

ON THE VARIETIES DEFINED BY RIEMANN-MUMFORD'S RELATIONS

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ABSTRACT. In this paper we consider varieties defined by Riemann-Mumford's relations. An irreducible component of these varieties is related to Siegel modular varieties. We prove that in most cases Riemann-Mumford varieties are not irreducible.

1. INTRODUCTION

1.1. Let \mathbb{H}_g be the Siegel upper half space, i.e., the set of complex symmetric matrices τ whose imaginary part is positive definite and let $Sp(2g, \mathbb{R})$ be the real symplectic group. $Sp(2g, \mathbb{R})$ acts transitively on \mathbb{H}_g via

$$\sigma \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$$

where $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is in $Sp(2g, \mathbb{R})$.

Let Γ be a subgroup of finite index of the integral symplectic group and k a half integer; thus a holomorphic function f defined on \mathbb{H}_g is called a modular form of weight k and multiplier χ for Γ if

$$f(\sigma \cdot \tau) = f((A\tau + B)(C\tau + D)^{-1}) = \chi(\sigma) \det(C\tau + D)^k f(\tau)$$

for all $\sigma \in \Gamma$. In the genus 1 case we require also the holomorphicity of f at the cusps. We denote by $[\Gamma, k, \chi]$ the vector space spanned by such forms.

Letting q denote an even positive integer and m an element of $q^{-1}\mathbb{Z}^g/\mathbb{Z}^g$ for some $g \geq 1$, a Thetanullwert is defined by

$$(1) \quad \theta \begin{bmatrix} m \\ 0 \end{bmatrix} (q\tau) = \sum_{p \in \mathbb{Z}^g} \exp(\pi i^t(p+m)q\tau(p+m)).$$

This is one of the simplest examples of modular forms of weight $\frac{1}{2}$ for $\Gamma_g(q, 2q)$ and a suitable multiplier.

These Thetanullwerte induce well-defined maps

$$(2) \quad \Theta_g(q) : \Gamma_g(q, 2q) \backslash \mathbb{H}_g \longrightarrow \mathbb{P}^{q^g-1}$$

that are embeddings for all g if $q \geq 4$ (cf. [6], [2]), and generically injective if $q = 2$ (cf. [8]).

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1.2. Really, we know more; in fact, let $R_g(q)$ be the ring generated by such Thetanullwerte. It is a well-known fact that its integral closure $S_g(q)$ is the ring of modular forms $S(\Gamma(q, 2q), \chi)$, with χ equal to the multiplier relative to the Thetanullwerte; cf. [2] and [6]. The map $\Theta_g(q)$ extends to the boundary of Satake's compactification of $\Gamma_g(q, 2q)\backslash\mathbb{H}_g$, that is, $Proj(S_g(q))$.

In the ring $R_g(q)$ there are some standard relations. They are the quartic Riemann's relations and linear equations $X_{-m} = X_m$ with $m \in q^{-1}\mathbb{Z}/\mathbb{Z}$.

Let $(Q_g(q))$ be the ring defined by the above equations. In this note we proceed to compare the associated projective varieties $Proj(R_g(q))$ and $Proj(Q_g(q))$. The final result of [7], page 202, states that $\Theta_g(q)(Proj(R_g(q)))$ is an irreducible component of $Proj(Q_g(q))$ when $q \geq 6$.

We shall show that, if $q \neq 2^s$, $Proj(Q_g(q))$ is not irreducible and hence it cannot be isomorphic to $\Theta_g(q)(Proj(R_g(q)))$.

In the last section, from a detailed analysis of $Proj(Q_1(6))$, we shall show how to reconstruct the ring of modular forms.

Finally, in this case we shall exhibit an explicit relation in the Thetanullwerte that is not a consequence of Riemann's relations.

2. RIEMANN-MUMFORD'S RELATIONS

2.1. We fix representatives for the characteristics. We choose the entries in the set

$$\mathcal{F}(q) = \left[0, \frac{1}{q}, \dots, \frac{q-1}{q} \right]$$

and we set $1 - m$ for the only characteristic n such that $m + n \equiv 0 \pmod 1$.

In this section we shall consider the projective variety $Proj(Q_g(q))$ defined in \mathbb{P}^{q^g-1} by the equations

$$\begin{aligned} (3) \quad & X_m = X_{1-m}, \\ (4) \quad & \left(\sum_{c \in \mathcal{F}(2)^g} \exp(4\pi i {}^t c' c) X_{a'+d+c} X_{b'+d+c} \right) \left(\sum_{c \in \mathcal{F}(2)^g} \exp(4\pi i {}^t c' c) X_{a+c} X_{b+c} \right) \\ & = \left(\sum_{c \in \mathcal{F}(2)^g} \exp(4\pi i {}^t c' c) X_{a+d+c} X_{b+d+c} \right) \left(\sum_{c \in \mathcal{F}(2)^g} \exp(4\pi i {}^t c' c) X_{a'+c} X_{b'+c} \right) \end{aligned}$$

with $c' \in \mathcal{F}(2)^g$ and $m, a, a', b, b', d \in \mathcal{F}(q)^g$ satisfying $a + b \equiv a' + b' \pmod 1$.

The link between these varieties and $Proj(R_g(q))$ is a consequence of the works of Mumford and Kempf; cf. [6] and [3]. In fact we have

Theorem 1. a) For all $q \geq 3$, $\Theta_g(q)$ is an immersion of $\Gamma_g(q, 2q)\backslash\mathbb{H}_g$ in \mathbb{P}^{q^g-1} .
 b) If q is even and $q \geq 6$, then $Im(\Theta_g(q))$ is a Zariski open subset of $Proj(Q_g(q))$.

A proof of the above theorem can be found in [7]. Really in [7] the first statement is for $q \geq 4$.

We recall that in the case $q = 3$ the injectivity of the map is proved in [4]. Then we proved in [10] the injectivity on the tangent spaces. Moreover the case $q = 2$ has been extensively studied in [8] and [9]. In [9] there are some inaccuracies, so at the moment we can say that the map $\Theta_g(2)$ is generically injective and it is injective when $g \leq 3$. Moreover we have to mention that when q is even the maps $\Theta_g(q)$ extend to the boundary of the Satake compactification.

2.2. According to the above facts, when q is even, $\Theta_g(q)(ProjS_g(q)) = ProjR_g(q)$ is an irreducible reduced component of $Proj(Q_g(q))$.

Clearly we would like to show that equations (3) and (4) define $(ProjR_g(q))$. Unluckily we will get a negative answer.

For example, from this labyrinth of polynomial relations, when $g = 1$ and $q = 6$, identifying X_1 with X_5 and X_2 with X_4 , we obtain 2 relations, namely

$$(5) \quad X_0^2 X_1 X_3 + X_0 X_2 X_3^2 = 2X_1^2 X_2^2$$

and

$$(6) \quad X_0^3 X_2 + X_1 X_3^3 = X_1^4 + X_2^4.$$

(We multiplied the indices by 6 to avoid heavier notation.)

The projective line of equations $X_1 = X_2 = 0$ is contained in $ProjQ_6$, which is not irreducible.

We shall prove that this is a general fact, when $q \equiv 2 \pmod{4}$. In fact, if we set $X_a = 0$ when $a \notin \mathcal{F}(2)^g$, the equations become trivial unless $\{a' + d, b' + d, a, b\}$ or $\{a', b', a + d, b + d\}$ are in $\mathcal{F}(2)^g$.

Each of these configurations implies that $2d \in \mathcal{F}(2)^g$, and, in these cases, we get $d \in \mathcal{F}(2)^g$.

We remark that these are exactly the quartic relations among Thetanullwerte with half integral characteristics and, it is a well-known fact, that these relations do not exist.

To be clearer, it can be easily verified that equation (4) becomes

$$(7) \quad \left(\sum_{c \in \mathcal{F}(2)^g} \exp(4\pi i {}^t c' c) X_{a'+c} X_{b'+c} \right) \left(\sum_{c \in \mathcal{F}(2)^g} \exp(4\pi i {}^t c' c) X_{a+c} X_{b+c} \right) \\ = \left(\sum_{c \in \mathcal{F}(2)^g} \exp(4\pi i {}^t c' c) X_{a+c} X_{b+c} \right) \left(\sum_{c \in \mathcal{F}(2)^g} \exp(4\pi i {}^t c' c) X_{a'+c} X_{b'+c} \right).$$

These are obviously tautological, so we proved the following.

Theorem 2. *Let us assume $q \equiv 2 \pmod{4}$. Then for any g the projective variety $Proj(Q_g(q))$ defined by the equations (3) and (4) has more than an irreducible component. In particular it contains a linear variety of dimension at least $2^g - 1$.*

Really, with some modifications we can prove more; in fact we have

Theorem 3. *Let us assume $q \neq 2^s$ for some positive integer $s > 1$. Then for any g the projective variety $Proj(Q_g(q))$ defined by the equations (3) and (4) has more than an irreducible component.*

Proof. Let us assume $q = 2^s o$, with o an odd number bigger than 1 and $Proj(Q_g(q))$ irreducible.

If we set $X_a = 0$ when $a \notin \mathcal{F}(2^s)^g$, also in these cases the equations (4) become trivial unless $\{a' + d, b' + d, a, b\}$ or $\{a', b', a + d, b + d\}$ are in $\mathcal{F}(2^s)^g$.

Each of these configurations implies that $2d \in \mathcal{F}(2^s)^g$, and we get $d \in \mathcal{F}(2^s)^g$. These are exactly the equations defining $Proj(Q_g(2^s))$.

Thus, if L_q is the linear space defined by $X_a = 0$ when $a \notin \mathcal{F}(2^s)^g$, we have

$$(8) \quad L_q \cap Proj(Q_g(q)) = Proj(Q_g(2^s)).$$

Now if $Proj(Q_g(q))$ is irreducible we know that its dimension is exactly $(1/2)g(g + 1)$ and in any case $Proj(Q_g(2^s))$ has dimension greater than or equal

to $(1/2)g(g + 1)$, since $ProjR_g(q)$ is an irreducible component. Thus we get a contradiction once we prove that the inclusion

$$L_q \cap Proj(Q_g(q)) \subset Proj(Q_g(q))$$

is proper.

Letting $\tau \in \mathbb{H}_g$ be purely imaginary, i.e. $\tau = iy$, we verify that $\Theta_g(q)(iy)$ in \mathbb{P}^{q^g-1} has all entries different from 0. This is an immediate consequence of the definition of Thetanullwerte, since

$$\theta \begin{bmatrix} m \\ 0 \end{bmatrix} (iqy) = \sum_{p \in \mathbb{Z}^g} \exp(-\pi qy[p + m])$$

is the convergent sum of positive terms.

This shows that

$$\Theta_g(q)(iy) \notin L_q.$$

These results are useful for a better understanding of the variety $ProjS_g(q)$ at least for small values of g . This will be discussed in the next section. \square

3. AN EXAMPLE

3.1. Using the results of the last section we will obtain a good description of $ProjS_1(6)$. We know that it has exactly 24 cusps. It is a Riemann surface of genus 13, since it is a Galois covering of degree 12 of $ProjS_1(2) \cong \mathbb{P}^1$ and ramifies only on the 6 cusps of $ProjS_1(2)$.

Let Y_1 and Y_2 be the quartics defined by (5) and (6); thus $ProjQ_1(6) = Y_1 \cap Y_2$ contains 4 lines L_1, L_2, L_3, L_4 of equations

$$\begin{aligned} X_1 = 0, X_2 = 0; \quad X_2 - X_0 = 0, X_3 - X_1 = 0; \\ X_2 - \phi^4 X_0 = 0, X_3 - \phi^8 X_1 = 0; \quad X_2 - \phi^8 X_0 = 0, X_3 - \phi^4 X_1 = 0 \end{aligned}$$

with $\phi = \exp\left(\frac{2\pi i}{12}\right)$. Thus we can write

$$ProjQ_1(6) = L_1 \cup L_2 \cup L_3 \cup L_4 \cup \mathcal{C}.$$

It has exactly 24 singular points. They are

$$\begin{aligned} [1, 0, 0, 1], [1, 0, 0, -1], [1, 0, 0, i], [1, 0, 0, -i], [0, 0, 0, 1], [1, 0, 0, 0], \\ [1, 0, 1, 0], [1, i, 1, i], [1, -i, 1, -i], [1, 1, 1, 1], [1, -1, 1, -1], [0, 1, 0, 1], \\ [1, 0, \phi^4, 0], [1, \phi, \phi^4, -i], [1, \phi^7, \phi^4, i], [1, \phi^{-2}, \phi^4, -1], [1, \phi^4, \phi^4, 1], [0, \phi^4, 0, 1], \\ [1, 0, \phi^8, 0], [1, \phi^{-1}, \phi^8, -i], [1, \phi^5, \phi^8, i], [1, \phi^2, \phi^8, -1], [1, \phi^8, \phi^8, 1], [0, \phi^8, 0, 1]. \end{aligned}$$

The first set of six points is contained in L_1 ; the second, the third and the fourth set in L_2, L_3, L_4 respectively.

Since $\Theta_g(q)$ is $\Gamma_g/\Gamma_g(q, 2q)$ -equivariant and it can be easily verified that the above points have nontrivial stabilizer for the action of $\Gamma_1(2, 4)/\Gamma_1(6, 12)$, they are the image of the cusps and have the same singularity.

Hence to prove that $ProjS_1(6) \cong ProjR_1(6)$ it is enough to check that a singularity is nodal.

This can be easily verified at the point $[1, 0, 0, 0]$. In fact, passing to affine coordinates, we have the following equations:

$$(9) \quad xz + yz^2 = 2x^2y^2$$

and

$$(10) \quad y + xz^3 = x^4 + y^4.$$

Then obtaining y in the second equation and substituting in the first we get that the principal tangent has equation $xz = 0$.

Thus $ProjS_1(6) \cong ProjR_1(6)$ is a curve in \mathbb{P}^3 whose automorphism group has order divisible by

$$|\Gamma_1 / \pm \Gamma_1(6, 12)| = 288.$$

3.2. A priori we cannot say that $ProjS_1(6) \cong \mathcal{C}$; in fact we have not shown that \mathcal{C} is irreducible.

Since \mathcal{C} is smooth, it is enough to show that it is connected.

We observe that the quartic Y_2 is smooth and the effective divisor \mathcal{C} induces a linear system on the quartic that is equivalent to

$$(4H - L_1 - L_2 - L_3 - L_4).$$

Here with H we denote the hyperplane section of \mathbb{P}^3 restricted to Y_2 .

Let us consider the linear system $H - L_i$, $i = 1, \dots, 4$. These induce maps $f_i : Y_2 \rightarrow \mathbb{P}^1$ that describe one-dimensional families of cubic curves in the quartic surface.

Each of these systems is without base points ; in fact this can be easily proved for points of the quartic that are not on the line and for the points on the line we remark that if one is a base point, then it should be a singular point of Y_2 , but this is impossible.

Consequently the linear system $(4H - L_1 - L_2 - L_3 - L_4)$ is without base points and thus it induces a morphism

$$f : Y_2 \rightarrow \mathbb{P}^n.$$

Moreover we have that $dim(f(Y_2)) = 2$; in fact for each of the maps f_i the generic fiber is a cubic and the generic fiber of the map

$$f_1 \times f_2 : Y_2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

is finite, since it is the intersection of two cubics contained in the quartic surface that are not in the same plane.

As a consequence of Zariski's Main Theorem (cf. [1], p.280, Ex.11.3), we get that all divisors in $|4H - L_1 - L_2 - L_3 - L_4|$, and in particular \mathcal{C} , are connected. Hence we get

$$ProjS_1(6) \cong \mathcal{C}.$$

We are grateful to Marco Manetti for suggesting this proof.

Really one could prove more; in fact with some computation it is possible to show that the divisor $(4H - L_1 - L_2 - L_3 - L_4)$ satisfies the condition of a criterion (Nakai-Moishezon) of ampleness.

3.3. Now we shall treat the relations in $R_1(6)$; in particular we look for relations that are not induced by Riemann's relations.

For this reason we analyze the graded rings $Q_1(6)$ and $S_1(6)$.

For the first graded ring we have that its Poincaré series is

$$(11) \quad P(t) = \sum_{k=0}^{\infty} dimQ_1(6)_k t^k = \frac{(1 - t^4)^2}{(1 - t)^4}.$$

Thus we have

$$(12) \quad \dim Q_1(6)_4 = 33, \quad \dim Q_1(6)_6 = 65.$$

Moreover from [5], p. 61, we get

$$(13) \quad P'(s) = \sum_{k=0}^{\infty} \dim S_1(6)_{2k} s^k = \frac{1 + 10s + 13s^2}{(1-s)^2}.$$

Consequently we get

$$\dim S_1(6)_4 = \dim [\Gamma_1(6, 12), 2, id] = 36, \quad \dim S_1(6)_6 = \dim [\Gamma_1(6, 12), 3, id] = 60.$$

Hence we have that the Thetanullwerte satisfy some relations in degree 6, in fact, using the decomposition of these spaces with respect to some characters of $\Gamma_1(2, 4)$.

Theorem 4. *The following relation holds:*

$$(14) \quad \begin{aligned} & 2 \left(\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}^4 (6\tau) - \theta \begin{bmatrix} 2 \\ 0 \end{bmatrix}^4 (6\tau) \right) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (6\tau) \theta \begin{bmatrix} 2 \\ 0 \end{bmatrix} (6\tau) \\ & - \left(\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4 (6\tau) - \theta \begin{bmatrix} 3 \\ 0 \end{bmatrix}^4 (6\tau) \right) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (6\tau) \theta \begin{bmatrix} 3 \\ 0 \end{bmatrix} (6\tau). \end{aligned}$$

This relation is not induced from Riemann's relations.

Proof. To avoid the problem induced by the multiplier we shall consider the modular form

$$(15) \quad \begin{aligned} g(\tau) = & 2 \left(\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}^4 - \theta \begin{bmatrix} 2 \\ 0 \end{bmatrix}^4 \right) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}^2 \theta \begin{bmatrix} 2 \\ 0 \end{bmatrix}^2 \\ & - \left(\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4 - \theta \begin{bmatrix} 3 \\ 0 \end{bmatrix}^4 \right) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 3 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \end{aligned}$$

where $g(\tau)$ is a cusp form, since it vanishes at all 24 cusps of $Proj S_1(6)$. An easy, but rather tedious computation, involving the Fourier coefficients of $g(\tau)$, shows that its vanishing at the cusps is so high that $g(\tau) \equiv 0$.

Elementary computations show that the projective lines L_1, L_2, L_3, L_4 are not contained in the surface Y_3 of degree 6 defined by equation (14); consequently the above relation is not induced from Riemann's relations. \square

We remark that the curve $Y_1 \cap Y_2 \cap Y_3$ is isomorphic to $Proj(S_1(6))$.

However we have numerical evidence that we did not give a complete description of all relations among the Thetanullwerte; however we can find all other relations using the action of $\Gamma_1/\Gamma_1(6, 12)$.

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