EXPONENTIAL GROWTH OF LIE ALGEBRAS
OF FINITE GLOBAL DIMENSION

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Abstract. Let $L$ be a connected finite type graded Lie algebra. If $\dim L = \infty$ and $\text{gldim } L < \infty$, then $\log \text{index } L = \alpha > 0$. If, moreover, $\alpha < \infty$, then for some $d$, $\sum_{i=1}^{d-1} \dim L_{k+i} = e^{k\alpha_k}$, where $\alpha_k \to \log \text{index } L$ as $k \to \infty$.

We work with graded vector spaces $V$ over a field $k$ of characteristic $\neq 2$ and denote by $V(p,q)$ the subspace $\{ V_i \mid p < i < q \}$. The logarithmic index of any graded vector space $V$ is defined by

$$\log \text{index } V = \limsup_k \frac{\log \dim V_k}{k},$$

and an infinite sequence $(q_i)$ is a quasi-geometric growth sequence for $V$ if for some fixed $n$, $q_i < q_{i+1} \leq nq_i$, for all $i$, and if

$$\frac{\log \dim V_{q_i}}{q_i} \to \log \text{index } V.$$

Now consider graded Lie algebras as defined in [4]; in particular we suppose $[x, [x, x]] = 0$, $x \in L_{\text{odd}}$, if char $k = 3$. $L$ is connected finite type (a cft graded Lie algebra) if $L = \{ L_i \}_{i \geq 1}$ and each $L_i$ is finite dimensional. The global dimension (gldim $L$) and depth of a cft graded Lie algebra, $L$, are defined, respectively, by

$$\text{gldim } L = \max \{ k \mid \text{Ext}^k_{UL}(k,k) \neq 0 \}$$

and

$$\text{depth } L = \min \{ k \mid \text{Ext}^k_{UL}(k,UL) \neq 0 \}.$$

It is easy to see that depth $L \leq \text{gl dim } L$.

Our main result reads:

Theorem. Suppose $L$ is a cft graded Lie algebra and $\dim L = \infty$. If $\text{gldim } L < \infty$, then $\log \text{index } L > 0$. If, moreover, $\log \text{index } L < \infty$, then for some $d$,

$$\sum_{i=1}^{d-1} \dim L_{k+i} = e^{k\alpha_k}, \text{ where } \alpha_k \to \log \text{index } L, \text{ as } k \to \infty.$$
Remark. Theorem 3 of [5] establishes the same conclusion when the hypothesis 
\[ \text{gldim } L < \infty \] 
is weakened to depth \( L < \infty \), but certain additional growth conditions 
on \( L \) are assumed.

Now suppose \( X \) is a simply connected topological space with each \( H_i(X; \mathbb{Q}) \) 
finite dimensional. Then the loop space homology, \( H_*(\Omega X; \mathbb{Q}) \), is the universal 
enveloping algebra of a cft graded Lie algebra \( L_X \), isomorphic to \( \pi_*(\Omega X) \otimes \mathbb{Q} \).

**Corollary.** If \( \dim L_X = \infty \), \( \text{gldim } L_X < \infty \) and \( \log \text{index } L_X < \infty \), then for some \( d \),

\[ \sum_{i=1}^{d-1} \dim \pi_{k+i}(X) \otimes \mathbb{Q} = e^{k \alpha_k}, \]

where \( \alpha_k \to \log \text{index } L_X \) as \( k \to \infty \). In particular \( \sum_{i=1}^{d-1} \dim \pi_{k+i}(X) \otimes \mathbb{Q} \) grows 
exponentially in \( k \).

**Proof of the Theorem.** First we establish

**Lemma 1.** An infinite-dimensional cft graded Lie algebra \( L \) of finite global dimension 
has a quasi-geometric growth sequence.

**Proof.** We use the same argument as in the proof of Theorem 2 in [5]: Put \( m = \text{gldim } L \), \( a = \left( \frac{1}{2(m+1)} \right)^{m+1} \) and \( \alpha = \log \text{index } L \).

The Cartan-Eilenberg-Serre cochain complex \( C^*(L) \) is in fact a Sullivan algebra 
([3]) of the form \( \bigwedge \langle sL \rangle^\# \), \( (sL)^\# \) denoting the dual of the suspension of \( L \) and \( \bigwedge V \) 
denoting the free graded commutative algebra on \( V \). The differential in \( \bigwedge (sL)^\# \) 
increases the length of word gradation by 1 and so gives a second gradation \( H^p(\bigwedge sL)^\# \) in \( H(\bigwedge (sL)^\#) \). As shown in [1], \( \text{Ext}_{UL}^p(k, k) \cong H^p(\bigwedge sL)^\# \), and so our hypothesis implies \( H^p(\bigwedge sL)^\# = 0, \ p > m \).

Note that for each \( k \), \( C^*(L) \geq k \) is obtained from \( C^*(L) \) by dividing by the ideal 
generated by elements in \( (sL)^\# \) of degree \( \leq k \). Since \( \text{gldim } L \geq k \) it follows 
that these quotient cochain complexes also satisfy \( H^p(\bigwedge sL)^\# = 0, \ p > m \). The argument 
of [2], section 4, can therefore be applied verbatim to \( \bigwedge (sL)^\# \) (with cat \( \bigwedge X \leq m \) replaced by \( \text{gldim } L \leq m \)) to conclude that \( \alpha > 0 \).

The same argument in the proof of Theorem 2 in [5] now shows that each \( L \) 
has a quasi-geometric growth sequence. Let \( n_i \) be an increasing sequence such that 
\( (\dim L_{n_i})^{\frac{1}{n_i}} \) converges to \( e^\alpha \). By starting the sequence at some \( n_j \) we may assume 
\( \dim L_{n_i} > \frac{1}{a} \), for all \( i \). Thus the formula \( (\dim L_{n_i})^{\frac{1}{n_i}} \geq (a \dim L_{n_i})^{\frac{1}{n_i}}, \ j < k \).

Hence interpolating the sequences \( n_i \) with the sequences \( q_j \) gives a quasi-geometric 
growth sequence \( (r_j) \).

We now revert to the proof of the Theorem. Since \( \text{gldim } L < \infty \) we may choose a 
non-zero element \( x \in L \) of even degree \( d \). Let \( N \) be the sub-Lie algebra of elements 
of degree \( > d \) that commute with \( x \). Then

\[ \text{gldim } L \geq \text{gldim } (kx + N) = 1 + \text{gldim } N. \]

If \( \log \text{index } N = \log \text{index } L = \alpha \), then certainly \( \dim N = \infty \), and so \( N \) 
satisfies the hypotheses of the Theorem. By induction on global dimension it satisfies the
conclusion. In particular, if \( \alpha < \infty \), then for some \( d \),
\[
\sum_{j=1}^{d-1} \dim N_{k+j} = e^{k \beta_k}
\]
with \( \beta_k \to \alpha \). Write
\[
\sum_{j=1}^{d-1} \dim L_{k+j} = e^{k \alpha_k}.
\]
Then \( \alpha_k \geq \beta_k \) and \( \lim sup \alpha_k = \alpha \) because \( \alpha = \log \text{dim} L \). Thus \( \alpha_k \to \alpha \), and the theorem holds in this case.

**Lemma 2.** There is a sequence of finitely generated sub-Lie algebras \( E(i) \subset L \) such that \( \log \text{dim} E(i) \to \alpha \).

**Proof.** Otherwise for some \( \varepsilon > 0 \) we have \( \log \text{dim} E \leq \alpha - \varepsilon \) for every finitely generated sub-Lie algebra \( E \subset L \). Construct an increasing sequence of finitely generated sub-Lie algebras, \( F(i) \), and increasing sequences \( (k_i) \) and \( (\ell_i) \), as follows. Set \( F(0) = 0 \), and if \( F(i) \) is constructed choose \( k_i \) and \( \ell_i \) so that
\[
(i) \quad \dim F(i)_{k} < e^{\ell_i(\alpha - \varepsilon/2)} , k \geq k_i ,
(ii) \quad \dim L_{\ell_i} \geq e^{\ell_i(\alpha - 1/i)} ,
(iii) \quad \ell_i > (m + 1)k_i .
\]
Then let \( F(i + 1) \) be the sub-Lie algebra generated by \( F(i) \) and \( L_{\ell_i} \).

Now let \( F = \bigcup_i F(i) \). Since \( \dim F_{\ell_i} \geq e^{\ell_i(\alpha - \frac{1}{4})} \) it follows that \( \log \text{dim} F = \alpha \). Moreover, because \( F \subset L \), \( \text{gldim} F \leq m \). Thus by Lemma 1 there is an infinite sequence \( q_j \) such that for all \( j \), \( q_j < q_{j+1} \leq (m + 1)q_j \) and \( \dim F_{q_j} \geq e^{q_j(\alpha - \varepsilon/2)} \). In particular we may choose \( i \) and \( j \) so that \( q_j \leq k_i < q_{j+1} \). But then \( q_{j+1} \leq (m + 1)q_j \leq (m + 1)k_i < \ell_i \), and it follows that \( F_{q_{j+1}} = F(q_{j+1}) \). This implies that \( \dim F_{q_{j+1}} < e^{q_{j+1}(\alpha - \varepsilon/2)} \), a contradiction.

Finally, we complete the proof of the theorem. It remains to consider the case \( \log \text{dim} N < \log \text{dim} L \). Let \( E(i) \subset L \) be finitely generated sub-Lie algebras such that \( \log \text{dim} E(i) \to \log \text{dim} L \). Moreover, \( \text{gldim} E(i) \leq m \) and, according to Lemma 1, each \( E(i) \) has a quasi-geometric growth sequence. Since \( E(i) \) is finitely generated, Theorem 3 of [5] applies and states that for some \( d_i \), \( \frac{\log \text{dim} E(i)(k,k+d_i)}{k} \) converges to \( \log \text{dim} E(i) \).

Fix \( \varepsilon > 0 \) and choose \( i \) so that \( \log \text{dim} E(i) \geq \alpha - \varepsilon/4 \). Then choose \( k_0 \) so that
\[
\frac{\log \text{dim} E(i)(k,k+d_i)}{k} \geq \alpha - \varepsilon/3 , \quad k \geq k_0 ,
\]
This implies that \( k_0 \) extends to an infinite sequence \( (k_\ell) \) such that \( k_\ell < k_{\ell+1} < k_\ell + d_i \) and such that
\[
\frac{\log \text{dim} L_{k_\ell}}{k_\ell} \geq \alpha - \varepsilon/2 , \quad \ell \geq 0 .
\]
On the other hand, since \( \log \text{dim} N < \log \text{dim} L \) we may assume (for \( k_0 \) sufficiently large and \( \varepsilon \) sufficiently small) that
\[
\sum_{j \leq d_i/k_\ell} \dim N_{k_\ell+jd} \leq \frac{1}{2} \dim L_{k_\ell} , \quad \text{for all } \ell .
\]
Since $N = (\ker \text{ad} x)_{>d}$ we have

$$\dim L_{k\ell + pd} \geq \dim L_{k\ell} - \sum_{j=0}^{p-1} \dim N_{k\ell + jd} \geq \frac{1}{2} \dim L_{k\ell}, \quad p \leq d_i/d.$$  

It follows that for $p \leq d_i/d$ and $k\ell$ sufficiently large

$$\log \dim L_{k\ell + pd} \geq \frac{\log \frac{1}{2} + \log \dim L_{k\ell}}{k\ell} \geq \frac{k\ell}{k\ell + pd} \geq \alpha - \varepsilon.$$  

This establishes the Theorem. \qed

References


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