POWERS AND ROOTS OF TOEPLITZ OPERATORS

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Abstract. We study the commutativity of two Toeplitz operators whose symbols are quasihomogeneous functions. We give a relationship between this commutativity and the roots (or powers) of the Toeplitz operators. We use this to characterize Toeplitz operators with symbols in $L^\infty(D)$ which commute with Toeplitz operators whose symbols are of the form $e^{ip\theta}r^m$.

1. Introduction

Let $D$ denote the open unit disk in the complex plane $\mathbb{C}$, and let $dA$ denote normalized Lebesgue area measure. The Bergman space, denoted by $L^2_a$, is the Hilbert space of analytic functions on $D$ that are square integrable with respect to $dA$. It is well known that $L^2_a$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$ and $(\sqrt{n+1}z^n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2_a$. Let $P$ be the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto $L^2_a$. For a function $\phi \in L^\infty(\mathbb{D}, dA)$, the Toeplitz operator with symbol $\phi$ is the operator $T_\phi$ from $L^2_a$ to $L^2_a$ defined by $T_\phi(f) = P(\phi f)$.

If $k_z(w) = \frac{1}{(1-\overline{w}z)^2} = \sum_{j=0}^{\infty} (1+j)w^j\overline{z}^j$ is the Bergman reproducing kernel, then $T_\phi(f)(z) = P(\phi f)(z) = \int_D \phi(w)f(w)k_z(w) \, dA(w)$.

The question to be studied in this paper is: When do two Toeplitz operators $T_\phi$ and $T_\psi$ commute? In 1964, Brown and Halmos [4] solved this problem for the analogously defined Toeplitz operators on the Hardy space. They showed that $T_\phi T_\psi = T_\psi T_\phi$ for some $\phi$ and $\psi \in L^\infty(\mathbb{T})$, where $\mathbb{T}$ is the unit circle of $\mathbb{C}$, if and only if either

(a) $\phi$ and $\psi$ are both analytic, or
(b) $\overline{\phi}$ and $\overline{\psi}$ are both analytic, or
(c) one of the two symbols is a linear function of the other.

We recall that a function in $L^\infty(\mathbb{T})$ is said to be analytic if all of its Fourier coefficients with negative indices are equal to 0.

The same question concerning Toeplitz operators on the Bergman space has a much more complicated answer. There are however some results which resemble those of [4]. In fact, Axler and Čučković proved in [2] that the condition that one of (a), (b) or (c) be true is still necessary and sufficient when the two symbols $\phi$ and $\psi$ are bounded harmonic functions on $\mathbb{D}$. Moreover, with Rao [3], they proved

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that if $\phi$ is a bounded analytic function and if $\psi$ is a bounded symbol such that $T_\phi$ and $T_\psi$ commute, then $\psi$ must be analytic too. When we consider arbitrary symbols, things are different. In [3] Ćučković and Rao used the Mellin transform to study the commutativity of multiplication of two Toeplitz operators $T_\phi$ and $T_\psi$ on the Bergman space and describe those operators which commute with $T_{e^{ip\theta}r^m}$ for $(m, p) \in \mathbb{N} \times \mathbb{N}$. In this paper we use our results from [7] to interpret and extend the results of [3]. We give some solutions in the case where the Toeplitz operators have symbols which are “quasihomogeneous” functions and show that these solutions are related to “$p^{th}$ roots” and powers of the Toeplitz operators.

As in [7] we say that a bounded symbol $f$ is quasihomogeneous of degree $k$ if it is of the form $e^{ik\theta} \phi$ where $\phi$ is a radial function. In this case we say that the Toeplitz operator $T_f$ is quasihomogeneous of degree $k$.

2. Preliminaries

The Mellin transform of a function $\psi \in L^1([0, 1], rdr)$ is defined by

$$\hat{\psi}(z) = \int_0^1 \psi(r)r^{z-1} dr.$$ 

It is easy to see that $\hat{\psi}$ is a bounded holomorphic function on the half-plane $\Pi = \{z : \Re z > 2\}$.

We denote the Mellin convolution of two functions $\phi$ and $\psi$ by $\phi \star_M \psi$ and we define it by the equation

$$(\phi \star_M \psi)(r) = \int_r^1 \phi(t)\psi(t)\frac{dt}{t}.$$ 

It is clear that the Mellin transform converts Mellin convolution into a pointwise product, i.e., that

$$\hat{(\phi \star_M \psi)}(r) = \hat{\phi}(r)\hat{\psi}(r).$$

We shall often use the following classical theorem (see [8, p. 102]).

**Theorem 1.** Suppose that $f$ is a bounded, holomorphic function on $\{z : \Re z > 0\}$ which vanishes at the pairwise distinct points $d_1, d_2, \ldots$, where

i) $\inf\{|d_n|\} > 0$ and

ii) $\sum_{n\geq 1} \Re(\frac{1}{d_n}) = \infty$.

Then $f$ vanishes identically on $\{z : \Re z > 0\}$.

**Remark 2.** We shall often apply this theorem to show that if $\psi \in L^1([0, 1], rdr)$ and if there exist $n_0 \in \mathbb{Z}_+, p \in \mathbb{N}$ such that

$$\hat{\psi}(n_0 + pk) = 0$$

for all $k \in \mathbb{N}$,

then $\hat{\psi}(z) = 0$ for all $z \in \{z : \Re z > 2\}$ and so $\psi = 0$.

3. Powers of Toeplitz operators

The following lemma determines the values of powers of a bounded quasihomogeneous Toeplitz operator evaluated at any element of the orthonormal basis of $L^2_a$.
Lemma 3. Let $n \in \mathbb{N}$, $s \in \mathbb{Z}_+$ and let $\psi$ be a bounded radial function on $\mathbb{D}$. Then, for all $k \in \mathbb{N}$ we have

$$\left(T_{e^{i\theta} \psi}\right)^n(\xi^k)(z) = \left[\prod_{j=0}^{n-1} 2(k+j s+s+1)\hat{\psi}(2k+2js+s+2)\right] z^{k+ns}$$

$$= \frac{\prod_{j=0}^{n-1} r^j(2k+2js+s+2)}{\prod_{j=0}^{n-1} \hat{r}(2k+2js+s+2)} z^{k+ns},$$

where $\mathbb{1}$ denotes the constant function with value one.

Proof. The lemma is a consequence of the following direct calculation. We write

$$T_{e^{i\theta} \psi}(\xi^k)(z) = \int_0^1 \int_0^{2\pi} \psi(r) \sum_{j=0}^{\infty} (j+1)e^{i(k+s-j) \theta} r^j z^{j\frac{1}{\pi}} r dr d\theta$$

and interchange the integral over $[0, 2\pi]$ and the sum to see that

$$T_{e^{i\theta} \psi}(\xi^k)(z) = 2(k+s+1)\hat{\psi}(2k+s+2)z^{k+s}$$

$$= \frac{\hat{\psi}(2k+s+2)}{\hat{r}(2k+2s+2)} z^{k+s}.$$\]

The lemma is proved by applying $T_{e^{i\theta} \psi}$ to $\xi^k$ $n$ times. \qed

We have the following decomposition of $L^2(\mathbb{D}, dA)$ as

$$L^2(\mathbb{D}, dA) = \bigoplus_{k \in \mathbb{Z}} e^{ik\theta} \mathcal{R},$$

where $\mathcal{R}$ is the space of functions on $[0, 1]$ that are square integrable with respect to the measure $r dr$. Thus every function $f \in L^2(\mathbb{D}, dA)$ has the decomposition

$$f(re^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r), \quad f_k \in \mathcal{R}.$$\]

Moreover, if $f \in L^\infty(\mathbb{D}, dA) \subset L^2(\mathbb{D}, dA)$, then for each $r \in [0, 1)$,

$$|f_k(r)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(re^{i\theta}) e^{-ik\theta} d\theta \right| \leq \sup_{z \in \mathbb{D}} |f(z)|, \quad \forall k \in \mathbb{Z},$$

and so the functions $f_k$ are bounded in the disk.

In [7] we proved the following results, which we will use in the proof of our main theorem.

Proposition 4. Let $\phi$ be a nonzero bounded radial function, let $p$ be a positive integer and let $f(re^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r) \in L^\infty(\mathbb{D}, dA)$. Then

a) $T_f$ commutes with $T_{e^{ip} \phi}$ if and only if $T_{e^{ik} f_k}$ commutes with $T_{e^{ip} \phi}$ for all $k \in \mathbb{Z}.$

b) If there exists $k \in \mathbb{Z}_{-}$ and a bounded radial function $f_k$ such that

$$T_{e^{ip} \phi} T_{e^{ik} f_k} = T_{e^{ik} f_k} T_{e^{ip} \phi},$$

then $f_k$ must be equal to zero.

c) If there exists $k \in \mathbb{Z}_{+}$ and a bounded radial function $f_k$ such that

$$T_{e^{ip} \phi} T_{e^{ik} f_k} = T_{e^{ik} f_k} T_{e^{ip} \phi},$$

then $f_k$ is unique up to a constant factor. In particular $f_0$ is a constant.
Thus if $p > 0$, $f(r e^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r)$ and $T_f$ commutes with $T_{e^{i\theta}}$, then each $f_k$ is uniquely determined up to multiplication by a constant and is equal to 0 for $k < 0$.

Next we present two technical but easy results which permit us to prove Propositions \ref{prop:7} and \ref{prop:9} the principal results of this section.

Remark 5. Let $(a_t)_{t \in \mathbb{N}}$ and $(b_t)_{t \in \mathbb{N}}$ be two nonvanishing sequences and $p$ and $s$ two positive integers such that

\begin{equation}
    a_{t+s} b_t = b_{t+p} a_t \quad \text{for all } t \in \mathbb{N}.
\end{equation}

Then if

\begin{equation}
    A_k = \prod_{j=0}^{s-1} a_{k+jp} \quad \text{and} \quad B_k = \prod_{j=0}^{p-1} b_{k+js},
\end{equation}

we have

\begin{equation}
    A_k B_{k+p} = A_{k+p} B_k \quad \text{for all } k \in \mathbb{N}.
\end{equation}

(Just multiply the $p$ equations obtained by taking $l = k, k+s, \ldots, k+(p-1)s$ in (2) together to see that, if (2) is true, then

\begin{equation}
    \frac{B_{k+p}}{B_k} = \frac{a_{k+ps}}{a_k} = \frac{A_{k+p}}{A_k} \quad \text{for all } k \in \mathbb{N}.
\end{equation}

Notation. Let $S$ and $T$ be two functions (resp. two operators). We will say that $S \equiv T$ if there exists a constant $c \neq 0$ such that $S = cT$.

Lemma 6. Let $F$ and $G$ be two nonzero bounded holomorphic functions on the half plane $\Pi = \{ z : \Re z > 2 \}$. If there exists $p \in \mathbb{N}$ such that

\begin{equation}
    F(z)G(z+p) = F(z+p)G(z) \quad \text{for all } z \in \Pi,
\end{equation}

then $F \equiv G$.

Proof. Suppose that (3) is true. Then, if (as above) we multiply the $k$ equations obtained by taking $z_n = z + np$ for $n = 0, \ldots, k-1$, we have

\begin{equation}
    F(z)G(z+kp) = F(z+kp)G(z) \quad \text{for all } k \in \mathbb{N}.
\end{equation}

Now, let $z_0 \in \Pi$ such that $G(z_0) \neq 0$ and let $E = \{ k \in \mathbb{N} : G(z_0+kp) = 0 \}$. If $\sum_{k \in E} \Re \left( \frac{1}{|z_0+kp|} \right) = \infty$, then Theorem \ref{thm:1} implies that $G = 0$. This contradicts the hypothesis of the lemma. Thus $\sum_{k \in E^c} \Re \left( \frac{1}{|z_0+kp|} \right) = \infty$, where $E^c$ is the complement in $\mathbb{N}$ of the set $E$.

Now, equation (4) implies that

\begin{equation}
    \frac{F(z_0+kp)}{G(z_0+kp)} = \frac{F(z_0)}{G(z_0)} \quad \text{for all } k \in E^c.
\end{equation}

So, applying Theorem \ref{thm:1} to the function $F - cG$, where $c = \frac{F(z_0)}{G(z_0)}$, completes the proof. \hfill \Box

Let $p$ and $s$ be two positive integers and $\psi$ a bounded radial function.

If $(T_{e^{i\theta}})^\ast$ is a Toeplitz operator, then it is the unique quasihomogeneous Toeplitz operator of degree $ps$ (see Proposition 3 and Proposition 4 of \cite{7}) which commutes with $T_{e^{i\theta}}$. It is natural to ask whether all nonzero Toeplitz operators which are of quasihomogeneous of degree a multiple of $s$ and which commute with $T_{e^{i\theta}}$ are of this form.
Proposition 7. Let \( p \) and \( s \) be two positive integers and \( \phi \) and \( \psi \) be two nonzero bounded radial functions such that

\[
T_{e^{ip\theta}} T_{e^{is\varphi}} = T_{e^{ip\theta}} T_{e^{is\varphi}}.
\]

Then

\[
(T_{e^{ip\theta}})^s = (T_{e^{is\varphi}})^p.
\]

Proof. For all \( k \in \mathbb{N} \), let

\[
a_k = \frac{\hat{\phi}(2k + p + 2)}{\hat{\psi}(2k + 2s + 2)} \quad \text{and} \quad b_k = \frac{\hat{\psi}(2k + s + 2)}{\hat{\psi}(2k + 2s + 2)},
\]

so that

\[
T_{e^{ip\theta}}(\xi^k)(z) = a_k z^{k+p} \quad \text{and} \quad T_{e^{is\varphi}}(\xi^k)(z) = b_k z^{k+s}.
\]

Then equation (5) shows that \( a_{k+s}b_k = b_{k+p}a_k \) for all \( k \in \mathbb{Z}_+ \), and so Remark (5) implies that

\[
\prod_{j=0}^{p-1} a_{k+jp} \prod_{j=0}^{p-1} b_{k+jp} = \prod_{j=0}^{s-1} a_{k+jp} \prod_{j=0}^{s-1} b_{k+jp}.
\]

Let \( F \) and \( G \) be the two bounded holomorphic functions defined for all \( z \in \mathbb{D} \) by

\[
F(z) = \prod_{j=0}^{p-1} \frac{\hat{\phi}(z + 2jp + p)}{\hat{\psi}(z + 2js + 2s)} \quad \text{and} \quad G(z) = \prod_{j=0}^{p-1} \frac{\hat{\psi}(z + 2jp + p)}{\hat{\psi}(z + 2js + 2s)}.
\]

Then equation (7) is equivalent to

\[
F(2k + 2)G(2k + 2p + 2) = F(2k + 2p + 2)G(2k + 2) \quad \text{for all} \quad k \in \mathbb{Z}_+.
\]

Now, applying Theorem (1) in the form of Remark (2) implies that

\[
F(z)G(z + 2p) = F(z + 2p)G(z) \quad \text{for all} \quad z \in \mathbb{D}.
\]

Finally, using Lemma (1) we obtain that

\[
\prod_{j=0}^{p-1} \frac{\hat{\phi}(z + 2jp + p)}{\hat{\psi}(z + 2js + 2s)} = \prod_{j=0}^{s-1} \frac{\hat{\psi}(z + 2jp + p)}{\hat{\psi}(z + 2js + 2s)} \quad \text{for all} \quad z \in \mathbb{D},
\]

and Lemma (2) completes the proof. \( \square \)

Remark 8. i) We will assume that \( (T_{e^{ip\theta}}) = I \), where \( I \) is the identity operator of \( L^2_a \) onto \( L^2_a \).

ii) If \( p \) and \( s \) are both negative integers and if \( T_{e^{ip\theta}} T_{e^{is\varphi}} = T_{e^{ip\theta}} T_{e^{is\varphi}} \), then by considering the adjoint operators we obtain

\[
T_{e^{-ip\theta}} T_{e^{-is\varphi}} = T_{e^{-ip\theta}} T_{e^{-is\varphi}}
\]

and so Proposition (7) implies that \( (T_{e^{ip\theta}})^s = (T_{e^{ip\theta}})^p \).

Now, by considering once again the adjoint operators we see that

\[
(T_{e^{ip\theta}})^s = (T_{e^{ip\theta}})^p.
\]
Proposition 9. Let $\phi$ and $\psi$ be two nonzero bounded radial functions and $n, p$ and $s$ be positive integers. Then

\[(T_{e^{i\theta} \phi})^n = (T_{e^{i\theta} \psi})^p \implies T_{e^{i\theta} \phi} = (T_{e^{i\theta} \psi})^p.\]

Proof. For all $k \in \mathbb{Z}_+$, let

\[a_k = 2(k + ps + 1)\widehat{\phi}(2k + ps + 2) \quad \text{and} \quad b_k = 2(k + s + 1)\widehat{\psi}(2k + s + 2),\]

so that

\[(T_{e^{i\theta} \phi})^n = (T_{e^{i\theta} \psi})^p \iff \prod_{j=0}^{n-1} a_{k+jps} = \prod_{j=0}^{np-1} b_{k+js} \quad \text{for all } k \in \mathbb{Z}_+\]

and

\[T_{e^{i\theta} \phi} = (T_{e^{i\theta} \psi})^p \iff a_k = \prod_{j=0}^{p-1} b_{k+js} \quad \text{for all } k \in \mathbb{Z}_+.\]

Suppose that

\[(8) \quad \prod_{j=0}^{n-1} a_{k+jps} = \prod_{j=0}^{np-1} b_{k+js} \quad \text{for all } k \in \mathbb{Z}_+.\]

We will prove that

\[(9) \quad a_{knps} = \prod_{j=0}^{p-1} b_{knps+js} \quad \text{for all } k \in \mathbb{Z}_+.\]

We prove (9) by induction on $k$. If we take $k = 0$ in equation (8), then we obtain

\[\prod_{j=0}^{n-1} a_{jps} = \prod_{j=0}^{np-1} b_{js} = \prod_{j=0}^{p-1} b_{js} \prod_{j=0}^{np-1} b_{js} = \prod_{j=0}^{p-1} b_{js} \prod_{j=0}^{np-1} b_{ps+js}.\]

Otherwise

\[\prod_{j=0}^{n-1} a_{jps} = a_0 \prod_{j=1}^{n-1} a_{jps} = a_0 \prod_{j=0}^{n-1} a_{ps+jps},\]

But equation (8) implies that

\[\prod_{j=0}^{n-1} a_{ps+jps} = \prod_{j=0}^{np-1} b_{ps+js},\]

Thus

\[a_0 = \prod_{j=0}^{p-1} b_{js}.\]
Now, assume (9) is true for \( knps \). We show it is true for \((k+1)\text{np}s\). We set \( k \) equal to \( \text{np}s \) in the left-hand side of (8) and obtain
\[
\prod_{j=0}^{n-1} a_{knps+jps} = a_{knps} \prod_{j=0}^{n-2} a_{knps+ps+jps}.
\]
Then
\[
a_{(k+1)\text{np}s} \prod_{j=0}^{n-1} a_{knps+jps} = a_{knps} \prod_{j=0}^{n-1} a_{knps+ps+jps}.
\]
But
\[
\prod_{j=0}^{n-1} a_{knps+ps+jps} = \prod_{j=0}^{n-1} b_{knps+ps+jps} \quad \text{and} \quad \prod_{j=0}^{np-1} b_{knps+ps+jps} = \prod_{j=0}^{p-1} b_{knps+ps+jsp} = \prod_{j=0}^{p-1} b_{(k+1)\text{np}s+ps+jsp}.
\]
Thus (9) is proved for \((\text{knps})_{k \in \mathbb{Z}_+}\). Hence, for all \( k \in \mathbb{Z}_+ \), we have
\[
\tilde{\phi}(2knps + ps + 2) \prod_{j=0}^{p-1} (2knps + 2js + 2s + 2) = \tilde{\phi}(2knps + p + 2) \prod_{j=0}^{p-1} \tilde{\psi}(2knps + 2s + 2)
\]
and, using equation (11) and Remark 2 we complete the proof. \( \square \)

**Remark 10.** In [7] (Proposition 6) we prove that if \( p > 0 \) and \( \phi \) is a nonzero bounded radial function and if there exists a bounded radial function \( \psi \) such that \( T_\psi \) commutes with \( T_{e^{ip\theta}\phi} \), then \( \psi \) must be a constant. Here is another proof of this proposition. In fact, using Proposition 7 we have \( (T_\psi)^p = I \), so Proposition 9 implies that \( T_\psi = I \), and so, that \( \psi = \tilde{1} \) since \( I \) is the Toeplitz operator of symbol 1.

### 4. Main result

Let \( p \) be a positive integer. We start this section with the definition of the \( T^{-p^{th}} \) root of a quasihomogeneous Toeplitz operator of degree \( p \) or \( -p \). This new notion plays an important role in the remainder of the paper.

**Definition 11.** Let \( \phi \) be a nonzero bounded radial function and \( p \) be a positive integer. We say that the Toeplitz operator \( T_{e^{ip\theta}\phi} \) has a \( T^{-p^{th}} \) root \( T_{e^{ip\theta}\psi} \) if and only if there exists a nonzero bounded radial function \( \psi \) such that
\[
T_{e^{ip\theta}\phi} = (T_{e^{ip\theta}\psi})^p.
\]

**Remark 12.**

i) The \( T^{-p^{th}} \) root of a quasihomogeneous Toeplitz operator is unique. In fact, suppose that \( T_{e^{ip\theta}\phi} \) has two \( T^{-p^{th}} \) roots \( T_{e^{ip\theta}\psi} \) and \( T_{e^{ip\theta}\tilde{\psi}} \). Then \( (T_{e^{ip\theta}\psi})^p = (T_{e^{ip\theta}\tilde{\psi}})^p \). Then, by Proposition 4 we have that \( T_{e^{ip\theta}\psi} = T_{e^{ip\theta}\tilde{\psi}} \), which implies that \( \psi = \tilde{\psi} \).

ii) If the quasihomogeneous degree is negative we have an analogous definition of the \( T^{-p^{th}} \) root. Let \( p \) be a positive integer and \( \phi \) be a bounded radial function. Then, we say that \( T_{e^{-ip\theta}\phi} \) has a \( T^{-p^{th}} \) root if there exists a bounded radial function \( \psi \) such that \( T_{e^{-ip\theta}\phi} = (T_{e^{-ip\theta}\psi})^p \). It is easy to see,
by taking adjoints, that $T_{e^{-ip\theta}}\phi$ has a $T$-$p^{th}$ root $T_{e^{-i\theta}}\psi$ if and only if $T_{e^{ip\theta}}\phi$ has a $T$-$p^{th}$ root $T_{e^{i\theta}}\psi$.

**Examples.**

i) $T_{e^{i\theta}\psi}\left(\frac{e^{2\pi i k} + 1}{2}\right)$ is the $T$-$2^{th}$ root of $T_{e^{i\theta}}\psi$.

ii) $T_{e^{i\theta}\psi}\left(\frac{e^{2\pi i k} + 1}{2}\right)$ is the $T$-$2^{th}$ root of $T_{e^{i\theta}}\psi$.

Now, if $T_{e^{i\theta}}\psi$ is the $T$-$p^{th}$ root of $T_{e^{i\theta}}\phi$ and if $(T_{e^{i\theta}}T_{e^{i\theta}})^k$ (for $k \in \mathbb{N}$) is a Toeplitz operator, then $(T_{e^{i\theta}}\phi)^k$ is the unique nonzero quasihomogeneous Toeplitz operator of degree $k$ which can commute with $T_{e^{i\theta}}\phi$. What we prove below is that if $T_{e^{i\theta}}\phi$ has a $T$-$p^{th}$ root $T_{e^{i\theta}}\phi$, then the only nonzero quasihomogeneous Toeplitz operator of degree $s$ which commutes with $T_{e^{i\theta}}\phi$ is an $s^{th}$ power of $T_{e^{i\theta}}\phi$, extending the result (Propositions 4 and 9) of section 3 in this case.

**Theorem 13.** Let $f$ be a nonzero radial bounded function and let $p$ be a positive integer. Assume that $T_{e^{i\theta}}\phi$ has a $T$-$p^{th}$ root $T_{e^{i\theta}}\psi$. Suppose that

$$f(re^{i\theta}) = \sum_{k=-\infty}^{\infty} e^{ik\theta} f_k(r) \in L^\infty(\mathbb{D}, dA)$$

is such that

$$T_f T_{e^{i\theta}}\phi = T_{e^{i\theta}}\phi T_f.$$

Then

i) $f_k = 0$ for $k < 0$.

ii) If $k \geq 0$ and $(T_{e^{i\theta}}\phi)^k$ is a Toeplitz operator, then either $T_{e^{i\theta}}f_k \equiv (T_{e^{i\theta}}\phi)^k$ or $f_k = 0$.

iii) If $k \geq 0$ and $(T_{e^{i\theta}}\phi)^k$ is not a Toeplitz operator, then $f_k = 0$.

**Proof.** Assertion a) of Proposition 4 implies that if equation (10) is true, then

$$T_{e^{ik\theta}} f_k T_{e^{i\theta}}\phi = T_{e^{i\theta}}\phi T_{e^{ik\theta}} f_k$$

for all $k \in \mathbb{Z}$.

Thus i) is a direct consequence of assertion b) of Proposition 4.

Now, to prove ii), let $k$ be a positive integer such that $(T_{e^{i\theta}}\phi)^k$ is a Toeplitz operator. Then $(T_{e^{i\theta}}\phi)^k$ is a quasihomogeneous Toeplitz operator of degree $k$ which commutes with $T_{e^{i\theta}}\phi$. So, if $f_k$ is not identically equal to zero, then $f_k$ is a bounded nonzero radial function such that $T_{e^{ik\theta}} f_k$ commutes with $T_{e^{i\theta}}\phi$. Thus, assertion c) of Proposition 4 implies that $T_{e^{ik\theta}} f_k \equiv (T_{e^{i\theta}}\phi)^k$.

Finally, let $k$ be a positive integer such that $(T_{e^{i\theta}}\phi)^k$ is not a Toeplitz operator and suppose that there exists a nonzero bounded radial function $f_k$ such that $T_{e^{ik\theta}} f_k$ commutes with $T_{e^{i\theta}}\phi$. Then Proposition 4 implies that

$$(T_{e^{ik\theta}} f_k)^p \equiv (T_{e^{i\theta}}\phi)^k.$$  

Thus $(T_{e^{ik\theta}} f_k)^p \equiv (T_{e^{i\theta}}\phi)^{kp}$ and Proposition 9 implies that $T_{e^{ik\theta}} f_k \equiv (T_{e^{i\theta}}\phi)^k$, which contradicts our hypothesis. This proves iii).

Before starting with corollaries, we state an interesting theorem which follows from [5] and give an idea of its proof. In fact we will apply this theorem to see that if $p$ is any positive integer and $m$ is any nonnegative integer, then the Toeplitz operator $T_{e^{ip\theta}e^{im\theta}}$ always has a $T$-$p^{th}$ root.
Theorem 14. Let \( p \geq 1 \) and \( m \geq 0 \) be two integers. For all integers \( s \), such that \( 1 \leq s < p \), there exists a unique bounded radial function \( \psi \) such that
\[
T_{e^{i s} r} T_{e^{i p} t} T_{e^{i s} r} = T_{e^{i p} t} T_{e^{i s} r} \psi .
\]

Proof. (This is a slight variation of the proof found in [5].) If \( m \geq 0, p \geq 1 \) and \( 1 \leq s < p \), we define the radial functions \( f \) and \( g \) by
\[
f(r) = 2p r^{2s} (1 - r^{2p})^{-s} \quad \text{and} \quad g(r) = 2p r^m + p (1 - r^{2p})^{s-1}.
\]
Let \( \psi \) be the radial function defined by
\[
r^s \psi = f * M g.
\]
Čučković and Rao prove, using a long rather technical calculation, that \( \psi \) is bounded. Here, we will show that \( \psi \) satisfies (11). To do this, we need only verify that for \( k \in \mathbb{Z}_+ \)
\[
\frac{2k + 2p + 2}{2k + m + p + 2} \hat{r}^s \psi (2k + 2p + 2) = \frac{2k + 2s + 2}{2k + m + p + 2s + 2} \hat{r}^s \psi (2k + 2).
\]
By (11), we have \( \hat{r}^s \psi (2k + 2) = \hat{f}(2k + 2) \hat{g}(2k + 2) \). A simple substitution \( t = r^{2p} \) shows that
\[
\hat{f}(2k + 2) = B \left( \frac{2k + 2s + 2}{2p} , 1 - \frac{s}{p} \right) \quad \text{and} \quad \hat{g}(2k + 2) = B \left( \frac{2k + m + p + 2}{2p} , \frac{s}{p} \right),
\]
where \( B \) denotes the beta function. Using the well-known identities \( B(z_1, z_2) = \frac{\Gamma(z_1) \Gamma(z_2)}{\Gamma(z_1 + z_2)} \) and \( \Gamma(1 + z) = z \Gamma(z) \), where \( \Gamma \) is the gamma function, it is easy to see that
\[
\hat{r}^s \psi (2k + 2) = \frac{(2k + 2s + 2)(2k + m + p + 2)}{(2k + 2p + 2)(2k + m + p + 2s + 2)} \hat{r}^s \psi (2k + 2),
\]
which finishes the proof. \( \square \)

Remark 15. i) It is trivial that \( T_{e^{i p} t} \) commutes with itself. So, if \( p = s \), assertion c) of Proposition 4 implies that \( \psi = r^m \).

ii) We wish to highlight the following case. If \( m = (2n + 1)p \) for \( n \in \mathbb{N} \), then the function \( \psi \) exists for all \( s \in \mathbb{N} \). In fact, if we substitute \( m = (2n + 1)p \) in (12) and use Theorem 4, we obtain for all \( z \in \Pi \)
\[
\frac{\hat{r}^s \psi (z + 2p)}{\hat{r}^s \psi (z)} = \frac{F(z + 2p)}{F(z)}, \quad \text{where} \quad F(z) = \frac{\Gamma \left( \frac{z + 2p}{2p} \right) \Gamma \left( \frac{z + n}{2p} \right)}{\Gamma \left( \frac{z}{2p} \right) \Gamma \left( \frac{z + 2p + n}{2p} \right)}.
\]
Now, using the identity \( \Gamma(1 + z) = z \Gamma(z) \) repeatedly, we have
\[
F(z) = 2p \prod_{j=0}^{n-1} \left( z + 2jp + 2p \right) \prod_{j=0}^{n} \left( z + 2jp + 2s \right),
\]
which is a proper fraction in \( z \) and can be written as
\[
F(z) = \sum_{j=0}^{n} \frac{a_j}{z + 2jp + 2s},
\]
Since \( \frac{1}{z + 2jp + 2s} = r^{2jp + 2s}(z) \), it follows by Lemma 4 that
\[
\hat{r}^s \psi(z) = \sum_{j=0}^{n} a_j r^{2jp + 2s}(z),
\]
where the $a_j$ are defined by (13), and so Theorem 1 implies that

$$\psi(r) \equiv \sum_{j=0}^{n} a_j r^{2jp+s}.$$ 

Next, we give some easy but interesting consequences of Theorem 14.

**Corollary 16.** For all integers $m \geq 0$, $p \geq 1$, and $s \geq 1$ there exists a bounded radial function $\psi$ such that $(T_{e^{ip\theta}})^p \equiv T_{e^{ip\theta}r^m}$.

**Proof.** Let $m \geq 0$, $p \geq 1$, and $s \geq 1$ be integers. Theorem 14 implies that there exists a bounded radial function $\psi$ such that

$$T_{e^{ip\theta}}T_{e^{ip\theta}r^m} = T_{e^{ip\theta}r^m}T_{e^{ip\theta}}.$$ 

Using Proposition 7 we have $(T_{e^{ip\theta}})^p \equiv (T_{e^{ip\theta}r^m})^s$ and so, an application of Proposition 9 finishes the proof. □

In [4], Brown and Halmos studied multiplicativity of Toeplitz operators on the Hardy space and showed that the product of two Toeplitz operators $T_f$ and $T_g$ is equal to a third Toeplitz operator $T_h$ for some $f, g$ and $h$ in $L^\infty(\mathbb{T})$ if and only if $f$ is conjugate analytic or $g$ is analytic, that is, hardly ever. The question of when the product of two Toeplitz operators on the Bergman space is equal to a third is much more complicated and still open. Most work on this question shows that it is not often true that the product of two Toeplitz operators is a Toeplitz operator (see [1] and [6]). But, below, we show that, for certain nontrivial Toeplitz operators $T_{e^{i\theta} \psi}$, not only is $(T_{e^{i\theta} \psi})^2$ equal to a Toeplitz operator, but there exists a positive integer $k$ such that $(T_{e^{i\theta} \psi})^i$ is a Toeplitz operator for all positive integers $i \leq k$.

**Corollary 17.** Let $m \geq 0$ and $p \geq 1$ be two integers. If $T_{e^{ip\theta}r^m}$ has a $T$-$p$th root $T_{e^{i\theta} \psi}$ then, for all integers $k$ with $1 \leq k \leq p$, the product $(T_{e^{i\theta} \psi})^k$ is a Toeplitz operator.

**Proof.** Let $k$ be an integer such that $1 \leq k \leq p$. By Theorem 14 we know that there exists a bounded radial function $\phi$ such that $T_{e^{ik\theta} \phi}$ commutes with $T_{e^{ip\theta}r^m}$. So, Proposition 7 implies that

$$(T_{e^{ik\theta} \phi})^p \equiv (T_{e^{ip\theta}r^m})^k.$$ 

Thus $(T_{e^{ik\theta} \phi})^p = (T_{e^{i\theta} \psi})^{kp}$ since $T_{e^{i\theta} \psi}$ is the $T$-$p$th root of $T_{e^{ip\theta}r^m}$, and so Proposition 9 finishes the proof. □

It is easily seen that if $f$ is a bounded analytic function on $\mathbb{D}$, then $T_f$ is just a multiplication operator. Thus for any integer $k \geq 1$, it is clear that $(T_f)^k$ is a Toeplitz operator of symbol $f^k$. By taking adjoints, we can see that the powers of a Toeplitz operator with conjugate analytic symbol are also Toeplitz operators. These are the trivial cases. The next corollary says there are nontrivial symbols $f$ such that $(T_f)^k$ is always a Toeplitz operator for all integers $k \geq 1$.

**Corollary 18.** There exist bounded radial functions $\psi$ such that for all integers $k \geq 1$ the product $(T_{e^{i\theta} \psi})^k$ is still a Toeplitz operator.
Proof. Let \( n \geq 0 \) and \( p \geq 1 \) be two integers. By Theorem 13 we know that the Toeplitz operator \( T_{e^{i\theta}r(2n+1)p} \) has a \( T \)-powers root \( T_{e^{i\psi^p}} \), where \( \psi \) is a bounded radial function. Moreover the assertion (ii) of Remark 15 tells us that, for all integers \( k \geq 1 \), there exists a bounded radial function \( \psi_k \) such that \( T_{e^{ik\theta}\psi_k} \) commutes with \( T_{e^{i\theta}r(2n+1)p} \). Thus Proposition 7 implies that \( (T_{e^{ik\theta}\psi_k})^p \equiv (T_{e^{i\theta}\psi})^k \), and again, Proposition 9 finishes the proof. □

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References


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