EXPONENTIAL GROWTH OF LIE ALGEBRAS
OF FINITE GLOBAL DIMENSION

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ABSTRACT. Let $L$ be a connected finite type graded Lie algebra. If $\dim L = \infty$ and $\text{gldim} L < \infty$, then $\log \text{index} L = \alpha > 0$. If, moreover, $\alpha < \infty$, then for some $d$, $\sum_{i=1}^{d-1} \dim L_{k+i} = e^{k\alpha_k}$, where $\alpha_k \to \log \text{index} L$ as $k \to \infty$.

We work with graded vector spaces $V$ over a field $k$ of characteristic $\neq 2$ and denote by $V(p,q)$ the subspace $\{V_i | p < i < q\}$. The logarithmic index of any graded vector space $V$ is defined by

$$\log \text{index} V = \limsup_k \frac{\log \dim V_k}{k},$$

and an infinite sequence $(q_i)$ is a quasi-geometric growth sequence for $V$ if for some fixed $n$, $q_i < q_i + 1 \leq nq_i$, for all $i$, and if

$$\frac{\log \dim V_{q_i}}{q_i} \to \log \text{index} V.$$

Now consider graded Lie algebras as defined in [4]; in particular we suppose $[x,[x,x]] = 0$, $x \in L_{\text{odd}}$, if char $k = 3$. $L$ is connected finite type (a cft graded Lie algebra) if $L = \{L_i\}_{i \geq 1}$ and each $L_i$ is finite dimensional. The global dimension ($\text{gldim} L$) and depth of a cft graded Lie algebra, $L$, are defined, respectively, by

$$\text{gldim} L = \max\{ k | \text{Ext}^k_{UL}(k,k) \neq 0 \}$$

and

$$\text{depth} L = \min\{ k | \text{Ext}^k_{UL}(k,UL) \neq 0 \}.$$ 

It is easy to see that depth $L \leq \text{gl dim} L$.

Our main result reads:

**Theorem.** Suppose $L$ is a cft graded Lie algebra and $\dim L = \infty$. If $\text{gldim} L < \infty$, then $\log \text{index} L > 0$. If, moreover, $\log \text{index} L < \infty$, then for some $d$,

$$\sum_{i=1}^{d-1} \dim L_{k+i} = e^{k\alpha_k}, \text{ where } \alpha_k \to \log \text{index} L, \text{ as } k \to \infty.$$
Remark. Theorem 3 of [5] establishes the same conclusion when the hypothesis
\( \text{gldim } L < \infty \) is weakened to depth \( L < \infty \), but certain additional growth conditions
on \( L \) are assumed.

Now suppose \( X \) is a simply connected topological space with each \( H_i(X; \mathbb{Q}) \)
finitely dimensional. Then the loop space homology, \( H_* (\Omega X; \mathbb{Q}) \), is the universal
enveloping algebra of a cft graded Lie algebra \( L_X \), isomorphic to \( \pi_* (\Omega X) \otimes \mathbb{Q} \).

**Corollary.** If \( \dim L_X = \infty \), \( \text{gldim } L_X < \infty \) and \( \log \text{index } L_X < \infty \), then for some \( d \),
\[
\sum_{i=1}^{d-1} \dim \pi_{k+i}(X) \otimes \mathbb{Q} = e^{k \alpha_k},
\]
where \( \alpha_k \to \log \text{index } L_X \) as \( k \to \infty \). In particular \( \sum_{i=1}^{d-1} \dim \pi_{k+i}(X) \otimes \mathbb{Q} \) grows
exponentially in \( k \).

**Proof of the Theorem.** First we establish

**Lemma 1.** An infinite-dimensional cft graded Lie algebra \( L \) of finite global dimension
has a quasi-geometric growth sequence.

**Proof.** We use the same argument as in the proof of Theorem 2 in [5]: Put \( m = \text{gldim } L \), \( a = \left( \frac{1}{2(m+1)} \right)^{m+1} \) and \( \alpha = \log \text{index } L \).

The Cartan-Eilenberg-Serre cochain complex \( C^*(L) \) is in fact a Sullivan algebra
(3) of the form \( \bigwedge (sL)^\# \) \((sL)^\# \) denoting the dual of the suspension of \( L \) and \( \bigwedge V \)
denoting the free graded commutative algebra on \( V \). The differential in \( \bigwedge (sL)^\# \)
increases the length of word gradation by 1 and so gives a second gradation \( H^{(p)} \)
in \( H(\bigwedge (sL)^\#) \). As shown in [1], \( \text{Ext}_{UL}^p (k, k) \cong H^{(p)} \), and so our hypothesis implies
\( H^{(p)} = 0 \), \( p > m \).

Note that for each \( k \), \( C^*(L_{\geq k}) \) is obtained from \( C^*(L) \) by dividing by the ideal
generated by elements in \( (sL)^\# \) of degree \( \leq k \). Since \( \text{gldim } L_{\geq k} \leq m \) it follows
that these quotient cochain complexes also satisfy \( H^{(p)} = 0 \), \( p > m \). The argument
of [2], section 4, can therefore be applied verbatim to \( \bigwedge (sL)^\# \) (with \( \text{cat } \bigwedge \leq m \)
replaced by \( \text{gldim } L \leq m \)) to conclude that \( \alpha > 0 \).

The same argument in the proof of Theorem 2 in [5] now shows that each \( L \)
has a quasi-geometric growth sequence. Let \( n_i \) be an increasing sequence such that
\( \text{dim } L_{n_i} \) converges to \( e^\alpha \). By starting the sequence at some \( n_j \) we may assume
\( \text{dim } L_{n_i} > \frac{1}{a} \) for all \( i \). Thus the formula in ([2], top of page 189) gives a sequence
\( n_i = q_0 < q_1 < \cdots < q_k = n_{i+1} \) such that \( q_{j+1} \leq (m+1)q_j \) and such that
\[
\left( \text{dim } L_{q_j} \right)^{1/(j+1)} \geq (a \text{ dim } L_{n_i})^{1/(i+1)}, \quad j < k.
\]
Hence interpolating the sequences \( n_i \) with the sequences \( q_j \) gives a quasi-geometric
growth sequence \( (r_j) \).

We now revert to the proof of the Theorem. Since \( \text{gldim } L < \infty \) we may choose a
non-zero element \( x \in L \) of even degree \( d \). Let \( N \) be the sub-Lie algebra of elements
of degree \( > d \) that commute with \( x \). Then
\[
\text{gldim } L \geq \text{gldim } (kx + N) = 1 + \text{gldim } N.
\]
If \( \log \text{index } N = \log \text{index } L = \alpha \), then certainly \( \text{dim } N = \infty \), and so \( N \) satisfies
the hypotheses of the Theorem. By induction on global dimension it satisfies the
conclusion. In particular, if $\alpha < \infty$, then for some $d$,

$$\sum_{j=1}^{d-1} \dim N_{k+j} = e^{k\beta_k}$$

with $\beta_k \to \alpha$. Write

$$\sum_{j=1}^{d-1} \dim L_{k+j} = e^{k\alpha_k}.$$ 

Then $\alpha_k \geq \beta_k$ and $\lim sup \alpha_k = \alpha$ because $\alpha = \log \text{index} L$. Thus $\alpha_k \to \alpha$, and the

Theorem holds in this case.

**Lemma 2.** There is a sequence of finitely generated sub-Lie algebras $E(i) \subset L$ such that $\log \text{index } E(i) \to \alpha$.

**Proof.** Otherwise for some $\varepsilon > 0$ we have $\log \text{index} E \leq \alpha - \varepsilon$ for every finitely generated sub-Lie algebra $E \subset L$. Construct an increasing sequence of finitely generated sub-Lie algebras, $F(i)$, and increasing sequences $(k_i)$ and $(\ell_i)$, as follows.

Set $F(0) = 0$, and if $F(i)$ is constructed choose $k_i$ and $\ell_i$ so that

(i) $\dim F(i) < e^{k_i(\alpha - \varepsilon/2)}$, $k_i \geq k$, 
(ii) $\dim L_{\ell_i} \geq e^{\ell_i(\alpha - 1/i)}$, 
(iii) $\ell_i > (m + 1)k_i$.

Then let $F(i + 1)$ be the sub-Lie algebra generated by $F(i)$ and $L_{\ell_i}$.

Now let $F = \bigcup_i F(i)$. Since $\dim F_{\ell_i} \geq e^{\ell_i(\alpha - 1/2)}$ it follows that $\log \text{index } F = \alpha$.

Moreover, because $F \subset L$, $\text{gldim } F \leq m$. Thus by Lemma 1 there is an infinite sequence $q_j$ such that for all $j$, $q_j < q_{j+1} \leq (m + 1)q_j$ and $\dim F_{q_j} \geq e^{q_j(\alpha - \varepsilon/2)}$.

In particular we may choose $i$ and $j$ so that $q_j \leq k_i < q_{j+1}$. But then $q_{j+1} \leq (m + 1)q_j \leq (m + 1)k_i < \ell_i$, and it follows that $F_{q_{j+1}} = F(i)_{q_{j+1}}$. This implies that $\dim F_{q_{j+1}} < e^{q_{j+1}(\alpha - \varepsilon/2)}$, a contradiction. 

Finally, we complete the proof of the theorem. It remains to consider the case $\log \text{index } N < \log \text{index } L$. Let $E(i) \subset L$ be finitely generated sub-Lie algebras such that $\log \text{index } E(i) \to \log \text{index } L$. Moreover, $\text{gldim } E(i) \leq m$ and, according to Lemma 1, each $E(i)$ has a quasi-geometric growth sequence. Since $E(i)$ is finitely generated, Theorem 3 of [5] applies and states that for some $d_i$, $\frac{\log \dim E(i)(k, k + d_i)}{k}$ converges to $\log \text{index } E(i)$.

Fix $\varepsilon > 0$ and choose $i$ so that $\log \text{index } E(i) \geq \alpha - \varepsilon/4$. Then choose $k_0$ so that

$$\frac{\log \dim E(i)(k, k + d_i)}{k} \geq \alpha - \varepsilon/3, \quad k \geq k_0.$$ 

This implies that $k_0$ extends to an infinite sequence $(k_\ell)$ such that $k_\ell < k_{\ell+1} < k_\ell + d_\ell$ and such that

$$\frac{\log \dim L_{k_\ell}}{k_\ell} \geq \alpha - \varepsilon/2, \quad \ell \geq 0.$$ 

On the other hand, since $\log \text{index } N < \log \text{index } L$ we may assume (for $k_0$ sufficiently large and $\varepsilon$ sufficiently small) that

$$\sum_{j \leq d_\ell/j} \dim N_{k_\ell+jd} \leq \frac{1}{2} \dim L_{k_\ell}, \quad \text{for all } \ell.$$
Since $N = (\ker \text{ad} x)_{>d}$ we have
\[
\dim L_{k^1 + pd} \geq \dim L_{k^1} - \sum_{j=0}^{p-1} \dim N_{k^1 + jd} \geq \frac{1}{2} \dim L_{k^1}, \quad p \leq d_i/d.
\]
It follows that for $p \leq d_i/d$ and $k^1$ sufficiently large
\[
\log \frac{\dim L_{k^1 + pd}}{k^1 + pd} \geq \log \left( \frac{k^1}{k^1 + pd} \right) \geq \frac{k^1}{k^1 + pd} \geq \alpha - \varepsilon.
\]
This establishes the Theorem. $\square$

References


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