MIDDLE POINTS, MEDIANS AND INNER PRODUCTS

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Abstract. Let $X$ be a real normed space with unit sphere $S$. Gurari and Sozonov proved that $X$ is an inner product space if and only if, for any $u, v \in S$, 
$$\inf_{t \in [0, 1]} \| tu + (1 - t)v \| = \| \frac{1}{2}u + \frac{1}{2}v \|.$$ 
We prove that it suffices to consider points $u, v \in S$ such that $\inf_{t \in [0, 1]} \| tu + (1 - t)v \| = \frac{1}{2}$.

Making use of the above result we also prove that if dim $X \geq 3$, $X$ is smooth, and $0$ is a Fermat-Torricelli median of any three points $u, v, w \in S$ such that $u + v + w = 0$, then $X$ is an inner product space.

1. Introduction

Let $X$ be a real normed space with unit sphere $S$. Gurari and Sozonov [8] proved that $X$ is an inner product space (i.p.s.) if and only if, for any $u, v \in S$, 
$$\inf_{t \in [0, 1]} \| tu + (1 - t)v \| = \| \frac{1}{2}u + \frac{1}{2}v \|$$ 
(see, e.g., [1], p. 29, where this result is used to establish many characterizations of i.p.s., especially in chapters 12 to 19). We prove in this paper that it suffices to consider pairs of points $u, v \in S$ such that 
$$\inf_{t \in [0, 1]} \| tu + (1 - t)v \| = \frac{1}{2},$$ 
i.e., we prove that $X$ is an i.p.s. if and only if

\begin{equation}
(1) \quad u, v \in S, \inf_{t \in [0, 1]} \| tu + (1 - t)v \| = \frac{1}{2} \Rightarrow u + v \in S.
\end{equation}

In geometrical terms, property (1) states that every chord of $S$ that supports $\frac{1}{2}S$ touches $\frac{1}{2}S$ at its middle point.

As a corollary of the above result, we obtain a new characterization of i.p.s. based on the location of the medians of three points.

By definition, the set $Z_L(u, v, w)$ of the Fermat-Torricelli medians of the points $u, v, w \in X$ from the set $L \subset X$ is formed by the points $z \in L$ such that
$$\| u - z \| + \| v - z \| + \| w - z \| = \inf_{x \in L} (\| u - x \| + \| v - x \| + \| w - x \|).$$

It is well known (see, e.g., [4], p. 274, or [6], p. 98) that if $X$ is either an i.p.s. or a two-dimensional space, then, for every $u, v, w \in X$,
$$Z_X(u, v, w) = Z_{\text{aff}}(u, v, w)(u, v, w) = Z_{\text{co}}(u, v, w)(u, v, w).$$

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where \( \text{aff}(u, v, w) \) and \( \text{co}(u, v, w) \) are the affine and the convex hull, respectively, of the points \( u, v, w \in X \).

However, the formally weaker property

\[
Z_X(u, v, w) \cap \text{co}(u, v, w) \neq \emptyset, \quad \text{for every } u, v, w \in X,
\]

is characteristic of real i.p.s. of dimension \( \geq 3 \) \cite{2}.

Also it is known (see, e.g., \cite{1}, p. 238, or the proof of Theorem 3.2 of this paper) that if \( X \) is an i.p.s., then

\[
(2) \quad u, v, w \in S, \quad u + v + w = 0 \implies 0 \in Z_X(u, v, w).
\]

We prove in the above-mentioned Theorem 3.2 that property (2), when \( X \) is smooth and of dimension \( \geq 3 \), is also characteristic of i.p.s..

2. Preliminary lemmas

It follows from the nature of property (1) and the fact that \( X \) is an i.p.s. when its two-dimensional subspaces are, that it suffices to consider the case in which \( X \) is two-dimensional, i.e., the space \( \mathbb{R}^2 \) endowed with a norm with unit sphere \( S \) and unit ball \( B \).

For given \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) in \( X \) we shall use the following notation:

- \( x \prec y \), when \( x \) precedes \( y \) in the positive orientation of \( X \), i.e.,

\[
x \land y = x_1y_2 - x_2y_1 > 0.
\]

- \( x \perp y \), when \( x \) is orthogonal to \( y \) in the sense of Birkhoff, \cite{3}, \cite{9}, i.e.,

\[
\|x\| = \|x + \lambda y\| \quad (\lambda \in \mathbb{R}),
\]

or, equivalently (see, e.g., \cite{7}),

\[
|x \land y| = \sup\{z \land y : \|z\| = \|x\|\}.
\]

Obviously, \( x \perp y \) means that the straight line \( L = \{x + \lambda y : \lambda \in \mathbb{R}\} \) supports the sphere \( S(0, \|x\|) \) at \( x \), i.e., \( x \in L \cap S(0, \|x\|) \) and \( L \cap \text{int} B(0, \|x\|) = \emptyset \).

**Lemma 2.1.** (i) For any \( u \in S \), there are unique \( u^*, u^{**} \in S \), \( u^{**} \prec u \prec u^* \), such that the segments \([u^{**}, u] \) and \([u, u^*] \) support \( \frac{1}{2}S \).

(ii) The map \( u \in S \to u^* \in S \) is a homeomorphism whose inverse is \( u \in S \to u^{**} \in S \).

(iii) If \( u, v \in S \) are such that \( u \prec v \), then, \( u^* \prec v^* \) and \( u + u^* \prec v + v^* \).

**Proof.** The proof is very intuitive and not difficult. \( \square \)

**Remark 2.2.** \([u, u^*] \) supports \( \frac{1}{2}S \) means that \([u, u^*] \cap \frac{1}{2}S \neq \emptyset \) and, for any \( x \in [u, u^*] \cap \frac{1}{2}S \), \( x \perp u^{**} - u \). In other words, \( x'(u^{**} - u) = 0 \) when \( x' \) is some linear functional which attains its norm at \( x \).

**Lemma 2.3.** If \( X \) fulfills (1), then it is regular (rotund and smooth).

**Proof.** Suppose that \( X \) is non-rotund. Then there exist \( u \in S \) and \( x, y \in \frac{1}{2}S \), \( x \prec y \), such that \([u, u^*] \cap \frac{1}{2}S = [x, y] \).

Property (1) states that \([u, u^*] \) supports \( \frac{1}{2}S \) at \( \frac{1}{2}(u + u^*) \) and, hence, \( x \leq u + u^* \leq y \). Suppose that \( u + u^* \prec y \). It follows from Lemma 2.1 that there exists \( v \in S \) such that \( u \prec v \) and \( u + u^* \prec v + v^* \prec y \). Then the fact that \([v, v^*] \) supports \( \frac{1}{2}S \) at \( \frac{1}{2}(v + v^*) \) leads to the absurdity \([v, v^*] \cap \frac{1}{2}S = [u, u^*] \cap \frac{1}{2}S = [x, y] \).
Suppose now that $X$ is non-smooth. Then there are $u, v \in S$, $u \prec v$, such that 
\[
\frac{1}{2}(u + u^*) = \frac{1}{2}(v + v^*) \text{ is the only point of } [u, u^*] \cap \frac{1}{2}S = [v, v^*] \cap \frac{1}{2}S.
\]
Hence, the segments $[u, v]$ and $[u^*, v^*]$ must be contained in $S$, in contradiction with the first part of this lemma.

\end{proof}

\begin{corollary}
If $X$ fulfills (1) and $u, v \in S$ are such that $u \prec v$ and $u + v \in S$, then $v = u^*$.

\begin{proof}
Let (other cases will be analogous) $u^* \prec v$ and $u \cup u^* \leq u \cup v$. Since 
\[(u + u^*) \cap u^* > 0, u^* \cap v > 0, \text{ and } (u + u^*) \cap v > 0 \text{ (i.e., } u^* \text{ is between } u + u^* \text{ and } v),\]
there exist $0 < t < 1$ and $\rho > 0$ such that 
\[\rho u^* = t(u + u^*) + (1 - t)v.
\]
Then, 
\[\rho u^* = t u^* + (1 - t)u \cap v \geq u \cup u^* \text{ and hence } \rho \geq 1.
\]
Therefore, it follows from $u + u^*, v \in S$ that $\rho = 1$ and the segment $[u + u^*, v]$ is in $S$, in contradiction with the rotundity of $X$.
\end{proof}

\begin{lemma}
Suppose that $X$ fulfills (1). Then, for every $u \in S$,
\begin{enumerate}
\item[(i)] $(u^*)^* = u^{**}$.
\item[(ii)] $u^* + u^{**} = 0$.
\item[(iii)] $u \cap u^* = u^* \cap u^{**} = u^* \cap u$.
\item[(iv)] $u + u^* \perp u^{-}, u^* + u^{**} \perp u^* - u^{**}, u^{**} + u \perp u^{**} - u$.
\end{enumerate}

\begin{proof}
(i) and (ii). It follows from (1) that $-(u + u^*) \in S$, and it is obvious that 
\[u - (u + u^*), u^* - (u + u^*) \in S.
\]
Then these properties are an immediate consequence of the above corollary.

(iii). The proof follows from $u \cap (u + u^* + u^{**}) = u^* \cap (u + u^* + u^{**}) = 0$.

(iv). It suffices to consider Remark 2.2 and the fact that the segment $[u, u^*]$ supports $\frac{1}{2}S$ at $-\frac{1}{2}(u + u^*)$.
\end{proof}

\begin{lemma}
Suppose that $X$ fulfills (1). Then:
\begin{enumerate}
\item[(i)] For any $u \in S$ there exist unique $u^1, u^1 \in S$, $u^1 \prec u \prec u^1$, such that $u^1 \perp u$ and $u \perp u^1$.
\item[(ii)] The map $u \in S \rightarrow u^1 \in S$ is a homeomorphism with inverse $u \in S \rightarrow u^1 \in S$.
\item[(iii)] If $u, v \in S$ are such that $u \prec v$, then $u^1 \prec v^1$.
\end{enumerate}

\begin{proof}
(i). It is easy to see and well known [11] that the uniqueness (for every $u \in S$) of $u^1$ and $u^1$ is equivalent to the rotundity and smoothness of $X$, respectively.

(ii) and (iii). As in Lemma 2.1, the proof is very intuitive and not difficult.
\end{proof}

In all that follows 
\[s : \theta \in [0, 2\pi] \rightarrow s(\theta) \in S\]
will be a “natural map” for $S$, i.e., a map such that $s(\theta) = (s_1(\theta), s_2(\theta))$ is the point of $S$ that makes an angle $\theta$ with a given point $(s_1(0), s_2(0))$ of $S$, measured with the positive orientation of the plane $X$. In other words, if $s(0) = ||(1, 0)||^{-1}(1, 0)$, then
\[s(\theta) = ||(\cos \theta, \sin \theta)||^{-1}(\cos \theta, \sin \theta).
\]

Since $S$ is a convex curve, the above map is continuous and of bounded variation, and, as a consequence of Lemma 2.1, also continuous and of bounded variation are the maps (non-natural, in general)
\[s^* : \theta \in [0, 2\pi] \rightarrow s^*(\theta) \in S, \quad s^{**} : \theta \in [0, 2\pi] \rightarrow s^{**}(\theta) \in S,
\]
where \( s^*(\theta) \) and \( s^{**}(\theta) \) are the unique points of \( S \) such that \( s^{**}(\theta) < s(\theta) < s^*(\theta) \) and the segments \([s^{**}(\theta), s(\theta)], [s(\theta), s^*(\theta)]\) support \( \frac{1}{2}S \).

Moreover, by virtue of Lemma 2.6, the same holds for

\[
  s^\updownarrow : \theta \in [0, 2\pi] \rightarrow s^\updownarrow(\theta) \in S, \quad \perp s : \theta \in [0, 2\pi] \rightarrow \perp s(\theta) \in S,
\]

when \( X \) fulfills (1).

Therefore all the Riemann-Stieltjes integrals that we shall write below make sense. For example, the well-known formula

\[
  (\alpha, \beta) \in \mathcal{A}_S \quad \Rightarrow \quad \int^\alpha_{s(\theta)} \int^\beta_{s(\theta)} \left[ s_1(\theta)ds_2(\theta) - s_2(\theta)ds_1(\theta) \right]
\]

correctly gives the area of the sector of the unit ball \( B \) that is between two points \( s(\alpha) \) and \( s(\beta) \), with \( 0 \leq \alpha < \beta \leq 2\pi \).

**Lemma 2.7.** If \( X \) fulfills (1), then:

(i) For any \( u \in S \), \( A(B_{uu^*}) = A(B_{u^*u^**}) = A(B_{u^**u}) \).

(ii) For any \( u, v \in S \), \( A(B_{uv}) = A(B_{u^*v^*}) \).

(iii) The function \( u \in S \rightarrow u \land u^* \) is constant.

**Proof.** (i). Let

\[
  s : \theta \in [0, 2\pi] \rightarrow s(\theta) \in S
\]

be a natural map for \( S \) such that \( u = s(\alpha) \), and let \( 0 \leq \alpha < \alpha^* \leq 2\pi \) be such that \( s^*(\alpha) = s(\alpha^*) \). Then, \( s^{**}(\alpha) = s^{**}(\alpha^*) \) and

\[
  \begin{align*}
  2s(\alpha) & = s(\alpha) + s^*(\alpha) + [s(\alpha) - s^*(\alpha)], \\
  2s^*(\alpha) & = s(\alpha) + s^*(\alpha) - [s(\alpha) - s^*(\alpha)] = s(\alpha^*) + s^*(\alpha^*) + [s(\alpha^*) - s^*(\alpha^*)], \\
  2s^{**}(\alpha) & = s(\alpha^*) + s^*(\alpha^*) - [s(\alpha^*) - s^*(\alpha^*)].
  \end{align*}
\]

Hence,

\[
  A(B_{uu^*}) - A(B_{u^**u})
\]

\[
  = \frac{1}{8} \int^\alpha_{s(\theta)} \left[ s(\theta) + s^*(\theta) + [s(\theta) - s^*(\theta)] \right] \land d\left\{ s(\theta) + s^*(\theta) + [s(\theta) - s^*(\theta)] \right\}
\]

\[
  - \frac{1}{8} \int^\alpha_{s(\theta)} \left[ s(\theta) + s^*(\theta) - [s(\theta) - s^*(\theta)] \right] \land d\left\{ s(\theta) + s^*(\theta) - [s(\theta) - s^*(\theta)] \right\}
\]

\[
  = \frac{1}{4} \int^\alpha_{s(\theta)} \left\{ [s(\theta) + s^*(\theta)] \land d[s(\theta) - s^*(\theta)] + [s(\theta) - s^*(\theta)] \land d[s(\theta) + s^*(\theta)] \right\}.
\]

Let \( \alpha = \theta_0 < \theta_1 < \ldots < \theta_n = \alpha^* \) be a partition of \([\alpha, \alpha^*]\). Since \( u + u^* \perp u - u^* \), (see Lemma 2.3 iv)), there exist \( \delta_0 \leq \delta_1 \leq \ldots \leq \delta_{n-1} \leq \delta_n \leq \theta_n \) such that, for \( k = 1, \ldots, n \),

\[
  \{s(\delta_k) - s^*(\delta_k)\} \land \{s(\theta_k) + s^*(\theta_k) - [s(\theta_{k-1}) + s^*(\theta_{k-1})]\} = 0
\]

and, hence,

\[
  \int^\alpha_{s(\theta)} [s(\theta) - s^*(\theta)] \land d[s(\theta) + s^*(\theta)] = 0.
\]
Therefore,

\[ A(B_{uv}) - A(B_{u^*v^*}) = \frac{1}{4} \int_{\alpha}^{\beta} [s(\theta) + s^*(\theta)] \wedge [s(\theta) - s^*(\theta)] = \frac{1}{2} \int_{\alpha}^{\beta} d[s(\theta) \wedge s^*(\theta)] \]

where the last equality is justified in Lemma 2.5(iii).

(ii). Suppose (other cases are analogous) that \( u, v \in S, u < v < u^* \). Then it suffices to consider (i),

\[ A(B_{uv}) = A(B_{u^*v^*}) = A(B_{u^*}^*) = \frac{1}{4} A(B) = A(B_{u^*v^*}) = A(B_{v^*v}) = A(B_{v^*}^*), \]

and the obvious fact that

\[ A(B_{uv}) = A(B_{uv}) + A(B_{uv}), \quad A(B_{uv^*}) = A(B_{uv}) + A(B_{u^*v^*}). \]

(iii). We have proved in (ii) that, for any \( 0 \leq \alpha < \beta \leq 2\pi, \)

\[ 0 = A(B_{s(\alpha)s(\beta)}) - A(B_{s^*(\alpha)s^*(\beta)}) \]

\[ = \frac{1}{8} \int_{\alpha}^{\beta} [s(\theta) + s^*(\theta)] \wedge d[s(\theta) + s^*(\theta)] \]

\[ - \frac{1}{8} \int_{\alpha}^{\beta} [s(\theta) + s^*(\theta)] \wedge d[s(\theta) + s^*(\theta)] \]

\[ = \frac{1}{4} \int_{\alpha}^{\beta} [s(\theta) + s^*(\theta)] \wedge d[s(\theta) + s^*(\theta)] = 2s(\alpha) \wedge s^*(\alpha) - 2s(\beta) \wedge s^*(\beta). \]

The same argument as in (i) shows that

\[ \int_{\alpha}^{\beta} [s(\theta) - s^*(\theta)] \wedge d[s(\theta) + s^*(\theta)] = 0, \]

and hence that

\[ 0 = \int_{\alpha}^{\beta} d[[s(\theta) + s^*(\theta)] \wedge [s(\theta) - s^*(\theta)]] = 2s(\alpha) \wedge s^*(\alpha) - 2s(\beta) \wedge s^*(\beta). \]

\[ \square \]

Lemma 2.8. Suppose that \( X \) fulfills (1) and that \( s : [0, 2\pi] \to S \) is a natural map for \( S \). Then:

(i) \( s \) is continuously differentiable, and there is a continuous function \( p : [0, 2\pi] \to \mathbb{R}_+ \) such that \( s'(\theta) = p(\theta)s^\perp(\theta) \).

(ii) For every \( \theta \in [0, 2\pi] \), \( s'(\theta) \wedge s^*(\theta) > 0 \).

(iii) \( s^* \) is continuously differentiable, and there is a continuous function \( q : [0, 2\pi] \to \mathbb{R}_+ \) such that \( s'''(\theta) = q(\theta)s^\perp(\theta) \).

Proof. (i). Since \( X \) fulfills (1) it is smooth, and it is well known that this is equivalent to the continuous differentiability of \( x \in X \to \|x\| \) at \( X \setminus \{0\} \). Then it follows from

\[ s(\theta) = \|(\cos \theta, \sin \theta)\|^{-1}(\cos \theta, \sin \theta) \]

that \( s \) is continuously differentiable.

Also, the fact that

\[ s(\theta) \wedge s^\perp(\theta) = \sup\{s(\lambda) \wedge s^\perp(\theta) : \lambda \in [0, 2\pi]\} \]

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shows that \( s'(\theta) \land s^\perp(\theta) = 0 \). Furthermore, it follows from
\[
\| (\cos \theta, \sin \theta) \|^2 s'_1(\theta) = - \sin \theta \| (\cos \theta, \sin \theta) \| - \cos \theta \| (\cos \theta, \sin \theta) \|',
\]
\[
\| (\cos \theta, \sin \theta) \|^2 s'_2(\theta) = \cos \theta \| (\cos \theta, \sin \theta) \| - \sin \theta \| (\cos \theta, \sin \theta) \|',
\]
that \( s'(\theta) \neq 0 \), and it is obvious that \( s(\theta) \prec s'(\theta) \). I.e., \( s'(\theta) = p(\theta)s^\perp(\theta) \), with \( p(\theta) > 0 \).

Finally, the continuity of \( p \) follows from the continuity of \( s' \) and \( s^\perp \).

(ii). We saw in Lemma 2.7 that, for any \( 0 \leq \alpha < \beta \leq 2\pi \),
\[
\int_{\alpha}^{\beta} [s^*(\theta) - s^{**}(\theta)] \land d[s^*(\theta) + s^{**}(\theta)] = - \int_{\alpha}^{\beta} [2s^*(\theta) + s(\theta)] \land ds(\theta) = 0,
\]
i.e.,
\[
2 \int_{\alpha}^{\beta} s^*(\theta) \land ds(\theta) = - \int_{\alpha}^{\beta} s(\theta) \land ds(\theta) < 0.
\]
Then, since \( s \) is continuously differentiable,
\[
\int_{\alpha}^{\beta} s^*(\theta) \land s'(\theta) d\theta < 0,
\]
as we wished to show.

(iii). It follows from Lemma 2.7(iii) that, for any \( \lambda \neq \theta \),
\[
0 = \frac{s(\lambda) - s(\theta)}{\lambda - \theta} \land s^*(\lambda) + s(\theta) \land \frac{s^*(\lambda) - s^*(\theta)}{\lambda - \theta}.
\]
Hence,
\[
\lim_{\lambda \to \theta} s(\theta) \land \frac{s^*(\lambda) - s^*(\theta)}{\lambda - \theta} = -s'(\theta) \land s^*(\theta) < 0.
\]
Since \( X \) is smooth and \( s^* \) is continuous, if \( (\lambda_n)_{n \in \mathbb{N}} \) and \( (\bar{\lambda}_n)_{n \in \mathbb{N}} \) are convergent to \( \theta \) sequences such that the sequences
\[
\left( \frac{s^*(\lambda_n) - s^*(\theta)}{\lambda_n - \theta} \right)_{n \in \mathbb{N}}, \quad \left( \frac{s^*(\bar{\lambda}_n) - s^*(\theta)}{\lambda_n - \theta} \right)_{n \in \mathbb{N}},
\]
are also convergent, then there exist two positive numbers \( q(\theta) \) and \( \bar{q}(\theta) \) such that
\[
\lim_{n \to \infty} \frac{s^*(\lambda_n) - s^*(\theta)}{\lambda_n - \theta} = q(\theta)s^*(\theta), \quad \lim_{n \to \infty} \frac{s^*(\bar{\lambda}_n) - s^*(\theta)}{\lambda_n - \theta} = \bar{q}(\theta)s^*(\theta),
\]
and it follows from the above result that \( q(\theta) = \bar{q}(\theta) \), i.e., that
\[
\lim_{\lambda \to \theta} \frac{s^*(\lambda) - s^*(\theta)}{\lambda - \theta} = q(\theta)s^*(\theta) = -s'(\theta) \land s^*(\theta).
\]
Finally, the continuity of \( s^{*'} \) and \( q \) follows from the continuity of \( s' \), \( s^* \), and \( s^{*'} \).
\( \square \)

3. Main results

**Theorem 3.1.** \( X \) is an inner product space if and only if
\[
(1) \quad u, v \in S, \quad \inf_{t \in [0,1]} \| tu + (1 - t)v \| = \frac{1}{2} \Rightarrow u + v \in S.
\]
Proof: It is easy to see (and well known) that if \( X \) is an i.p.s., i.e. \( \|x\|^2 = (x|x) \), then it fulfills (1).

It suffices to consider that, for any \( u, v \in S \) the convex function

\[
F(t) = \|(1 - t)u + tv\|^2 = 1 - 2t + 2t^2 + 2t(1 - t)(u|v)
\]

is such that

\[
F'(t) = 2(1 - 2t)[(u|v) - 1],
\]

and hence, when \((u|v) < 1 (i.e. when u and v are linearly independent)\), \( F \) attains its minimum at \( t = \frac{1}{2} \).

By virtue of the above lemmas, to prove the converse we can take \( X \) endowed with a norm, and we can denote by \( u \) and \( u^* \) the two points \( u \) and \( v \) of hypothesis (1).

Let \( s : [0, 2\pi] \to S \) be a natural map for \( S \). It follows from Lemma 2.7(ii) that, for any \( \alpha \in [0, 2\pi] \),

\[
\int_0^\alpha s(\theta) \wedge ds(\theta) = \int_0^\alpha s^*(\theta) \wedge ds^*(\theta),
\]

from Lemma 2.5(ii) and (iv) that

\[
s(\theta) \perp s(\theta) + 2s^*(\theta), \quad s^*(\theta) \perp -2s(\theta) - s^*(\theta),
\]

and from Lemma 2.8 that \( s \) and \( s^* \) are continuously differentiable and such that

\[
s'(\theta) = p(\theta)s^\perp(\theta), \quad s^*(\theta) = q(\theta)s^\perp(\theta),
\]

where \( p \) and \( q \) are positive and continuous functions.

Then it follows from the uniqueness of \( s^\perp(\theta) \) that there exist two continuous functions \( k : [0, 2\pi] \to \mathbb{R}_+ \) and \( l : [0, 2\pi] \to \mathbb{R}_+ \) such that

\[
s'(\theta) = k(\theta)[s(\theta) + 2s^*(\theta)], \quad s^*(\theta) = l(\theta)[-2s(\theta) - s^*(\theta)],
\]

and the first equality between the Riemann-Stieltjes integrals can be reduced to the following equality between ordinary Riemann integrals:

\[
\int_0^\alpha k(\theta)s(\theta) \wedge [s(\theta) + 2s^*(\theta)]d\theta = \int_0^\alpha l(\theta)s^*(\theta) \wedge [-2s(\theta) - s^*(\theta)]d\theta,
\]

i.e. (see Lemma 2.7(iii)),

\[
\int_0^\alpha k(\theta)d\theta = \int_0^\alpha l(\theta)d\theta \quad (\alpha \in [0, 2\pi]),
\]

from which it follows (\( k \) and \( l \) are continuous) that \( k = l \).

Hence, we have the following system of differential equations:

\[
\begin{align*}
    s_1'(\theta) &= k(\theta)[s_1(\theta) + 2s_1^*(\theta)], \\
    s_2'(\theta) &= k(\theta)[s_2(\theta) + 2s_2^*(\theta)], \\
    s_1^*(\theta) &= -k(\theta)[2s_1(\theta) + s_1^*(\theta)], \\
    s_2^*(\theta) &= -k(\theta)[2s_2(\theta) + s_2^*(\theta)].
\end{align*}
\]

The first and third give that

\[
2s_1(\theta)s_1'(\theta) + s_1^2(\theta) + s_1(\theta)s_1^*(\theta) + 2s_1^2(\theta)s_1^*(\theta) = 0,
\]

i.e. that \( s_1^2(\theta) + s_1^2(\theta) + s_1(\theta)s_1^*(\theta) \) is constant.

Analogously, the second and fourth give that \( s_2^2(\theta) + s_2^2(\theta) + s_2(\theta)s_2^*(\theta) \) is also constant.
Then, for the (non-restrictive) initial data
\[(s_1(0), s_2(0)) = (1, 0), \quad (s_1^*(0), s_2^*(0)) = (-\frac{1}{2}, \frac{\sqrt{2}}{2}),\]
we have that
\[s_1^2(\theta) + s_2^2(\theta) + s_1(\theta)s_1^*(\theta) = s_2^*(\theta) + s_2^2(\theta) + s_2(\theta)s_2^*(\theta) = \frac{3}{4}.\]

This, together with
\[s_1(\theta)s_2^2(\theta) - s_2(\theta)s_1^*(\theta) = \frac{\sqrt{2}}{2} \tag{iii}\]
(see Lemma 2.7 (iii)), leads to
\[s_1^2(\theta) + s_2^2(\theta) = 1,\]
i.e., to the fact that \(S\) is a circumference (an ellipse), as we wished to show. \(\square\)

**Theorem 3.2.** Suppose that \(X\) is smooth and of dimension \(\geq 3\). Then \(X\) is an inner product space if and only if
\[(2) \quad u, v, w \in S, \quad u + v + w = 0 \Rightarrow 0 \in Z_X(u, v, w).\]

**Proof.** It is known (see, e.g., [4], p. 238) that if \(X\) is an inner product space (of any dimension), i.e. \(\|x\|^2 = \langle x|x \rangle\), then it fulfills (2). Indeed, let \(u, v, w \in S\) be such that \(u + v + w = 0\). Then for
\[F(x) = \|u - x\| + \|v - x\| + \|w - x\|\]
we have that
\[F'(x)(t) = \frac{(x - u|t)}{\sqrt{(u - x|u - x)}} + \frac{(x - v|t)}{\sqrt{(v - x|v - x)}} + \frac{(x + u + v|t)}{\sqrt{(u + v + x|u + v + x)}},\]
from which it follows that
\[F'(0)(t) = -(u|t) - (u|t) + (u + v|t) = 0,\]
i.e., the convex function \(F\) attains its minimum at 0.

To prove the converse we may assume \(\dim X = 3\).

Since \(X\) is smooth, for any \(u \in S\) there is a unique \(u' \in S'\) (unit sphere of the dual space \(X'\)) such that \(u'(u) = 1\).

Let \(u, v, w \in S\) be such that \(u + v + w = 0\). Then \(0 \in Z_X(u, v, w),\) and a corollary of the Hahn-Banach theorem (see, e.g., [2], Proposition 1) says that this is equivalent to \(u' + v' + w' = 0\).

So we have that
\[u'(u) = v'(v) = w'(w) = 1,\]
\[u'(u + v + w) = v'(u + v + w) = w'(u + v + w) = 0,\]
\[(u' + v' + w')(u) = (u' + v' + w')(v) = (u' + v' + w')(w) = 0,\]
from which it follows that
\[u'(u) + u'(w) = v'(u) + v'(w) = w'(u) + w'(v) = -1,\]
\[v'(u) + w'(u) = u'(v) + w'(v) = u'(w) + v'(w) = -1,\]
and, hence,
\[u'(u) = v'(w) = w'(u), \quad u'(w) = v'(u) = w'(v).\]
Let $L$ be the 2-dimensional subspace $\text{span}(u, v, w)$, let $(L_n)$ be a sequence of 2-dimensional subspaces of $X$ that contain $u$ and converges (in the obvious sense) to $L$, and let $v_n, w_n \in L_n \cap S$ be such that

$$u + v_n + w_n = 0.$$

Since the sequence

$$(\tau_n, -\tau_n) = \left( \frac{v - v_n}{\|v - v_n\|}, \frac{w - w_n}{\|w - w_n\|} \right) = \left( \frac{v - v_n}{\|v - v_n\|}, \frac{v_n - v}{\|v_n - v\|} \right)$$

is in the compact set $S \times S$ it has a subsequence that converges to a point $(\tau, -\tau) \in S \times S$ such that $v \perp \tau$ and $w \perp \tau$, i.e. (see Remark 2.2) $u'(\tau) = w'(\tau) = 0$ and thus $u'(\tau) = 0$.

Moreover, since $\dim(\ker u' \cap \ker v' \cap \ker w') = 1$, every convergent subsequence of $(\tau_n, -\tau_n)$ converges to either $(\tau, -\tau)$ or $(-\tau, \tau)$, and hence

$$\lim_{n \to \infty} u' \left( \frac{v - v_n}{\|v - v_n\|} \right) = \lim_{u + v \in S, v \to v} u' \left( \frac{v - \bar{v}}{\|v - \bar{v}\|} \right) = 0.$$

Since this is valid for every $v \in S$ such that $u + v \in S$, we get that $\{v \in S : u + v \in S\}$ is a differentiable curve that is contained in a plane parallel to $\ker u'$ (i.e. it is a tangent vector at $v$ and $u'(\tau) = 0$). Specifically,

$$\{v \in S : u + v \in S\} = \left( -\frac{1}{2}u + \ker u' \right) \cap S,$$

i.e. $u'(v) = u'(w) = -1/2$, for every $u, v, w \in S$ such that $u + v + w = 0$.

We have, finally, that for every $\lambda \in \mathbb{R}$

$$\|v + w\| = -u'(v + w + \lambda(v - w)) \leq \|v + w + \lambda(v - w)\|,$$

i.e., $v + w \perp v - w$.

In other words, $w = v^*$ and the segment $[v, w]$ supports $\frac{1}{2}S$ at its middle point. Since this is true for every $v \in S$, Theorem 3.1 shows that $X$ is an inner product space. □

Remark 3.3. We presume that in the above theorem the smoothness of $X$ is unnecessary, but the corresponding proof appears to be much more involved.

Remark 3.4. In a first version of this paper we said that $u + v + w = 0$ and $u' + v' + w' = 0$ imply $u'(v) = u'(w) = -1/2$, and then we concluded that if $X$ (of any dimension) fulfills

$$u, v, w \in S, u + v + w = 0 \Rightarrow 0 \in Z_{\text{co}(u, v, w)}(u, v, w),$$

then $X$ is an inner product space.

But for $\dim X = 2$ we only have $u'(v) + u'(w) = -1$, and, furthermore, the following counterexample shows that not only our old proof was wrong.

Example 3.5 ([5]). Let $X$ be the space $\mathbb{R}^2$ endowed with a norm whose unit sphere $S$ is a (rectilinear or curvilinear) regular hexagon, i.e. a convex curve that is invariant under rotations of $\pi/3$. Then, it is easy to see that if $u, v, w \in S$ are such that $u + v + w = 0$, then they are vertices of an equilateral triangle inscribed in $S$.

Furthermore, if, for the above three points, $z \in Z_X(u, v, w)$, then either $z = 0$ or $z$ is a vertex of an equilateral triangle, centered at 0, whose other vertices are also in $Z_X(u, v, w)$, and, since this set is convex, $0 \in Z_X(u, v, w)$. 


Note finally that if $S$ is a rectilinear regular hexagon, then $X$ is neither smooth nor rotund, but for other curvilinear regular hexagons $X$ may be smooth and rotund.

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