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ON THE FINITENESS PROPERTIES OF EXTENSION AND TORSION FUNCTORS OF LOCAL COHOMOLOGY MODULES

KAZEM KHASHYARMANESH

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Abstract. Let R be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$, and M a finitely generated R-module. In this paper, for fixed integers j and n, we study the finiteness of $\mathrm{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ and $Tor^R_j(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ in several cases.

1. Introduction

Throughout this paper, R will denote a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ an ideal of R. Also M will denote a finitely generated R-module. Our terminology follows the textbook [3] on local cohomology.

It is a well-known result that if R is a complete local ring with maximal ideal $\mathfrak{m}$, then the R-module M is Artinian if and only if $\mathrm{Supp}_R(M) \subseteq \{\mathfrak{m}\}$ and $\mathrm{Ext}^i_R(R/\mathfrak{m}, M)$ is finitely generated for all $i \in \mathbb{N}_0$ (cf. [9, Proposition 1.1]). (We use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers.) The following conjecture was made by Grothendieck (see [8, Expose XIII, Conjecture 1.2]).

1.1. For any ideal $\mathfrak{a}$ and any finitely generated R-module, the module

$$\mathrm{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$$

is finitely generated for all $n \geqslant 0$, where $H^n_{\mathfrak{a}}(M)$ is the n-th local cohomology module of M with respect to $\mathfrak{a}$.

Hartshorne has produced in [9] a counterexample which shows that this conjecture is false even when R is regular. Hartshorne asked the following question.

1.2. If $\mathfrak{a}$ is an ideal of R and M is a finitely generated R-module, when are $\mathrm{Ext}^j_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$ finitely generated for all n and j?

Hartshorne defined a module N to be $\mathfrak{a}$-cofinite if the support of N is contained in $V(\mathfrak{a})$ and $\mathrm{Ext}^j_R(R/\mathfrak{a}, N)$ is finitely generated for all $j \in \mathbb{N}_0$. By working in the

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derived category, he also showed that if M is a finitely generated R-module, where R is a complete regular local ring, then $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite in two cases:

(i) $\mathfrak{a}$ is a non-zero principal ideal [9, Corollary 6.3];
(ii) $\mathfrak{a}$ is a prime ideal with dimension one [9, Corollary 7.7].

There are several papers devoted to the extension of Hartshorne's second result to more general situations: We refer the reader to the papers of Huneke and Koh [10], Delfino [4], Delfino and Marley [5], and Yoshida [19].

In view of (1.2), we consider the following questions.

1.3. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R and M is a finitely generated R-module, for fixed integers j and n, when is $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ finitely generated?

1.4. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R and M is a finitely generated R-module, for fixed integers j and n, when is $Tor^R_j(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ finitely generated?

One of the main results in this paper is the following theorem.

Theorem 1.5. *Fix $j \in \mathbb{N}_0, n \in \mathbb{N}$, the ideals $\mathfrak{a}$ and $\mathfrak{b}$ with $\mathfrak{a} \subseteq \mathfrak{b}$, and a finitely generated R-module M of dimension d. Assume that*

(i) $\operatorname{Ext}^{j+t+1}_R(R/\mathfrak{b}, H^{n-t}_{\mathfrak{a}}(M))$ *is finitely generated for $t = 1, \dots, n$, and*
(ii) $\operatorname{Ext}^{j-k-1}_R(R/\mathfrak{b}, H^{n+k}_{\mathfrak{a}}(M))$ *is finitely generated for $k = 1, \dots, d-n$.*

Then $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ is finitely generated.

Also, we show that generalizations of theorems in [6, Theorem 2.1], [7, Theorem A and Theorem B], [1, Theorem 1.2], [13, Proposition 2.5], [2, Theorem 2.2] and [12, Theorem B(β)] are immediate consequences from the above theorem. Note that Proposition 2.5 in [13] and Theorem 1.2 in [1] were proved by using spectral sequences. As we point out in Remark 3.4, the proof of Theorem 2.1 in [6] contains some flaws, but we will show that the statement of Theorem 2.1 in [6] is true (see Corollary 3.5(i)). Moreover, we can establish an analogue of the above results for $Tor^R_j(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$.

2. Exact sequences of local cohomology modules

The concept of a filter regular sequence plays an important role in this paper. We say that a sequence $x_1, \dots, x_n$ of elements of $\mathfrak{a}$ is an $\mathfrak{a}$-filter regular sequence on M, if

$$\operatorname{Supp}_R\left(\frac{(x_1, \dots, x_{i-1})M :_M x_i}{(x_1, \dots, x_{i-1})M}\right) \subseteq V(\mathfrak{a})$$

for all $i = 1, \dots, n$, where $V(\mathfrak{a})$ denotes the set of prime ideals of R containing $\mathfrak{a}$. Also, we say that an element $x \in \mathfrak{a}$ is an $\mathfrak{a}$-filter regular element on M if $\operatorname{Supp}_R(0 :_M x) \subseteq V(\mathfrak{a})$. The concept of an $\mathfrak{a}$-filter regular sequence on M is a generalization of the concept of a filter regular sequence which has been studied in [15], [16] and has led to some interesting results. Both concepts coincide if $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal of a local ring with maximal ideal $\mathfrak{m}$. Note that $x_1, \dots, x_n$ is a weak M-sequence if and only if it is an R-filter regular sequence on M. It is easy to see that the analogue of [16, Appendix 2(ii)] holds true whenever R is Noetherian, M is finitely generated and $\mathfrak{m}$ is replaced by $\mathfrak{a}$, so that, if $x_1, \dots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M, then there is an element $y \in \mathfrak{a}$ such that $x_1, \dots, x_n, y$ is an $\mathfrak{a}$-filter regular sequence on M. Thus, for any positive integer n, there exists an $\mathfrak{a}$-filter regular sequence on M of length n (also see [1, Proposition 2.2]).

Proposition 2.1 (See [11, Proposition 1.2] and [1, Proposition 2.3])**.** *Let $x_1, \dots, x_n$ ($n > 0$) be an $\mathfrak{a}$-filter regular sequence on M. Then there are the following isomorphisms:*

$$H^i_{\mathfrak{a}}(M) \cong \begin{cases} H^i_{(x_1,\dots,x_n)}(M) & for \quad 0 \le i < n, \\ H^{i-n}_{\mathfrak{a}}(H^n_{(x_1,\dots,x_n)}(M)) & for \quad n \le i. \end{cases}$$

The most important technical part is the following result.

Proposition 2.2. *Let M be a finitely generated R-module and let $\mathfrak{a}, \mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. Then*

(i) *for any $\mathfrak{a}$-filter regular element x on M, there exist a long exact sequence*

$$\begin{aligned} 0 &\longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{b}, H^0_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{b}, M) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{b}, H^1_{(x)}(M)) \\ &\longrightarrow \operatorname{Ext}^2_R(R/\mathfrak{b}, H^0_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ext}^2_R(R/\mathfrak{b}, M) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{b}, H^1_{(x)}(M)) \\ &\longrightarrow \cdots \\ &\longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{b}, H^0_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{b}, M) \longrightarrow \operatorname{Ext}^{j-1}_R(R/\mathfrak{b}, H^1_{(x)}(M)) \\ &\longrightarrow \operatorname{Ext}^{j+1}_R(R/\mathfrak{b}, H^0_{\mathfrak{a}}(M)) \longrightarrow \cdots \end{aligned}$$

and the isomorphism

$$\operatorname{Hom}_R(R/\mathfrak{b}, H^0_{\mathfrak{a}}(M)) \cong \operatorname{Hom}_R(R/\mathfrak{b}, H^1_{(x)}(M)), \ \textit{and}$$

(ii) *for any positive integer n and any $\mathfrak{a}$-filter regular sequence $x_1, \dots, x_{n+1} \in \mathfrak{a}$ on M, there exist a long exact sequence*

$$\begin{aligned} 0 &\longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{b}, H^n_I(M)) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{b}, H^{n+1}_J(M)) \\ &\longrightarrow \operatorname{Ext}^2_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ext}^2_R(R/\mathfrak{b}, H^n_I(M)) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{b}, H^{n+1}_J(M)) \\ &\longrightarrow \cdots \\ &\longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_I(M)) \longrightarrow \operatorname{Ext}^{j-1}_R(R/\mathfrak{b}, H^{n+1}_J(M)) \\ &\longrightarrow \operatorname{Ext}^{j+1}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M)) \longrightarrow \cdots \end{aligned}$$

and the isomorphism

$$\operatorname{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M)) \cong \operatorname{Hom}_R(R/\mathfrak{b}, H^n_J(M)),$$

where $I := (x_1, \dots, x_n)$ and $J := (x_1, \dots, x_{n+1})$.

Proof. Let $n \in \mathbb{N}_0$ and $x_1, \dots, x_{n+1} \in \mathfrak{a}$ be an $\mathfrak{a}$-filter regular sequence on M. (Note that the existence of such a sequence is explained in the beginning of this section.) Put $U_0 := M$ and $U_i := H^i_{(x_1,\dots,x_i)}(M)$ for $i = 1, \dots, n+1$. Note that $x_1, \dots, x_i$ is an $(x_1, \dots, x_{i+1})$-filter regular sequence on M. Then, in view of [3, Exercises 1.1.2 and 2.1.4] and Proposition 2.1, we have

$$\begin{aligned} H^0_{(x_{i+1})}(U_i) &\cong H^0_{(x_{i+1})}(H^0_{(x_1,\dots,x_i)}(U_i)) \cong H^0_{(x_1,\dots,x_{i+1})}(U_i) \\ &\cong H^i_{(x_1,\dots,x_{i+1})}(M) \cong H^i_{\mathfrak{a}}(M) \quad \text{and} \\ H^1_{(x_{i+1})}(U_i) &\cong H^1_{(x_1,\dots,x_{i+1})}(U_i) \cong H^{i+1}_{(x_1,\dots,x_{i+1})}(M). \end{aligned}$$

So, by [3, Remark 2.2.17], for each $i = 0, 1, \dots, n$, we obtain the following exact sequence:

$$0 \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow U_i \xrightarrow{f_i} (U_i)_{x_{i+1}} \longrightarrow U_{i+1} \longrightarrow 0.$$

Put $L_i := \mathrm{Im} f_i$ for $i = 0, 1, \ldots, n$. Since the multiplication by x_{i+1} is an automorphism on $(U_i)_{x_{i+1}}$ and $x_{i+1} \in \mathfrak{b}$, it follows from the exact sequence

$$0 \longrightarrow L_i \longrightarrow (U_i)_{x_{i+1}} \longrightarrow U_{i+1} \longrightarrow 0$$

that

$$\mathrm{Hom}_R(R/\mathfrak{b}, L_i) = 0 \qquad (*) \qquad \text{and}$$

$$\mathrm{Ext}_R^j(R/\mathfrak{b}, L_i) \cong \mathrm{Ext}_R^{j-1}(R/\mathfrak{b}, U_{i+1}) \qquad (**)$$

for all $i = 0, 1, \ldots, n$ and $j \in \mathbb{N}$. Hence, for $i = 0, 1, \ldots, n$, by applying the functor $\mathrm{Hom}_R(R/\mathfrak{b}, -)$ on the exact sequence

$$0 \longrightarrow H_{\mathfrak{a}}^i(M) \longrightarrow U_i \longrightarrow L_i \longrightarrow 0,$$

in conjunction with $(*)$ and $(**)$, one can obtain an exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}_R^1(R/\mathfrak{b}, H_{\mathfrak{a}}^i(M)) \longrightarrow \mathrm{Ext}_R^1(R/\mathfrak{b}, U_i) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{b}, U_{i+1}) \\ &\longrightarrow \mathrm{Ext}_R^2(R/\mathfrak{b}, H_{\mathfrak{a}}^i(M)) \longrightarrow \mathrm{Ext}_R^2(R/\mathfrak{b}, U_i) \longrightarrow \mathrm{Ext}_R^1(R/\mathfrak{b}, U_{i+1}) \\ &\longrightarrow \cdots \\ &\longrightarrow \mathrm{Ext}_R^j(R/\mathfrak{b}, H_{\mathfrak{a}}^i(M)) \longrightarrow \mathrm{Ext}_R^j(R/\mathfrak{b}, U_i) \longrightarrow \mathrm{Ext}_R^{j-1}(R/\mathfrak{b}, U_{i+1}) \\ &\longrightarrow \mathrm{Ext}_R^{j+1}(R/\mathfrak{b}, H_{\mathfrak{a}}^i(M)) \longrightarrow \cdots \end{aligned}$$

and the isomorphism

$$\mathrm{Hom}_R(R/\mathfrak{b}, H_{\mathfrak{a}}^i(M)) \cong \mathrm{Hom}_R(R/\mathfrak{b}, U_i).$$

Now, the results follow from the cases $i = n = 0$ and $i = n > 0$. $\square$

By making a straightforward modification to the arguments in the proof of Proposition 2.2, one can obtain the following proposition.

Proposition 2.3. *Let M be a finitely generated R-module and let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. Then*

(i) *for any $\mathfrak{a}$-filter regular element x on M, there exists a long exact sequence*

$$\begin{aligned} R/\mathfrak{b} \otimes_R H_{\mathfrak{a}}^0(M) &\longrightarrow R/\mathfrak{b} \otimes_R M \longrightarrow Tor_1^R(R/\mathfrak{b}, H_{(x)}^1(M)) \longrightarrow 0 \\ Tor_1^R(R/\mathfrak{b}, H_{\mathfrak{a}}^0(M)) &\longrightarrow Tor_1^R(R/\mathfrak{b}, M) \longrightarrow Tor_2^R(R/\mathfrak{b}, H_{(x)}^1(M)) \longrightarrow \\ &\qquad \cdots \longrightarrow \\ Tor_{j-1}^R(R/\mathfrak{b}, H_{\mathfrak{a}}^0(M)) &\longrightarrow Tor_{j-1}^R(R/\mathfrak{b}, M) \longrightarrow Tor_j^R(R/\mathfrak{b}, H_{(x)}^1(M)) \longrightarrow \\ &\qquad \cdots \longrightarrow Tor_{j+1}^R(R/\mathfrak{b}, H_{(x)}^1(M)) \longrightarrow \end{aligned}$$

and $R/\mathfrak{b} \otimes_R H_{(x)}^1(M) = 0$, and

(ii) *for any positive integer n and any $\mathfrak{a}$-filter regular sequence $x_1, \ldots, x_{n+1} \in \mathfrak{a}$ on M, there exists a long exact sequence*

$$\begin{aligned} R/\mathfrak{b} \otimes_R H_{\mathfrak{a}}^n(M) &\longrightarrow R/\mathfrak{b} \otimes_R H_I^n(M) \longrightarrow Tor_1^R(R/\mathfrak{b}, H_J^{n+1}(M)) \longrightarrow 0 \\ Tor_1^R(R/\mathfrak{b}, H_{\mathfrak{a}}^n(M)) &\longrightarrow Tor_1^R(R/\mathfrak{b}, H_I^n(M)) \longrightarrow Tor_2^R(R/\mathfrak{b}, H_J^{n+1}(M)) \longrightarrow \\ &\qquad \cdots \longrightarrow \\ Tor_{j-1}^R(R/\mathfrak{b}, H_{\mathfrak{a}}^n(M)) &\longrightarrow Tor_{j-1}^R(R/\mathfrak{b}, H_I^n(M)) \longrightarrow Tor_j^R(R/\mathfrak{b}, H_J^{n+1}(M)) \longrightarrow \\ &\qquad \cdots \longrightarrow Tor_{j+1}^R(R/\mathfrak{b}, H_J^{n+1}(M)) \longrightarrow \end{aligned}$$

and $R/\mathfrak{b} \otimes_R H_J^{n+1}(M) = 0$, where $I := (x_1, \ldots, x_n)$ and $J := (x_1, \ldots, x_{n+1})$.

3. Finiteness properties of extension functors of local cohomology modules

Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. In this section, we are interested in conditions which ensure that, for fixed integers j and n, $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ is finitely generated. To this end, in the first lemma, for a fixed $j \in \mathbb{N}_0$, we determine conditions on $H^i_{\mathfrak{a}}(M)$ $(i < n)$ which ensure that $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^n_{(x_1,\ldots,x_n)}(M))$ is finitely generated for every $\mathfrak{a}$-filter regular sequence $x_1, \ldots, x_n$ on M. Moreover, in the second lemma, we are interested in seeking conditions on $H^i_{\mathfrak{a}}(M)$ $(i > n)$ which again are sufficient for the finiteness of $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{(x_1,\ldots,x_n)}(M))$, for every $\mathfrak{a}$-filter regular sequence $x_1, \ldots, x_n$ on M. Note that, in general, $H^n_{(x_1,\ldots,x_n)}(M)$ is not finitely generated, unless $H^i_{(x_1,\ldots,x_n)}(M) = 0$ for all $i > 0$ (cf. [19, Proposition 3.1]).

Lemma 3.1. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. Let $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$ be fixed integers such that $\operatorname{Ext}^{j+t+1}_R(R/\mathfrak{b}, H^{n-t}_{\mathfrak{a}}(M))$ is finitely generated for all $t = 1, \ldots, n$. Then, for any $\mathfrak{a}$-filter regular sequence $x_1, \ldots, x_n$ on M, $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{(x_1,\ldots,x_n)}(M))$ is finitely generated.*

Proof. Let $x_1, \ldots, x_n$ be an $\mathfrak{a}$-filter regular sequence on M. We use induction on n, the length of the sequence. When $n = 1$, the result is immediate from Proposition 2.2(i). Assume, inductively, that $n \geqslant 2$, and the result has been proved for positive integers smaller than n. By the inductive hypothesis, $\operatorname{Ext}^{j+1}_R(R/\mathfrak{b}, H^{n-1}_{(x_1,\ldots,x_{n-1})}(M))$ is finitely generated. Also, in view of Proposition 2.2(ii), there exists a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^{j+1}_R(R/\mathfrak{b}, H^{n-1}_{(x_1,\ldots,x_{n-1})}(M)) \longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{(x_1,\ldots,x_n)}(M)) \longrightarrow \operatorname{Ext}^{j+2}_R(R/\mathfrak{b}, H^{n-1}_{\mathfrak{a}}(M)) \longrightarrow \cdots .$$

Now, the result follows from the assumption that $\operatorname{Ext}^{j+2}_R(R/\mathfrak{b}, H^{n-1}_{\mathfrak{a}}(M))$ is finitely generated. $\square$

Lemma 3.2. *Let M be a finitely generated R-module of dimension d. Let $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$ be fixed integers such that $\operatorname{Ext}^{j-k+1}_R(R/\mathfrak{b}, H^{n+k-1}_{\mathfrak{a}}(M))$ is finitely generated for all $k = 1, \ldots, d-n+1$. Then, for any $\mathfrak{a}$-filter regular sequence $x_1, \ldots, x_n$ on M, $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{(x_1,\ldots,x_n)}(M))$ is finitely generated.*

Proof. By the Grothendieck vanishing theorem (cf. [3, Theorem 6.1.2]), we can assume that $n \leqslant d$. Let $x_1, \ldots, x_n$ be an $\mathfrak{a}$-filter regular sequence on M. Then there exists $x_{n+1} \in \mathfrak{a}$ such that $x_1, \ldots, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M. We use (descending) induction on n. When $n = d$, in view of Proposition 2.2(ii) and [3, Theorem 6.1.2], $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^d_{(x_1,\ldots,x_d)}(M))$ is a homomorphic image of $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^d_{\mathfrak{a}}(M))$ and, by our assumption, the latter module is finitely generated. Assume, inductively that $0 \leqslant n < d$ and the result has been proved for integers greater than n. By inductive hypothesis, $\operatorname{Ext}^{j-1}_R(R/\mathfrak{b}, H^{n+1}_{(x_1,\ldots,x_{n+1})}(M))$ is finitely generated. Moreover, in view of Proposition 2.2(ii), there exists an exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{(x_1,\ldots,x_n)}(M)) \longrightarrow \operatorname{Ext}^{j-1}_R(R/\mathfrak{b}, H^{n+1}_{(x_1,\ldots,x_{n+1})}(M)) \longrightarrow \cdots .$$

The result now follows from the assumption that $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ is finitely generated. $\square$

Now we are ready to present one of our main results.

Theorem 3.3. *Fix $j \in \mathbb{N}_0$, $n \in \mathbb{N}$, the ideals $\mathfrak{a}$ and $\mathfrak{b}$ with $\mathfrak{a} \subseteq \mathfrak{b}$, and a finitely generated R-module M of dimension d. Assume that*

(i) $\operatorname{Ext}^{j+t+1}_R(R/\mathfrak{b}, H^{n-t}_{\mathfrak{a}}(M))$ *is finitely generated for all $t = 1, \ldots, n$, and*
(ii) $\operatorname{Ext}^{j-k-1}_R(R/\mathfrak{b}, H^{n+k}_{\mathfrak{a}}(M))$ *is finitely generated for all $k = 1, \ldots, d-n$.*

Then $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ is finitely generated.

Proof. Let $x_1, \ldots, x_{n+1} \in \mathfrak{a}$ be an $\mathfrak{a}$-filter regular sequence on M. By Lemma 3.1, condition (i) implies that $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{(x_1,\ldots,x_n)}(M))$ is finitely generated. Also, in view of Lemma 3.2, our hypothesis in condition (ii) ensures that

$$\operatorname{Ext}^{j-2}_R(R/\mathfrak{b}, H^{n+1}_{(x_1,\ldots,x_{n+1})}(M))$$

is finitely generated. Next, in view of Proposition 2.2(ii), we have the following exact sequence:

$$\begin{aligned}\operatorname{Ext}^{j-2}_R(R/\mathfrak{b}, H^{n+1}_{(x_1,\ldots,x_{n+1})}(M)) &\longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M)) \\ &\longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{(x_1,\ldots,x_n)}(M)).\end{aligned}$$

Both end terms are finitely generated, so $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ is also finitely generated, as required. $\square$

Remark 3.4. Although Theorem 2.1 in [6] is true, the proof of it holds some flaws. In the proof of Theorem 2.1 in [6], the R-module $N = E(M)/M$ is not necessarily finitely generated, where $E(M)$ is the injective hull of M. So, in the inductive hypotheses, we cannot assume that M is finitely generated. Also, the assumption that M is an R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{b}, M)$ is a finitely generated R-module for every $i \leqslant s$, does not assert that $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite (see [14, Corollary 1.8]).

The first part of the following corollary establishes the statement of Theorem 2.1 in [6]. The second part is the main result of [7].

Corollary 3.5 (Compare [7, Theorem A])**.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. Let n be a non-negative integer such that $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^i_{\mathfrak{a}}(M))$ is finitely generated for all $i < n$ and all $j \in \mathbb{N}_0$. Then*

(i) $\operatorname{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ *is finitely generated, and so $\operatorname{Ass}_R(H^n_{\mathfrak{a}}(M)) \cap V(\mathfrak{b})$ is finite, and*
(ii) $\operatorname{Ext}^1_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ *is finitely generated.*

Proof. (i) Apply Theorem 3.3 with $j = 0$ and note that $\operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))) = \operatorname{Ass}_R(H^n_{\mathfrak{a}}(M)) \cap V(\mathfrak{b})$.

(ii) Apply Theorem 3.3 with $j = 1$. $\square$

Note that the first part of Corollary 3.5 is a generalization of the main results of [2] and [12].

Asadollahi and Schenzel, in [1, Theorem 1.2], by using spectral sequences, proved that over a local ring $(R, \mathfrak{m})$ if M is a Cohen-Macaulay R-module and $t = \operatorname{grade}(\mathfrak{a}, M)$, then $\operatorname{Ext}^2_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is finitely generated if and only if $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t+1}_{\mathfrak{a}}(M))$ is finitely generated. The following corollary which is a generalization of [1, Theorem 1.2] is a main result of [7].

Corollary 3.6 (Compare [7, Theorem B])**.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. Let s be a non-negative integer such that $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^i_{\mathfrak{a}}(M))$ is finitely generated for all $i < s$ and all $j \in \mathbb{N}_0$. Then the following statements are equivalent:*

(i) $\operatorname{Hom}_R(R/\mathfrak{b}, H^{s+1}_{\mathfrak{a}}(M))$ *is finitely generated.*
(ii) $\operatorname{Ext}^2_R(R/\mathfrak{b}, H^s_{\mathfrak{a}}(M))$ *is finitely generated.*

Proof. (i)$\Longrightarrow$(ii) Apply Theorem 3.3 with $j = 2$ and $n = s$.
(ii)$\Longrightarrow$(i) Apply Theorem 3.3 with $j = 0$ and $n = s + 1$. $\square$

Delfino and Marley, in [5], proved that, over a local ring R, $H^d_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite, where $d = \dim M$. In the following corollary, for a fixed integer j, we investigate the finiteness of $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^{d-1}_{\mathfrak{a}}(M))$.

Corollary 3.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module of dimension d. Let j be a non-negative integer such that $\operatorname{Ext}^{j+t+1}_R(R/\mathfrak{a}, H^{d-t-1}_{\mathfrak{a}}(M))$ is finitely generated for all $t = 1, \ldots, d-1$. Then $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^{d-1}_{\mathfrak{a}}(M))$ is finitely generated.*

Proof. By [5, Theorem 3], $H^d_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. The result now follows from Theorem 3.3. $\square$

By using Theorem 3.3, in conjunction with [3, Corollary 3.3.3], we have the following corollary.

Corollary 3.8. *Let $\mathfrak{b}$ be an ideal of R. Fix $j \in \mathbb{N}_0$ and $x, y \in \mathfrak{b}$. Then the following statements are equivalent:*

(i) $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^2_{(x,y)}(M))$ *is finitely generated.*
(ii) $\operatorname{Ext}^{j+2}_R(R/\mathfrak{b}, H^1_{(x,y)}(M))$ *is finitely generated.*

Marley and Vassilev, in [13], by using the Grothendieck spectral sequence established the following corollary in the case $\mathfrak{a} = \mathfrak{b}$. Now, Theorem 3.3 provides a slight generalization of [13, Proposition 2.5].

Corollary 3.9. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. Suppose there exists an integer $n \in \mathbb{N}_0$ such that $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^i_{\mathfrak{a}}(M))$ is finitely generated for all $i \neq n$ and all $j \in \mathbb{N}_0$. Then $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ is finitely generated for all j.*

4. Finiteness properties of torsion functors of local cohomology modules

Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. In this section, for fixed integers j and n, we study the finiteness of $Tor^R_j(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$. Using Proposition 2.3 together with straightforward modification to the arguments in the proof of Theorem 3.3, one can obtain the following theorem.

Theorem 4.1. *Fix $j \in \mathbb{N}_0$, $n \in \mathbb{N}$, the ideals $\mathfrak{a}$ and $\mathfrak{b}$ with $\mathfrak{a} \subseteq \mathfrak{b}$, and a finitely generated R-module M of dimension d. Assume that*

(i) $Tor^R_{j-t-1}(R/\mathfrak{b}, H^{n-t}_{\mathfrak{a}}(M))$ *is finitely generated for all $t = 1, \ldots, n$, and*
(ii) $Tor^R_{j+k+1}(R/\mathfrak{b}, H^{n+k}_{\mathfrak{a}}(M))$ *is finitely generated for all $k = 1, \ldots, d-n$.*

Then $Tor^R_j(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ is finitely generated.

Definition 4.2 (See [18])**.** Let $(R, \mathfrak{m})$ be a local ring, M an R-module and $E := E_R(R/\mathfrak{m})$ the injective hull of $R/\mathfrak{m}$. We define a prime ideal $\mathfrak{p}$ of R to be a coassociated prime of M if $\mathfrak{p}$ is an associated prime of $\mathrm{Hom}_R(M, E)$. We denote the set of coassociated primes of M by $\mathrm{Coass}_R M$ (or simply $\mathrm{Coass}M$, if there is no ambiguity about the underlying ring).

Note that $\mathrm{Coass}M = \emptyset$ if and only if $M = 0$. Also, for a finitely generated R-module M and arbitrary R-module N, by [17, Theorem 1.22] and [5, Remark p. 50], we have that $\mathrm{Coass}_R(M \otimes_R N) = \mathrm{Supp}_R M \cap \mathrm{Coass}_R N$. In the following corollary, we provide a dual version of Corollary 3.5 "in some sense".

Corollary 4.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. Let n be a non-negative integer such that $Tor_j^R(R/\mathfrak{b}, H_{\mathfrak{a}}^i(M))$ is finitely generated for all $i > n$ and all $j \in \mathbb{N}_0$. Then*

(i) *$R/\mathfrak{b} \otimes_R H_{\mathfrak{a}}^n(M)$ is finitely generated and so, whenever R is local, the set $V(\mathfrak{b}) \cap \mathrm{Coass}_R(H_{\mathfrak{a}}^n(M))$ is finite, and*
(ii) *$Tor_1^R(R/\mathfrak{b}, H_{\mathfrak{a}}^n(M))$ is finitely generated.*

For an R-module M, the cohomological dimension of M with respect to $\mathfrak{a}$ is defined as

$$\mathrm{cd}(\mathfrak{a}, M) := max\{i \in \mathbb{Z},\ H_{\mathfrak{a}}^i(M) \neq 0\}.$$

Now, it follows from Corollary 4.3 that, for any ideals $\mathfrak{a}$ and $\mathfrak{b}$ with $\mathfrak{a} \subseteq \mathfrak{b}$, the R-modules $R/\mathfrak{b} \otimes_R H_{\mathfrak{a}}^h(M)$ and $Tor_1^R(R/\mathfrak{b}, H_{\mathfrak{a}}^h(M))$ are finitely generated, where h is the cohomological dimension of M with respect to $\mathfrak{a}$.

Corollary 4.4. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. Let s be a non-negative integer such that $Tor_j^R(R/\mathfrak{b}, H_{\mathfrak{a}}^i(M))$ is finitely generated for all $i > s$ and all $j \in \mathbb{N}_0$. Then the following statements are equivalent:*

(i) *$R/\mathfrak{b} \otimes_R H_{\mathfrak{a}}^{s-1}(M)$ is finitely generated.*
(ii) *$Tor_2^R(R/\mathfrak{b}, H_{\mathfrak{a}}^s(M))$ is finitely generated.*

Proof. (i)$\Longrightarrow$(ii) Apply Theorem 4.1 with $j = 2$ and $n = s$.
(ii)$\Longrightarrow$(i) Apply Theorem 4.1 with $j = 0$ and $n = s - 1$. □

The next corollary follows from Theorem 4.1 and [3, Corollary 3.3.3].

Corollary 4.5. *Let $\mathfrak{b}$ be an ideal of R. Fix $j \in \mathbb{N}_0$ and $x, y \in \mathfrak{b}$. Then the following statements are equivalent:*

(i) *$Tor_j^R(R/\mathfrak{b}, H_{(x,y)}^2(M))$ is finitely generated.*
(ii) *$Tor_{j-2}^R(R/\mathfrak{b}, H_{(x,y)}^1(M))$ is finitely generated.*

Corollary 4.6. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. Suppose there exists an integer $n \in \mathbb{N}_0$ such that $Tor_j^R(R/\mathfrak{b}, H_{\mathfrak{a}}^i(M))$ is finitely generated for all $i, j \in \mathbb{N}_0$ with $i \neq n$. Then $Tor_j^R(R/\mathfrak{b}, H_{\mathfrak{a}}^n(M))$ is finitely generated for all j.*

Proof. It is immediate from Theorem 4.1. □

We note an easy consequence of Corollary 4.6.

Corollary 4.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$ and $h \in \mathbb{Z}$. Suppose $Tor_j^R(R/\mathfrak{b}, H_{\mathfrak{a}}^i(M))$ is finitely generated for all $j \in \mathbb{N}_0$ and all $i < h$ and $H_{\mathfrak{a}}^i(M) = 0$ for all $i > h$. Then $Tor_j^R(R/\mathfrak{b}, H_{\mathfrak{a}}^i(M))$ is finitely generated for all i and j. In particular, if $H_{\mathfrak{a}}^i(M) = 0$ for all $i > 1$, then $Tor_j^R(R/\mathfrak{b}, H_{\mathfrak{a}}^i(M))$ is finitely generated for all i and j.*

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Department of Mathematics, Mashhad University, P.O. Box 1159-91775, Mashhad, Iran; Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran

E-mail address: `Khashyar@ipm.ir`