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A MINIMUM FIXED POINT THEOREM FOR SMOOTH FIBER PRESERVING MAPS

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(Communicated by Paul Goerss)

This paper is dedicated to my advisor, Robert F. Brown

ABSTRACT. Let $p : E \to B$ be a smooth fiber bundle. Given a smooth fiber preserving map $f : E \to E$, we will show that f can be deformed by a smooth, fiber preserving homotopy to a smooth map $g : E \to E$ such that the number of fixed points of g is equal to the fiberwise Nielsen number of f.

For a given map $f : X \to X$, where X is a compact ANR, the Nielsen number of f, denoted $N(f)$, is a lower bound for the number of fixed points of maps homotopic to f. Wecken proved that if X is a triangulated manifold of dimension greater than or equal to 3, there is a map g homotopic to f that has $N(f)$ fixed points [6]. A space is said to be *Wecken* if every self map of it has this property. Brown later proved that topological manifolds of dimension at least 3 are Wecken [1]. The corresponding theorem in the smooth category was proved by Jiang in [4]. He showed that for a smooth manifold M of dimension ≥ 3, if $f : M \to M$ is a smooth map, then f can always be smoothly deformed to a map g with exactly $N(f)$ fixed points.

The goal of this note is to apply Jiang's smooth Wecken theorem to prove a smooth version of a Wecken-type theorem for fiber preserving maps of Heath, Keppelmann and Wong [3]. In the setting of this theorem, $p : E \to B$ is a fibration of compact connected ANR's. Then the pair $(f, \bar{f})$ is called a fiber preserving map of p if $f : E \to E$, $\bar{f} : B \to B$ and the condition $\bar{f}p = pf$ is satisfied. The fiberwise Nielsen number $N_{\mathcal{F}}(f, p)$ of $(f, \bar{f})$, also known as the naive addition formula, is then defined to be $N_{\mathcal{F}}(f, p) = \sum_{x \in \xi} N(f_x)$, where ξ is a set consisting of one point from each essential fixed point class of $\bar{f}$. If $g : E \to E$ is homotopic to f by a fiber preserving homotopy, then g has at least $N_{\mathcal{F}}(f, p)$ fixed points.

Theorem 1 (Heath, Keppelmann, Wong)**.** *Let $(f, \bar{f})$ be a fiber preserving map from p to itself with the property that $\bar{f}$ is homotopic to a map $\bar{g}$ that has exactly $N(\bar{f})$ fixed points. Suppose further that every fiber over the unique set of essential representatives for $\bar{g}$ is a Wecken space. Then there is a fiber preserving map $(g, \bar{g})$ that is fiber homotopic to $(f, \bar{f})$ with the property that g has exactly $N_{\mathcal{F}}(f, p)$ fixed points.*

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For the fiber Wecken theorem in the smooth category, we must assume that we have a smooth fiber bundle. This consists of a smooth surjective map $p : E \to B$, where E, B and the fiber, F, are smooth compact manifolds with or without boundary. Furthermore, B can be covered by a system of local coordinate neighborhoods $\{U_\alpha\}$ such that there are diffeomorphisms $\phi_{U_\alpha} : U_\alpha \times F \to p^{-1}(U_\alpha)$ satisfying $p\phi_{U_\alpha}(x, y) = x$.

We will prove the following.

Theorem 2. *If $p : E \to B$ is a smooth fiber bundle with dim $B, F \geq 3$ and $(f, \bar{f})$ is a smooth fiber preserving map of p, then there exists a smooth fiber preserving map $(g, \bar{g})$ smoothly fiber preserving homotopic to $(f, \bar{f})$ such that g has $N_{\mathcal{F}}(f, p)$ fixed points.*

Proof. First, we can directly apply Jiang's smooth Wecken theorem to $\bar{f}$. This gives us a smooth map $\bar{h} : B \to B$ that is smoothly homotopic to $\bar{f}$, by a smooth homotopy α_t, where $\alpha_0 = \bar{f}$ and $\alpha_1 = \bar{h}$, such that $\bar{h}$ has $N(\bar{f})$ fixed points. By the smooth covering homotopy theorem [2], there exists a smooth lift $\widetilde{\alpha_t \circ p} : E \to E$ of $\alpha_t \circ p$ since $(f, \bar{f})$ is a smooth fiber preserving map of p. Let $h = \widetilde{\alpha_1 \circ p}$; then $(h, \bar{h})$ is a fiber preserving map of p that is smoothly fiber homotopic to $(f, \bar{f})$.

Suppose b is a fixed point of $\bar{h}$ and that b is contained in some local coordinate chart V that has the local trivialization property. Since $\bar{h}$ has isolated fixed points, we can find an open neighborhood U of b where $U \subset \bar{h}^{-1}(V) \cap V$ and U has no additional fixed points of $\bar{h}$. We may assume that U is the interior of a geodesic ball with center b and radius 1 (we can always rescale). This implies that any two points in U can be joined by a unique arc length geodesic that is contained in U.

Let $h_b = h|_{p^{-1}(b)} : p^{-1}(b) \to p^{-1}(b)$. Applying the smooth Wecken theorem to h_b, there exists a smooth map g_b smoothly homotopic to h_b, by a homotopy we will call h_t, where $h_0 = g_b$ and $h_1 = h_b$, such that g_b has $N(h_b)$ fixed points. Since h_b is homotopic to f_b by the homotopy h_t above, it follows that g_b has $N(f_b)$ fixed points. The local triviality conditions on U and V give us a homotopy

$$\phi_V|^{-1}_{\{b\}\times F} \circ h_t \circ \phi_U|_{\{b\}\times F} : \{b\} \times F \to \{b\} \times F$$

such that

$$\phi_V|^{-1}_{\{b\}\times F} \circ h_t \circ \phi_U|_{\{b\}\times F}(b, y) = (b, \tilde{h}_t(b, y)),$$

where $\tilde{h}_t$ is a smooth homotopy on $\{b\} \times F$. Let $\tilde{h}_0(b, y) = \tilde{g}(b, y)$ and $\tilde{h}_1(b, y) = \tilde{h}(b, y)$.

Since $(h, \bar{h})$ is a fiber preserving map of p, we have that $\phi_V^{-1} \circ h \circ \phi_U : U \times F \to V \times F$ is of the form

$$\phi_V^{-1} \circ h \circ \phi_U(x, y) = (\bar{h}(x), \tilde{h}(x, y)),$$

where $\tilde{h}$ is a smooth map on $U \times F$. For each $x \in U$, there exists a unique arc-length parameter geodesic $\gamma_x : I \to U$, where $\gamma_x(0) = b$, $\gamma_x(1) = x$, γ_x depends smoothly on its endpoints and varies continuously with x. Define $k_t : U \times F \to F$ by $k_t(x, y) = \tilde{h}(\gamma_x(t), y)$. Then k_t is a continuous homotopy, where $k_0(x, y) = \tilde{h}(b, y)$ and $k_1(x, y) = \tilde{h}(x, y)$. We can now define a homotopy $c_t : U \times F \to F$ as follows:

$$c_t(x, y) = \begin{cases} \tilde{h}_{2t}(b, y), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ k_{2t-1}(x, y), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

By standard smooth approximation techniques, c_t can be approximated by a smooth homotopy $\tilde{c}_t$ with $\tilde{c}_0(x,y) = \tilde{g}(b,y)$ and $\tilde{c}_1(x,y) = \tilde{h}(x,y)$.

Consider a smooth monotone increasing function $B : I \to I$, such that B equals 0 on the interval $[0, \frac{1}{2}]$ and B equals 1 on the interval $[\frac{3}{4}, 1]$. Define $\tilde{l}_t : U \times F \to V \times F$ by

$$\tilde{l}_t(x,y) = (\bar{h}(x), \tilde{c}_{t+(1-t)B(dist(b,x))}(x,y)),$$

where $dist(b,x)$ is the length of the unique minimal geodesic γ_x joining b to x. If $\phi_U^{-1}(z) = (x,y)$, we use $\tilde{l}_t$ to define a smooth homotopy $l_t : p^{-1}(U) \to p^{-1}(V)$ by

$$l_t(z) = \phi_V \circ \tilde{l}_t \circ \phi_U^{-1}(z) = \phi_V \circ \tilde{l}_t(x,y) = \phi_V(\bar{h}(x), \tilde{c}_{t+(1-t)B(dist(b,x))}(x,y)).$$

Consider

$$l_0(z) = \phi_V(\bar{h}(x), \tilde{c}_{B(dist(b,x))}(x,y)).$$

When $z \in p^{-1}(b)$, then $l_0(z) = g_b(z)$. If $dist(b,x) \geq \frac{3}{4}$, then $l_0(z) = h(z)$. Note that l_t is a homotopy ending at $h(z)$.

Extend l_t to a smooth fiber preserving homotopy, which we will call L_t, defined on all of E by taking defining L_t to be h outside of the neighborhood of $p^{-1}(b)$. Define g to be L_0. Now g is a smooth self map of E that is smoothly homotopic to h, where $g|_{p^{-1}(b)} = g_b$ and $g = h$ outside a neighborhood of $p^{-1}(b)$. The map g has $N(f_b)$ fixed points on the fiber $p^{-1}(b)$ over the fixed point b. Repeated application of this process produces a map g that has $N_{\mathcal{F}}(f,p) = \sum_{x \in \xi} N(f_x)$ fixed points, where ξ is any set of representatives for the essential fixed point classes of $\bar{f}$. $\square$

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