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TORNADO SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^2$: APPLICATIONS

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Abstract. We show partial regularity of bounded positive solutions of some semilinear elliptic equations $\Delta u = f(u)$ in domains of $\mathbb{R}^2$. As a consequence, there exists a large variety of nonnegative singular solutions to these equations. These equations have previously been studied from the point of view of free boundary problems, where solutions additionally are stable for a variational problem, which we do not assume.

1. Introduction

In this paper we study positive solutions to the equations $\Delta u = f(u)$ on domains of $\mathbb{R}^n$ where $f(u)$ is like $u^{-\alpha}$ near zero. For the main result, when $0 < \alpha < 1$, we show the nonexistence of "tornado sequences" of solutions. If additionally $n = 2$, then it follows by results of the first paper [8], that uniformly bounded solutions of this equation are equicontinuous. Existence of "singular solutions" can be proved then using a degree argument. The bulk of the proof lies in understanding the radially symmetric singular solutions to $\Delta u = u^{-\alpha}$. We establish the existence of a continuous family of solutions that can be used as barriers for the maximum principle. Important remaining open questions are what happens when $\alpha \geq 1$ and when $n \geq 3$. Some work in this direction was done in [7].

We recall the definition of tornado sequences.

Definition 1. A tornado sequence of solutions to $\Delta u = f(u)$ is given by a number $\epsilon > 0$, a sequence $r_j > 0$, $j = 1, 2, \ldots$, with $r_j \to 0$, and positive solutions u_j defined on balls B_{r_j} of radius r_j such that $u_j > \epsilon$ on ∂B_{r_j} and $\min_{B_{r_j}} u_j \to 0$.

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We will assume that Ω is Lipschitz and that $f(u) = g(u)u^{-\alpha}$, where g satsifies the following:

$$\begin{aligned} &0 < \alpha < 1, \\ &g(u) \text{ is Hölder continuous on } (\delta, \infty), \forall \delta > 0, \\ &0 \le C_1 < g(u) < C_2 < \infty. \end{aligned} \tag{1}$$

Note that any solution u is a subsolution to $\Delta u = C_1 u^{-\alpha}$ and a supersolution to $\Delta u = C_2 u^{-\alpha}$. We may now state the main results. Note that Theorem 1 holds in any dimension while Corollary 1 and Theorem 4 hold only in $\mathbb{R}^2$.

Theorem 1. *If f satisfies the above assumptions, there does not exist a tornado sequence of solutions to $\Delta u = f(u)$.*

Corollary 1. *If f satisfies the same assumptions, then a sequence of positive solutions u_k to $\Delta u = f(u)$ on a domain $\Omega \subset \mathbb{R}^2$ with $u_k \le M$ is equicontinuous on any compact subdomain.*

Theorem 2. *A sequence of positive solutions u_j to $\Delta u = f(u)$ on a disk $B \subset \mathbb{R}^2$ with $|u_j|_{C^1(\partial B)} \le M$ and $u_j\,|_{\partial B} > \epsilon$ is equicontinuous on $\bar{B}$.*

We call a nonnegative function u_0 a singular solution to $\Delta u = f(u)$ if u_0 is a limit of a sequence of positive smooth solutions and $\min u_0 = 0$. The next result shows that, under the conditions of Corollary 1, there is a large variety of nonnegative singular solutions to $\Delta u = f(u)$ on any disc in $\mathbb{R}^2$.

Theorem 3. *Let B be any disc in $\mathbb{R}^2$, and let $\varphi_t \colon B \longrightarrow \mathbb{R}$ be continuous functions defined on the boundary, continuously parametrized by $t \in [0,1]$. There are constants M and $\epsilon > 0$ depending only on f and the radius of the disc so that if $\varphi_0 > M$ and $\varphi_1 < \epsilon$, there is a $t_0 \in [0,1]$ and a singular solution u_0 on B with $u_0 = \varphi_{t_0}$ on ∂B.*

So, for example, consider any boundary function φ_0 with $\varphi_0 > M$, and let $\varphi_t = (1-t)\varphi_0$. Then φ_{t_0} will have the same relative sizes of Fourier coefficients as φ_0. So, for each distinct set of Fourier coefficients with constant coefficient 1 which can produce a positive function on ∂B, there corresponds a distinct singular solution on the disc B, whose boundary data has the same relative sizes of Fourier coefficients.

In [9] and [10], Phillips studied the free boundary problem for $\Delta u = (1-\alpha)u^{-\alpha}$ for $0 < \alpha < 1$. In this case, the free boundary solution with boundary data $u = \psi$ on $\partial\Omega$ is the function $u \ge 0$ which minimizes the integral $\int_\Omega |Du|^2 + u^{1-\alpha}$. This solution is allowed to be identically zero (and thus not satisfy the differential equation) on a positive measure subset $\tilde{\Omega} \subset \Omega$. The minimizer is locally $C^{1,\frac{1-\alpha}{1+\alpha}}$, and the free boundary $\partial\tilde{\Omega}$ has locally finite $(n-1)$-dimensional Hausdorff measure. We note that our results do not apply to the free boundary case, but do apply to solutions to the PDE which are not minimizers of a variational integral, nor even stable with respect to a variational integral. In our case, the variational integral would be

$$\mathcal{I}(u) = \int_\Omega \frac{1}{2}|Du|^2 + F(u), \qquad F(u) = \int_1^u f(z)\,dz.$$

Solutions of the PDE $\Delta u = f(u)$ are stationary, and they are stable if the second variation of $\mathcal{I}$ is nonnegative. The equation $\Delta u = u^{-\alpha}$ also arises in relation to

chemical catalyst kinetics (see [1] and [5]). Similar equations were studied in [3] and [4]. Singular and stable solutions of the equation $\Delta u = \frac{1}{u}$ were studied in [7], where the work was inspired by a similar study of the singular minimal surface equation, as in [11]. The major results, which also apply to $\Delta u = u^{-\alpha}$, rely heavily on the assumption that solutions are stable with respect to the corresponding variational integral.

2. Basic facts

We are particularly interested in "singular solutions" of $\Delta u = f(u)$, i.e. those that are not differentiable to second order. We construct such solutions as nonnegative functions which are limits of positive smooth solutions. In the following, we show that nonnegative weak solutions which achieve the value zero are indeed singular.

Definition 2. By a *weak solution* to $\Delta u = f(u)$, we mean a function $u \in L^2_{\mathrm{loc}}(\Omega)$, $\Omega \subset \mathbb{R}^n$, such that $f(u) \in L^2_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega u\Delta\zeta = \int_\Omega f(u)\zeta\,, \qquad \text{for all } \zeta \in C_c^\infty(\Omega).$$

Assuming f satisfies the assumptions (1), with $C_1 > 0$, weak solutions which have a zero in the interior of a domain have limited regularity.

Lemma 1. *Let $u \in C^{1,\beta}(B_\delta(0))$ be a weak solution of $\Delta u = f(u)$ such that $u \geq 0$ and $u(0) = 0$. Then $\beta \leq \frac{1-\alpha}{1+\alpha}$.*

Proof. Let $0 < \rho < \delta$ be arbitrary. Since $u \geq 0$ and $u \in C^1$, $\nabla u(0) = 0$, and for any $x \in B_\delta(0)$, $|\nabla u(x)| \leq C|x|^\beta$, and thus $u(x) \leq C|x|^{\beta+1}$.

We now let ζ be a radially symmetric function in $C_c^\infty(\Omega)$ with $\zeta = 1$ on $B_{\rho/2}(0)$, $|\nabla\zeta| \leq \frac{C_1}{\rho}$, and $|\Delta\zeta| \leq \frac{C_2}{\rho^2}$. Then

$$\int_{B_\rho} u\Delta\zeta \leq C\frac{C_2}{\rho^2}\int_{B_\rho} |x|^{\beta+1} \leq \tilde{C}_2\rho^{\beta+n-1}, \tag{2}$$

while by the weak equation,

$$\int_{B_\rho} u\Delta\zeta = \int_{B_\rho} f(u)\zeta \geq C\int_{B_{\rho/2}} \left(|x|^{\beta+1}\right)^{-\alpha} \geq \tilde{C}_1\rho^{n-\alpha\beta-\alpha}. \tag{3}$$

Thus, $\rho^{n-\alpha\beta-\alpha} \leq \tilde{C}\rho^{\beta+n-1}$ for arbitrarily small ρ, from which necessarily $\beta \leq \frac{1-\alpha}{1+\alpha}$. □

Note. Setting $C_{\alpha,n} = \left[\frac{(1+\alpha)^2}{2n(1+\alpha)-4\alpha}\right]^{\frac{1}{1+\alpha}}$, the function $U_{\alpha,n}(x) = C_{\alpha,n}|x|^{\frac{2}{1+\alpha}}$ is a weak solution of $\Delta u = u^{-\alpha}$ achieving the maximum allowed regularity.

The same argument as above can be applied to show a lack of boundary regularity.

Lemma 2. *Suppose $\Omega \subset \mathbb{R}^n$ is a domain with an interior cone condition, and suppose $0 \in \partial\Omega$. Let $u \in C^{1,\beta}(\bar{\Omega})$ be a weak solution of $\Delta u = f(u)$ such that $u \geq 0$ and $u(0) = 0$. Then $\beta \leq \frac{1-\alpha}{1+\alpha}$.*

Proof. The above integrals (2) and (3), taken instead over $B_\rho \cap \Omega$, can be estimated in exactly the same way.

There are some additional positivity results that will be useful in the next section.

Lemma 3. *Suppose u is a positive and smooth subsolution to $\Delta u = C_1 u^{-\alpha}$ on a domain Ω which includes a ball $B_{2\rho}$ of radius 2ρ. Then*

(1) $\int_{B_{2\rho}\setminus B_\rho} u^{1+\alpha} \geq C_1 \alpha \omega_n \rho^{n+2}$,

(2) $\sup_{\partial\Omega} u \geq \left(\frac{C_1\alpha}{2^n-1}\right)^{\frac{1}{1+\alpha}} \rho^{\frac{2}{1+\alpha}}$.

Here ω_n is the volume of the unit ball in $\mathbb{R}^n$. Note that the second property states that positive solutions on Ω with very small boundary values do not exist, and that solutions defined on all of $\mathbb{R}^n$ must be unbounded.

Proof. We use the weak inequality

$$\int_\Omega Du \cdot D\zeta \leq -C_1 \int_\Omega \zeta u^{-\alpha} \quad \text{for all } \zeta \in C_c^1(\Omega)$$

with $\zeta = u^\alpha \varphi^2$ and φ a Lipschitz function which is equal to zero outside $B_{2\rho}$. Then

$$\int_\Omega \alpha u^{\alpha-1}\varphi^2 |Du|^2 + 2\int_\Omega u^\alpha \varphi Du \cdot D\varphi \leq -C_1 \int_\Omega \varphi^2.$$

Using the Cauchy–Schwarz inequality,

$$\begin{aligned}
\int_\Omega \varphi^2 \left(\alpha u^{\alpha-1}|Du|^2 + C_1\right) &\leq 2\int_\Omega u^\alpha \varphi |Du||D\varphi| \\
&\leq 2\int_\Omega \left(u^{\frac{\alpha-1}{2}}\varphi|Du|\right)\left(u^{\frac{\alpha+1}{2}}|D\varphi|\right) \\
&\leq \alpha \int_\Omega u^{\alpha-1}\varphi^2|Du|^2 + \frac{1}{\alpha}\int_\Omega u^{\alpha+1}|D\varphi|^2.
\end{aligned}$$

Thus,

$$C_1\alpha \int_\Omega \varphi^2 \leq \int_\Omega u^{\alpha+1}|D\varphi|^2. \tag{4}$$

We choose φ to be radially symmetric on $B_{2\rho}$ so that, using the radial variable r, $\varphi(r) = 1$ for $0 \leq r \leq \rho$, and $\varphi(r) = 2 - r/\rho$ for $\rho \leq r \leq 2\rho$. Then $|D\varphi| = 1/\rho$ on $B_{2\rho} \setminus B_\rho$. So, (4) becomes

$$C_1\alpha\omega_n\rho^n \leq C_1\alpha \int_\Omega \varphi^2 \leq \frac{1}{\rho^2}\int_{B_{2\rho}\setminus B_\rho} u^{1+\alpha},$$

which implies the first result. Now suppose $u \leq M$ on $\partial\Omega$, so by the maximum principle, $u \leq M$ on $B_{2\rho}$. The first result implies

$$M^{\alpha+1}\omega_n((2\rho)^n - \rho^n) \geq C_1\alpha\omega_n\rho^{n+2},$$

which completes the proof.

3. Radially symmetric solutions

The analysis of solutions which are radially symmetric will be very important to us.

Definition 3. A *radial solution* to $\Delta u = f(u)$ is a solution $u(x)$ which only depends on $|x|$, i.e. $u(x) = u(r)$, $r = |x|$.

Radial solutions satisfy the ordinary differential equation $u_{rr} + \frac{n-1}{r}u_r = f(u)$. We present the existence and asymptotics of radial solutions to $\Delta u = u^{-\alpha}$ with various initial-boundary conditions. Some analysis of this ODE was done by Brauner–Nicolaenko in [1]. We also investigate the existence of radial solutions to $\Delta u = f(u)$.

Lemma 4. *For any $\epsilon > 0$, there is a radial solution to $\Delta u = f(u)$ defined on $\mathbb{R}^n$ with $u(0) = \epsilon$. In case f is locally Lipschitz near $u = \epsilon$, the solution is unique.*

Proof. If we assume f is Lipschitz on $[\frac{\epsilon}{2}, 2\epsilon]$, this can be done by the Contraction Mapping Theorem. Setting $v = u - \epsilon$, we seek a solution v to the equation

$$\left(r^{n-1}v_r\right)_r = r^{n-1}f(\epsilon + v). \tag{5}$$

We consider the map T defined on $\{v \in C^0[0,\delta] \,:\, |v| < \delta\}$, $\delta < \epsilon/2$, by

$$T(v)(r) = \int_0^r \frac{1}{t^{n-1}} \int_0^t s^{n-1} f(\epsilon + v(s))\, ds\, dt.$$

Then

$$\sup |T(v)| \le \frac{\delta^2}{2n} f(\epsilon - \delta)$$

and

$$\sup |T(v_1) - T(v_2)| \le \frac{\delta^2 L}{2n} \sup |v_1 - v_2|,$$

where L is the Lipschitz constant. So, for small enough δ, T is a contraction mapping. Now $T(v) \in C^2[0,\delta]$ with $T(v)'(0) = 0$ and the unique fixed point satisfies (5). Since $f > 0$, the solution v cannot have a local maximum; thus $v \ge 0$, and then by integration, $v(r) \le Cr^2$. By the ODE existence and uniqueness theorem, this solution can be continued to all of $[0,\infty)$. Thus, $u = \epsilon + v$ is a radial solution on $\mathbb{R}^n$. In the case that f is not locally Lipschitz, local existence follows from the Schauder fixed point theorem applied as in the proof of the next lemma. □

The following lemma, which we state in generality, will be applied in the somewhat simpler case that $g(u) = C_1 = C_2$ is a single constant.

Lemma 5. *For any $a \ge 0$, there is a radial solution to $\Delta u = f(u)$ defined on $\mathbb{R}^n \setminus B_a(0)$ with $u(a) = 0$ and $u_r(a) = 0$.*

Proof. For $\delta > 0$, let K be the closed convex subset of $X = C^0[a, a+\delta]$ defined by

$$K = \left\{ \; v \in C^0[a, a+\delta] \,:\, A_1(r-a)^{\frac{2}{1+\alpha}} \le v(r) \le A_2(r-a)^{\frac{2}{1+\alpha}} \; \right\}.$$

Here the A_i and δ are constants to be chosen. We define $T : K \to X$ by

$$T(v)(r) = \int_a^r \frac{1}{t^{n-1}} \int_a^t s^{n-1} f(v(s))\, ds\, dt.$$

We will show for small enough δ that $T : K \to K$ and that T is continuous and compact in order to apply the Schauder Fixed Point Theorem.

Now if $v \in K$,

$$\begin{aligned} T(v)(r) &= \int_a^r \frac{1}{t^{n-1}} \int_a^t s^{n-1} g(v(s)) v(s)^{-\alpha}\, ds\, dt \\ &\le \int_a^r \int_a^t g(v(s)) v(s)^{-\alpha}\, ds\, dt \\ &\le C_2 \int_a^r \int_a^t A_1^{-\alpha} (s-a)^{\frac{-2\alpha}{1+\alpha}}\, ds\, dt \\ &\le \frac{C_2 A_1^{-\alpha}(1+\alpha)^2}{2(1-\alpha)} (r-a)^{\frac{2}{1+\alpha}}. \end{aligned}$$

For the lower bound, we use a large integer m, depending on a and α. We do this so that δ is independent of a, and allows $a = 0$.

$$\begin{aligned} T(v)(r) &= \int_a^r \frac{1}{t^{n-1}} \int_a^t s^{n-1} g(v(s)) v(s)^{-\alpha}\, ds\, dt \\ &\ge \int_{\frac{(m-1)a+r}{m}}^r t^{1-n} \int_{\frac{(m-1)a+r}{m}}^t s^{n-1} g(v(s)) v(s)^{-\alpha}\, ds\, dt \\ &\ge m^{1-n} C_1 A_2^{-\alpha} \int_{\frac{(m-1)a+r}{m}}^r t^{1-n} \int_{\frac{(m-1)a+r}{m}}^t (s-a)^{\frac{-2\alpha}{1+\alpha}}\, ds\, dt \\ &= \frac{m^{1-n} C_1 A_2^{-\alpha}(1+\alpha)}{1-\alpha} \left(\frac{(1+\alpha) m^{2/(1+\alpha)} - 2m + 1 - \alpha}{2m^{2/(1+\alpha)}} \right) (r-a)^{\frac{2}{1+\alpha}}. \end{aligned}$$

We choose m depending on α so that

$$\frac{(1+\alpha) m^{2/(1+\alpha)} - 2m + 1 - \alpha}{2m^{2/(1+\alpha)}} \ge \frac{1+\alpha}{4},$$

and so

$$T(v)(r) \ge \frac{m_\alpha^{1-n} C_1 A_2^{-\alpha}(1+\alpha)^2}{4(1-\alpha)} (r-a)^{\frac{2}{1+\alpha}}.$$

For T to map K to itself, we need the inequalities

$$A_1 \le \frac{m^{1-n} C_1 A_2^{-\alpha}(1+\alpha)^2}{4(1-\alpha)} \le \frac{C_2 A_1^{-\alpha}(1+\alpha)^2}{2(1-\alpha)} \le A_2,$$

which are satisfied if we assume (without loss of generality) that $C_1 C_2 \ge 1$ and set the constants

$$\begin{aligned} A_1 &= \left(\frac{m^{1-n} C_1}{2C_2^\alpha} \right)^{\frac{1}{1-\alpha^2}} \left(\frac{(1+\alpha)^2}{2(1-\alpha)} \right)^{\frac{1}{1+\alpha}}, \\ A_2 &= \left(C_2 m^{(n-1)\alpha} \left(\frac{2}{C_1} \right)^\alpha \right)^{\frac{1}{1-\alpha^2}} \left(\frac{(1+\alpha)^2}{2(1-\alpha)} \right)^{\frac{1}{1+\alpha}}. \end{aligned}$$

To see that T is continuous, we note that

$$|T(v_1)(r) - T(v_2)(r)| \le \int_a^r t^{1-n} \int_a^t s^{n-1} \left| f(v_1(s)) - f(v_2(s)) \right| ds\, dt$$

and we estimate

$$|f(v_1(s)) - f(v_2(s))| \le v_1(s)^{-\alpha} \left| g(v_1(s)) - g(v_2(s)) \right| + g(v_2(s)) \left| v_1(s)^{-\alpha} - v_2(s)^{-\alpha} \right|$$

and

$$\left|v_1(s)^{-\alpha} - v_2(s)^{-\alpha}\right| \le \frac{\alpha}{\delta} \min\{v_1(s), v_2(s)\}^{-\alpha-\delta} |v_1(s)^\delta - v_2(s)^\delta|.$$

Since T maps K into C^1, it is compact, and by the Schauder Fixed Point Theorem, there exists a solution in K.

Remark. By a similar method, there exist solutions with $u(a) = 0$ and $u_r(a) = A$ for any $A > 0$.

Remark. In contrast, for $\alpha \ge 1$, there exists no radial solution u to $\Delta u = f(u)$ on $1 \le r < 1 + \epsilon$ with $u > 0$ on $(1, 1 + \epsilon)$ and $u(1) = 0$. For example in case $n = 2$, we change to the variable $t = \log r$ so that $u_{tt} = e^{2t} f(u)$ on an interval $[0, \epsilon')$ with $u(0) = 0$. Since $u_t > 0$, we have $u_t \le u_t(\epsilon') = C$, so $u \le Ct$. But then $u_{tt} \ge \tilde{C} t^{-\alpha}$, and so $u_t \le C + \tilde{C} \int_{\epsilon'}^t s^{-\alpha}\, ds$. So, for $\alpha \ge 1$, u_t is unbounded below on approach to $t = 0$, a contradiction. A similar argument holds for $n \ge 3$.

We now turn to the asymptotics of radial solutions to $\Delta u = u^{-\alpha}$. We will show that solutions are asymptotic to the solution $U_{\alpha,n}(x) = C_{\alpha,n}|x|^{\frac{2}{1+\alpha}}$, where as before,

$$C_{\alpha,n}^{-1-\alpha} = \frac{2 - 2\alpha}{(1+\alpha)^2} + \frac{2(n-1)}{1+\alpha}.$$

First, by Lemma 3, all solutions are eventually bounded below by $B_1 r^{\frac{2}{1+\alpha}}$ and thus bounded above by $B_2 r^{\frac{2}{1+\alpha}}$ for $r > r_0$. Here the constants B_1, B_2, and r_0 may depend on ϵ for the solutions of Lemma 4 and on a for the solutions of Lemma 5.

We consider the function v where $u = C_{\alpha,n} r^{\frac{2}{1+\alpha}}(1 + v)$. The equation for v is

$$v_{rr} + \frac{\frac{4}{1+\alpha} + n - 1}{r} v_r + \frac{(1+\alpha) C_{\alpha,n}^{-1-\alpha}}{r^2} v = \frac{C_{\alpha,n}^{-1-\alpha}}{r^2} \left[(1+v)^{-\alpha} - 1 + \alpha v\right].$$

In order to make the equation autonomous, we change to the variable $t = \log r$ and we have the equation

$$v_{tt} + \beta v_t + \gamma v = C_{\alpha,n}^{-1-\alpha} \left[(1+v)^{-\alpha} - 1 + \alpha v\right], \tag{6}$$

where

$$\beta = \frac{n + 2 + (n-2)\alpha}{1+\alpha} \qquad \text{and} \qquad \gamma = \frac{2n + 2(n-2)\alpha}{1+\alpha}.$$

Now we need to show that as $t \to \infty$, all solutions v converge to the solution $v = 0$. Note that solutions v now satisfy $0 < B_1 < 1 + v < B_2$. Equation (6) may be written as an autonomous system for $\vec{v} = (v, v_t)$:

$$\begin{bmatrix} v \\ v_t \end{bmatrix}_t = \begin{bmatrix} 0 & 1 \\ -\gamma & -\beta \end{bmatrix} \begin{bmatrix} v \\ v_t \end{bmatrix} + C_{\alpha,n}^{-1-\alpha} \begin{bmatrix} 0 \\ (1+v)^{-\alpha} - 1 + \alpha v \end{bmatrix},$$

which we will write briefly as $\vec{v}_t = A\vec{v} + \vec{h}(\vec{v})$.

Note that $(0, 0)$ is the only critical point of this system among $\vec{v}$ with $1 + v > 0$. For small v, the nonlinear part $\vec{h}(\vec{v})$ is on the order of v^2. The linear part with matrix A has eigenvalues

$$\lambda = \frac{-\beta}{2} \pm \sqrt{\frac{\beta^2}{4} - \gamma}.$$

For $n = 2$, $\frac{\beta^2}{4} - \gamma = \frac{4}{(1+\alpha)^2} - \frac{4}{1+\alpha} < 0$, and the eigenvalues are complex. For $n \ge 7$, the eigenvalues are real, and for intermediate n, the situation depends on α. From

now on we will assume that $n = 2$. In this case the real part of the eigenvalues is $-\frac{2}{1+\alpha}$.

First we will show that for any $\epsilon > 0$, eventually $|\vec{v}(t_0)| < \epsilon$ for some t_0. For notational convenience, we will substitute $x = v, y = v_t$ and write the autonomous system as

$$\begin{aligned} x_t &= P(x,y) = y, \\ y_t &= Q(x,y) = -\gamma x - \beta y + C_{\alpha,2}^{-1-\alpha}((1+x)^{-\alpha} - 1 + \alpha x). \end{aligned}$$

First, there are no closed cycles of this system in the half-plane $1 + x > 0$ since by Green's Theorem, if Γ were a cycle of period T enclosing a region Ω, then

$$\begin{aligned} 0 &= \int_0^T x_t y_t \, dt - y_t x_t \, dt = \int_\Gamma x_t \, dy - y_t \, dx = \int_\Gamma P \, dy - Q \, dx \\ &= \int_\Omega (P_x + Q_y) \, dA = \int_\Omega -\beta \, dA. \end{aligned}$$

Now consider the region D consisting of a rectangle with a small disk about zero removed:

$$D = \{B_1 - 1 < x < B_2 - 1, -R < y < R\} \setminus \left\{x^2 + y^2 < \epsilon^2\right\}.$$

Since there are no critical points or cycles in D, by the Poincaré–Bendixson Theorem, any solution trajectory in D must leave in finite time. By our estimates, no trajectory may leave through either side $x = B_1 - 1$ or $x = B_2 - 1$. On the other hand, on the side $y = R$, we have $y_t = -\gamma x - \beta R + C_{\alpha,2}^{-1-\alpha}((1+x)^{-\alpha} - 1 + \alpha x) < 0$ for R large enough, and similarly $y_t > 0$ along the side $y = -R$ for R large enough. Thus, the trajectory must leave through the circle $x^2 + y^2 = \epsilon^2$ at some time t_0.

Now the solution satisfies

$$\vec{v} = e^{A(t-t_0)}\vec{v}(t_0) + \int_{t_0}^t e^{A(t-s)}\vec{h}(\vec{v}(s)) \, ds$$

and so

$$|\vec{v}| \le e^{-\frac{\beta}{2}(t-t_0)}|\vec{v}(t_0)| + \int_{t_0}^t e^{-\frac{\beta}{2}(t-s)}|\vec{h}(\vec{v}(s))| \, ds.$$

Suppose at some time t_1 after t_0 the trajectory grows to where $|\vec{v}(t_1)| = 2\epsilon$. Then we have

$$2\epsilon \le e^{-\frac{\beta}{2}(t_1-t_0)}\epsilon + \int_{t_0}^{t_1} e^{-\frac{\beta}{2}(t_1-s)} C_0 \epsilon^2 \, ds < \epsilon + C\epsilon^2,$$

a contradiction for small ϵ. Thus, replacing ϵ by $\epsilon/2$, we have

$$e^{\frac{\beta}{2}t}|\vec{v}| \le e^{\frac{\beta}{2}t_0}|\vec{v}(t_0)| + \epsilon \int_{t_0}^t e^{\frac{\beta}{2}s}|\vec{v}(s)| \, ds,$$

and so by Gronwall's Inequality,

$$|\vec{v}| < Ce^{(\epsilon - \beta/2)t}$$

for $t > t_0$.

Now, the homogeneous equation

$$w_{tt} + \beta w_t + \gamma w = 0$$

has solutions $w_1 = e^{\lambda_1 t}$ and $w_2 = e^{\lambda_2 t}$, where λ_i are the eigenvalues of A given above. The Wronskian W of these has order $e^{-\beta}$. By the variation of parameters formula, we may now write the solution v as

$$v = A_1 w_1 + A_2 w_2 + w_1 \int_0^t -h(v)\frac{w_2}{W} + w_2 \int_0^t -h(v)\frac{w_1}{W},$$

where $h(v) = (1+v)^{-\alpha} - 1 + \alpha v$. Since the integrals in this formula are convergent, we may rewrite it as

$$v = \tilde{A}_1 w_1 + \tilde{A}_2 w_2 + O\left(e^{(\epsilon-\beta)t}\right).$$

We have shown the following.

Lemma 6. *Let u be a radial solution to $\Delta u = u^{-\alpha}$ in dimension 2. Then*

$$u - C_{\alpha,2} r^{\frac{2}{1+\alpha}} = A\cos(a\log r) + B\sin(a\log r) + O\left(r^{\epsilon - \frac{2}{1+\alpha}}\right)$$

for any ϵ and suitable a.

The equation $\Delta u = u^{-\alpha}$ has a natural scaling so that if u is any solution and $C > 0$ any constant, then the function $\tilde{u}_C(x) = C^{-\frac{2}{1+\alpha}} u(Cx)$ is also a solution. The radially symmetric singular solution $U_{\alpha,2}$ is invariant under this scaling. The above asymptotics show that any radially symmetric solution u satisfies $|u - U_{\alpha,2}| < k$ for some constant k. Thus, for any constant C, $|\tilde{u}_C - U_{\alpha,2}| < C^{-\frac{2}{1+\alpha}} k$, and so as $C \to \infty$, the scalings of any radially symmetric solution converge uniformly to $U_{\alpha,2}$. So, if we choose all the scalings of a radial solution with $u(1) = u'(1) = 0$, and all scalings of a radial solution with $u(0) = 1, u'(0) = 0$, these form a continuous family of solutions including $U_{\alpha,2}$.

Corollary 2. *There exists a continuous family $W_s, s \in \mathbb{R}$ of radially symmetric solutions to $\Delta u = u^{-\alpha}$, where*

- *$W_0 = U_{\alpha,2}$,*
- *for $s < 0$, W_s is defined on $(-s, \infty)$ with $W_s(-s) = W_s'(-s) = 0$, and*
- *for $s > 0$, W_s is complete with $W_s(0) = t, W_s'(0) = 0$.*

Also, because of the oscillatory asymptotics in Lemma 6, we have

Corollary 3. *The Dirichlet problem*

$$\begin{cases} \Delta u = u^{-\alpha} \text{ on } B_1 \subset \mathbb{R}^2, \\ u = C_{\alpha,2} \text{ on } \partial B_1 \end{cases}$$

has infinitely many distinct positive solutions. The free boundary problem

$$\begin{cases} \Delta u = u^{-\alpha} \text{ on } \Omega \subset B_1 \subset \mathbb{R}^2, \\ u = C_{\alpha,2} \text{ on } \partial B_1, \\ u = |\nabla u| = 0 \text{ on } \partial\Omega \cap B_1 \end{cases}$$

also has infinitely many positive solutions.

Similar statements hold for suitable α in dimensions 3 through 6.

4. Tornados

We are now ready to prove Theorem 1.

Proof of Theorem 1. Suppose $\{u_j\}$ is a tornado sequence of solutions to $\Delta u = f(u)$ as in Definition 1, where $f(u)$ satisfies the hypotheses in (1). Consider the continuous family of functions $\tilde{W}_s(x) = W_s(\sqrt{C_2}x)$, which satisfy $\Delta\tilde{W}_s = C_2\tilde{W}_s^{-\alpha}$, and are thus supersolutions to the equation. For each j, we consider $s_j := \sup\{s : \tilde{W}_s < u_j\}$. Then $-\infty < s_j < \infty$ and $\tilde{W}_{s_j} \leq u_j$ with $\tilde{W}_{s_j}(x_j) = u_j(x_j)$ for some $x_j \in \bar{B}_{r_j}$. Now for very large j, $\tilde{W}_{\inf u_j} < \epsilon$ on B_{r_j}. Thus, $\tilde{W}_{s_j} < \epsilon$ on B_{r_j} and $x_j \in B_{r_j}$. Also, $0 < \tilde{W}_{s_j} \leq u_j$ on $\tilde{\Omega} = \{x \in B_{r_j} : \tilde{W}_{s_j}(x) > 0\}$. But $\Delta(\tilde{W}_{s_j} - u_j) > C_2\left(\tilde{W}_{s_j}^{-\alpha} - u_j^{-\alpha}\right) > 0$. Since $\tilde{W}_{s_j} - u_j$ has an interior zero maximum at x_j, by the Hopf Maximum Principle, $\tilde{W}_{s_j} = u_j$, a contradiction.

Corollary 1 now follows from the result of [8].

Remark. The same proof holds showing there is no tornado sequence of class μ with $\mu(\delta) = |\log\delta|^{-m}$ for any $0 < m < 1$ as in [8].

We can also modify the proof of the tornado theorem to show boundary continuity on the disk.

Theorem 4. *A sequence of positive solutions u_j to $\Delta u = f(u)$ on a disk $B \subset \mathbb{R}^2$ with $|u_j|_{C^1(\partial B)} \leq M$ and $u_j\,|_{\partial B} > \epsilon$ is equicontinuous on $\bar{B}$.*

Proof. Suppose not. Then there is a fixed $\delta > 0$ and a sequence of solutions u_j and points $x_j, y_j \in \bar{B}$ with $u_j(x_j) - u_j(y_j) > \delta$ and $|x_j - y_j| \to 0$. Using Corollary 1 and passing to a subsequence, then $x_j, y_j \to x_0 \in \partial B$. By standard elliptic regularity (for instance Theorem 9.14 in [6]), then there are points z_j with $z_j \to z_0 \in \partial B$ and $u_j(z_j) \to 0$. Now we use the functions $\tilde{W}_s$ as in the above proof to get a contradiction.

5. Singular solutions

The construction of singular solutions to $\Delta u = f(u)$ follows the Leray–Schauder degree scheme used in [7] and [11]. We are able here to get better results than those achieved for the equation $\Delta u = \frac{1}{u}$ in [7].

We will consider the Banach space $\mathcal{B} = C^{2,\mu}(\Omega)$ of functions with Hölder continuous second derivatives on a domain $\Omega \subset \mathbb{R}^2$ with norm $|\cdot|_{2,\mu}$. First, by Schauder estimates, for any $\delta > 0$, there is a number M_δ such that any solution u of $\Delta u = f(u)$ with $\delta \leq u \leq \frac{1}{\delta}$ satisfies $|u|_{2,\mu} < M_\delta$. We set

$$\mathcal{U}_\delta = \{u \in \mathcal{B} : \, u > \delta, \, |u|_{2,\alpha} < M_\delta\}.$$

Given boundary data φ_0 with $\delta < \varphi_0 < \frac{1}{\delta}$, we consider the nonlinear operator $T_0 : \mathcal{U}_\delta \to \mathcal{B}$, defined by $T_0(u) = v$, where v is the solution of

$$\begin{cases} \Delta v = f(u) & \text{on } \Omega, \\ v = \varphi_0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 7. *For each $\Omega \subset \mathbb{R}^2$, there exists $M(\Omega)$ such that if $\varphi_0 > M(\Omega)$, then the Leray–Schauder degree* $\deg(I - T_0, \mathcal{U}_\delta, 0) = 1$ *for all* $0 < \delta < 1/M(\Omega)$.

Proof. We choose an origin $0 \in \Omega$. We again use the family $\tilde{W}_s$ of solutions to $\Delta \tilde{W}_s = C_2 \tilde{W}_s^{-\alpha}$. We choose s_0 such that $\tilde{W}_{s_0} > 1$, and set

$$M(\Omega) = \sup_{s \le s_0, x \in \partial\Omega} \tilde{W}_s(x).$$

Now consider $T_t : \mathcal{U}_\delta \to \mathcal{B}$, for $0 \le t \le 1$, defined by $T_t(u) = v$, where v is the solution to

$$\begin{cases} \Delta v = (1-t) f(u) & \text{on } \Omega, \\ v = \varphi_0 & \text{on } \partial\Omega. \end{cases}$$

Since T_1 is a constant function with image in $\mathcal{U}_\delta$, $\deg(I - T_1, \mathcal{U}_\delta, 0) = 1$. So, the result holds unless there is a fixed point of T_t on $\partial\mathcal{U}_\delta$ for some $0 \le t < 1$. But any such fixed point u satisfies $|u|_{2,\mu} < M_\delta$ by the Schauder estimate, and u is also a subsolution of $\Delta u = C_2 u^{-\alpha}$, and so the continuous family of barrier functions $\tilde{W}_s$ force u to be greater than $\tilde{W}_{s_0}$ and thus greater than δ on Ω. So, there are no fixed points on $\partial\mathcal{U}_\delta$ and the result is proved.

Conversely, it follows from Lemma 3 that there is $\epsilon(\Omega) > 0$ such that if $\varphi_0 < \epsilon$, then $\deg(I - T_0, \mathcal{U}_\delta, 0) = 0$. Now we suppose φ_t is a smooth family of boundary data with $\varphi_0 > M(\Omega)$ and $\varphi_1 < \epsilon(\Omega)$. We consider $T_t : \mathcal{U}_\delta \to \mathcal{B}$ defined by $T_t(u) = v$, the solution of

$$\begin{cases} \Delta v = f(u) & \text{on } \Omega, \\ v = \varphi_t & \text{on } \partial\Omega. \end{cases}$$

By the properties of the degree, there must be an intermediate t_δ and a fixed point u_δ of T_{t_δ} on $\partial\mathcal{U}_\delta$. Thus, u_δ satisfies

$$\begin{cases} \Delta u_\delta = f(u_\delta) & \text{on } \Omega, \\ u = \varphi_{t_\delta} & \text{on } \partial\Omega, \end{cases}$$

with $u_\delta \ge \delta$ and $\min_\Omega u_\delta = \delta$. We choose a sequence $\delta_j \to 0$ and construct corresponding $u_j = u_{\delta_j}$. By Corollary 1, there is a subsequence $u_j \to u_0$ on any compact subset of Ω, where u_0 is a singular solution to $\Delta u = f(u)$. If Ω is a disc B, there is a subsequence $u_j \to u_0$ on $\bar{B}$ with $u_0 = \varphi_{t_0}$ on ∂B. The large variety of possible families φ_t implies a large variety of singular solutions to the equation $\Delta u = f(u)$. This completes the proof of Theorem 3.

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References

[1] Brauner, C.M. and Nicolaenko, B., "On nonlinear eigenvalue problems which extend into free boundaries problems," *Bifurcation and nonlinear eigenvalue problems (Proc., Session, Univ. Paris XIII, Villetaneuse, 1978)*, pp. 61–100, *Lecture Notes in Math., 782*, Springer, Berlin-New York, 1980. MR0572251 (83e:35127)

[2] Caffarelli, Luis and Salsa, Sandro, *A geometric approach to free boundary problems*, Graduate Studies in Mathematics, 68, Amer. Math. Soc., 2005. MR2145284

[3] Dávila, Juan, "Global regularity for a singular equation and local H^1 minimizers of a non-differentiable functional," *Commun. Contemp. Math.*, vol. 6, no. 1, pp. 165–193, 2004. MR2048779 (2005e:35061)

[4] Dávila, Juan and Montenegro, Marcelo, "Positive versus free boundary solutions to a singular elliptic equation," *J. Anal. Math.*, vol. 90, pp. 303–335, 2003. MR2001074 (2005d:35278)

[5] Diaz, J.I., Morel, J.M. and Oswald, L., "An elliptic equation with singular nonlinearity," *Comm. P.D.E.*, vol. 12, pp. 1333–1344, 1987. MR0912208 (89m:35077)

[6] Gilbarg, David and Trudinger, Neil S., *Elliptic Partial Differential Equations of Second Order*, Springer, New York, revised 2nd editon, 2001. MR1814364 (2001k:35004)

[7] Meadows, Alexander, "Stable and singular solutions of the equation $\Delta u = \frac{1}{u}$," *Indiana Univ. Math. J.*, 2005. MR2106341 (2005i:35094)

[8] Meadows, Alexander, "Tornado solutions for semilinear elliptic equations in $\mathbb{R}^2$: regularity," *Proc. Amer. Math. Soc.*, vol. 135, no. 5, pp. 1411–1417, 2007.

[9] Phillips, Daniel, "A minimization problem and the regularity of solutions in the presence of a free boundary," *Indiana Univ. Math. J.*, vol. 32, no. 1, pp. 1–17, 1983. MR0684751 (84e:49012)

[10] Phillips, Daniel, "Hausdorff measure estimates of a free boundary for a minimum problem," *Comm. P.D.E.*, vol. 8, pp. 1409–1454, 1983. MR0714047 (85b:35068)

[11] Simon, Leon, "Some examples of singular minimal hypersurfaces," *in preparation*, 2001.

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