). Then \( k_t \) is a continuous homotopy, where \( k_0(x, y) = \tilde{h}(b, y) \) and \( k_1(x, y) = \tilde{h}(x, y) \). We can now define a homotopy \( c_t : U \times F \to F \) as follows:

\[
c_t(x, y) = \begin{cases} 
\tilde{h}_2t(b, y), & \text{if } 0 \leq t \leq \frac{1}{2}, \\
k_{2t-1}(x, y), & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]
By standard smooth approximation techniques, $c_t$ can be approximated by a smooth homotopy $\tilde{c}_t$ with $\tilde{c}_0(x, y) = \tilde{g}(b, y)$ and $\tilde{c}_1(x, y) = \tilde{h}(x, y)$.

Consider a smooth monotone increasing function $B : I \to I$, such that $B$ equals 0 on the interval $[0, \frac{1}{2}]$ and $B$ equals 1 on the interval $[\frac{3}{4}, 1]$. Define $\tilde{l}_t : U \times F \to V \times F$ by

$$\tilde{l}_t(x, y) = (\tilde{h}(x), \tilde{c}_{t+(1-t)B(\text{dist}(b, x))}(x, y)),$$

where $\text{dist}(b, x)$ is the length of the unique minimal geodesic $\gamma_t$ joining $b$ to $x$. If $\phi_U^{-1}(z) = (x, y)$, we use $\tilde{l}_t$ to define a smooth homotopy $l_t : p^{-1}(U) \to p^{-1}(V)$ by

$$l_t(z) = \phi_V \circ \tilde{l}_t \circ \phi_U^{-1}(z) = \phi_V \circ \tilde{l}_t(x, y) = \phi_V(\tilde{h}(x), \tilde{c}_{t+(1-t)B(\text{dist}(b, x))}(x, y)).$$

Consider

$$l_0(z) = \phi_V(\tilde{h}(x), \tilde{c}_{B(\text{dist}(b, x))}(x, y)).$$

When $z \in p^{-1}(b)$, then $l_0(z) = g_b(z)$. If $\text{dist}(b, x) \geq \frac{1}{4}$, then $l_0(z) = h(z)$. Note that $l_t$ is a homotopy ending at $h(z)$.

Extend $l_t$ to a smooth fiber preserving homotopy, which we will call $L_t$, defined on all of $E$ by taking defining $L_t$ to be $h$ outside of the neighborhood of $p^{-1}(b)$.

Define $g$ to be $L_0$. Now $g$ is a smooth self map of $E$ that is smoothly homotopic to $h$, where $g|_{p^{-1}(b)} = g_b$ and $g = h$ outside a neighborhood of $p^{-1}(b)$. The map $g$ has $N(f_b)$ fixed points on the fiber $p^{-1}(b)$ over the fixed point $b$. Repeated application of this process produces a map $g$ that has $N_{t}(f, p) = \sum_{x \in \xi} N(f_x)$ fixed points, where $\xi$ is any set of representatives for the essential fixed point classes of $f$. □

References


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