OPERATORS THAT ADMIT A MOMENT SEQUENCE, II

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Abstract. As the title indicates, this note is a continuation of a paper by Foias, Jung, Ko and Pearcy, in which it was shown that certain classes of operators on a Hilbert space admit moment sequences. Herein we extend these results.

1. Introduction

In this note \( \mathcal{H} \) will always be a separable, infinite-dimensional, complex Hilbert space, and \( \mathcal{L}(\mathcal{H}) \) will denote the algebra of all bounded linear operators on \( \mathcal{H} \). As usual, \( \mathbf{K} = \mathbf{K}(\mathcal{H}) \) will denote the ideal of compact operators in \( \mathcal{L}(\mathcal{H}) \), and we write \( \mathbb{N}[\mathbb{N}_0] \) for the set of positive [nonnegative] integers. Following \( [1] \) and \( [7] \), we say that an operator \( T \) in \( \mathcal{L}(\mathcal{H}) \) admits a moment sequence if there exist nonzero vectors \( x \) and \( y \) in \( \mathcal{H} \) and a (finite, regular) Borel measure \( \mu \) supported on the spectrum \( \sigma(T) \) of \( T \) such that for every complex polynomial \( p \),

\[
\langle p(T)x, y \rangle = \int_{\sigma(T)} p(\lambda) \ d\mu(\lambda).
\]

(We use the term measure here in the usual sense of a nonnegative-valued set function.)

The motivation for \( [7] \) (and this continuation) is the following nice theorem of Atzmon and Godefroy \( [1] \).

Theorem 1.1. Suppose \( \mathcal{X} \) is a real separable Banach space and \( T \) is a bounded linear operator on \( \mathcal{X} \) that admits a moment sequence (with associated Borel measure \( \mu \) supported on \( \sigma(T) \subset \mathbb{R} \)). Then \( T \) has a nontrivial invariant subspace.

It is obvious that every \( T \) in \( \mathcal{L}(\mathcal{H}) \) that has a nontrivial invariant subspace (n.i.s.) admits a moment sequence (associated with the measure \( \mu \equiv 0 \) on \( \sigma(T) \)), and Theorem 1.1 points in the direction of the possible equivalence of the two concepts. Thus the authors believe that the question of which operators in \( \mathcal{L}(\mathcal{H}) \) can be shown to have a moment sequence is worth further exploration.

The basic tool used in \( [7] \) was a rather deep theorem of Foias-Pasnicu-Voiculescu \( [5] \), together with the theory of quasitrivial operators (cf., e.g., \( [5] \)), and its main theorem was the following, where \( (\mathbb{N} + \mathbf{K}) \) denotes the set of all operators \( T \)
in $\mathcal{L}(\mathcal{H})$ that can be written as a sum $T = N + K$, where $N$ is a normal operator and $K$ is compact.

**Theorem 1.2** ([7]). Every $T \in (N + K)$ admits a moment sequence.

In this note we first show that Theorem 1.2 has a somewhat shorter proof that can be based on a theorem of Lomonosov [6], and then we modestly enlarge the class of operators known to admit moment sequences.

2. A NEW PROOF

We write, as usual, $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/K$ for the Calkin map, and $\sigma_e(T) := \sigma(\pi(T))$, $\|T\|_e := \|\pi(T)\|$. The above-mentioned result of Lomonosov is the following.

**Theorem 2.1** ([6]). Let $A$ be a proper (i.e., $A \neq \mathcal{L}(\mathcal{H})$) subalgebra of $\mathcal{L}(\mathcal{H})$ that is closed in the weak operator topology (WOT) and contains the identity operator $1_{\mathcal{H}}$. Then there exist nonzero vectors $x$ and $y$ in $\mathcal{H}$ such that

1) $(x, y) \geq 0$, and

2) the linear functional $\varphi \in A^*$ defined by $\varphi(A) = \langle Ax, y \rangle$ satisfies $|\varphi(A)| \leq \langle x, y \rangle \|A\|_e$ for every $A \in A$ (and therefore also satisfies $\varphi(1_{\mathcal{H}}) = \langle x, y \rangle = \|\varphi\|$).

Following [6], we say that a functional $\varphi \in A^*$ with the above properties is a positive, vector functional on $A$. Recall that an operator $T$ in $\mathcal{L}(\mathcal{H})$ is called essentially normal if $T^*T - TT^* \in K$, i.e., if $\pi(T)$ is normal. We begin our program with the following new proof of Theorem 1.2.

**New proof of Theorem 1.2.** Let $A_T$ be the unital, WOT-closed subalgebra of $\mathcal{L}(\mathcal{H})$ generated by $T$. Since $A_T$ is abelian, $A_T \neq \mathcal{L}(\mathcal{H})$. We apply Theorem 2.1 to obtain nonzero vectors $x, y$ in $\mathcal{H}$ and the positive vector functional $\varphi$ on $A_T$ satisfying $\varphi(A) = \langle Ax, y \rangle$ and $|\varphi(A)| \leq \langle x, y \rangle \|A\|_e$ for every $A \in A_T$. Then, as above, $\varphi(1_{\mathcal{H}}) = \langle x, y \rangle = \|\varphi\|$, and we also have $\varphi(A_T \cap K) = 0$. Thus, by standard facts about factoring through quotient algebras, there exists $\hat{\varphi} \in (\pi(A_T))^*$ such that $\varphi = \hat{\varphi} \circ \pi$ and $\hat{\varphi}(\pi(1_{\mathcal{H}})) = \langle x, y \rangle = \|\hat{\varphi}\|$. In particular, we have, for every (complex) polynomial $p$,

$$\varphi(p(T)) = \hat{\varphi}(\pi(p(T))) = \hat{\varphi}(p(\pi(T))).$$

(2.1)

From the hypothesis, we know that $\pi(T)$ is normal, and writing $P(\pi(T))$ for the polynomial algebra generated by $\pi(T)$, the Hahn-Banach theorem yields the fact that $\hat{\varphi}|_{P(\pi(T))}$ has an extension $\hat{\varphi}_{\text{ext}}$ to the abelian unital $C^*$-algebra $C^*(\pi(T))$ generated by $\pi(T)$ satisfying

$$\|\hat{\varphi}_{\text{ext}}\| = \|\hat{\varphi}_{\text{ext}}(\pi(1_{\mathcal{H}}))\| = \langle x, y \rangle.$$

In other words, $\hat{\varphi}_{\text{ext}} \in (C^*(\pi(T)))^*$, and since this dual space is isometrically isomorphic to the Banach space $\mathcal{M}(\sigma_e(T))$ of all complex, regular, Borel measures on $\sigma(\pi(T)) = \sigma_e(T)$, we obtain, finally, that there exists $\mu \in \mathcal{M}(\sigma_e(T))$ such that for every complex polynomial $p$,

$$\langle p(T) x, y \rangle = \varphi(p(T)) = \hat{\varphi}(\pi(p(T))) = \hat{\varphi}_{\text{ext}}(p(\pi(T))) = \int_{\sigma_e(T)} p(\lambda) \, d\mu(\lambda).$$

Moreover, since $\varphi, \hat{\varphi}$, and $\hat{\varphi}_{\text{ext}}$ are positive linear functionals, the corresponding complex measure $\mu$ is, in fact, a measure, so the proof is complete. $\square$
Using Theorem 1.2, the BDF-theory of essentially normal operators (cf. [3]), and the characterization of quasitriangular operators from [2], as well as the Berger-Shaw theorem [4], we now obtain, as in [7], the following.

**Corollary 2.2** ([7]). Every $T$ in $\mathcal{L}(\mathcal{H})$ that is either nonbiquasitriangular, essentially normal, or hyponormal admits a moment sequence.

### 3. Some new results

In this section we enlarge the class of operators in $\mathcal{L}(\mathcal{H})$ known to have a moment sequence. We first recall from [9] that an operator $T$ in $\mathcal{L}(\mathcal{H})$ is called *almost hyponormal* if $T^*T - TT^*$ can be written as $P + K$, where $P \geq 0$ and $K \in C_1(\mathcal{H})$, the ideal of trace-class operators in $\mathcal{L}(\mathcal{H})$. A little-known theorem from [9] is the following.

**Theorem 3.1** ([9]). Suppose $T \in \mathcal{L}(\mathcal{H})$ is almost hyponormal, and let $X$ be any Hilbert-Schmidt operator in $\mathcal{L}(\mathcal{H})$ (i.e., $X \in C_2(\mathcal{H})$). Then, if $T^*T - TT^* \not\in C_1(\mathcal{H})$, the operator $T + X$ has an n.i.s.

Our first new result partially generalizes Corollary 2.2.

**Theorem 3.2.** Every operator in $\mathcal{L}(\mathcal{H})$ of the form $T + X$, where $T$ is almost hyponormal and $X \in C_2(\mathcal{H})$, admits a moment sequence.

**Proof.** If $T^*T - TT^* \not\in C_1(\mathcal{H})$, then by Theorem 3.1, $T + X$ has an n.i.s. and thus admits a moment sequence. On the other hand, if $T^*T - TT^* \in C_1(\mathcal{H})$, then $T + X$ is essentially normal, and thus admits a moment sequence via Corollary 2.2.

Our second new result shows that, indeed, the properties of having an n.i.s. and admitting a moment sequence are equivalent for a very special class of operators.

**Theorem 3.3.** Suppose $T \in \mathcal{L}(\mathcal{H})$ and $\sigma(T)$ contains at least one isolated point. Then $T$ has an n.i.s. if and only if $T$ admits a moment sequence.

**Proof.** If $\sigma(T) \neq 0$, then $\sigma(T)$ is disconnected, and hence $T$ has an n.i.s. and admits a moment sequence. The case remaining is that in which $\sigma(T) = \{\lambda_0\}$ for some $\lambda_0 \in \mathbb{C}$. Note first that if $\lambda_0 = 0$ and $T$ admits a moment sequence, say

$$\langle T^n x, y \rangle = \int_{\{\lambda_0\}} \lambda^n \, d\mu, \quad n \in \mathbb{N}_0,$$

for some nonzero vectors $x$ and $y$ and some (necessarily atomic) measure $\mu$ supported on $\{0\}$, then $\langle T^n x, y \rangle = 0$ for $n \in \mathbb{N}$, and the vector $Tx$ is not cyclic for $T$. Thus

$$\mathcal{M} = \bigvee_{n \in \mathbb{N}} T^n x$$

is an n.i.s. for $T$. Therefore to complete the proof it suffices to show that translation of an arbitrary operator $A$ in $\mathcal{L}(\mathcal{H})$ by an arbitrary scalar preserves the property of admitting a moment sequence. But this is immediate from (1.1) and the obvious fact that $\{p(A) : p \in \mathbb{C}[x]\} = \{q(A + \gamma \cdot 1_{\mathcal{H}}) : q \in \mathbb{C}[x]\}$ after making the appropriate change of variables.

We next recall that a corollary of the proof of Theorem 1.2 given in [7] was the following.
Corollary 3.4. Suppose $T \in (N + K)$ and $(0) \neq M$ is an n.i.s. for $T$. Then $T\vert_M$ admits a moment sequence.

It is worthwhile to consider the structure of such $T\vert_M$. If dim $M = n \in \mathbb{N}$, then $T\vert_M$ is essentially an $n \times n$ complex matrix, and nothing more need be said. On the other hand, if dim $(H \ominus M) \in \mathbb{N}$, then a matricial calculation and use of the well-known fact that $T \in (N + K)$ if and only if $T$ is essentially normal and biquasitriangular yields the fact that $T\vert_M \in (N + K)(M)$. Thus we now consider the structure of $T\vert_M$ when both $M$ and $H \ominus M$ are infinite dimensional. Upon identifying $H \ominus M$ with $M$, we may suppose that $T \in (N + K)(M \oplus M)$, and thus may be written matricially as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where the $T_{ij} \in \mathcal{L}(M)$. Let us write $T = N + K$ with $N$ normal and $K \in K$, and write the corresponding $2 \times 2$ matrices for $N$ and $K$ as

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}, \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},$$

where again the $N_{ij}$ and $K_{ij} \in \mathcal{L}(M)$. Obviously, $K_{ij} \in K(M)$ for $i, j = 1, 2$ and $N_{21} = -K_{21} \in K$. Thus $\pi_M(N_{21}) = 0$, $\pi_{M \oplus M}(T)$ is normal, and $\pi_M(T_{11})$ is subnormal. Since $T_{11} = T\vert_M$ has a moment sequence, some essentially subnormal operators have moment sequences. This raises the following interesting problem.

Problem 3.5. Does every essentially subnormal or essentially hyponormal operator in $\mathcal{L}(H)$ admit a moment sequence?

Theorem 3.2 above gives a partial answer to this question, and this next proposition gives another.

Proposition 3.6. Every $T = S + K \in \mathcal{L}(H)$ with $S$ subnormal and $K \in K$ has a moment sequence.

Proof. Let $N$ be a minimal normal extension of $S$ acting on a Hilbert space $K \supset H$ and let $J = K \oplus 0_{K \supset H}$ be a (compact) extension of $K$. Then clearly $N + J \in (N + K)(K)$, $(N + J)H \subset H$, and $(N + J)\vert_H = S + K$. So the result follows from Corollary 3.4. \qed

We close this note with some additional interesting problems in this area.

Problem 3.7. Let $T$ be an invertible operator in $\mathcal{L}(H)$ admitting a moment sequence. Does $T^{-1}$ admit a moment sequence?

Problem 3.8. Suppose that for $n \in \mathbb{N}$, $T_n \in \mathcal{L}(H)$ admits a moment sequence and $\|T_n - T_0\| \to 0$. Does $T_0$ admit a moment sequence?

Problem 3.9. Does every quasinilpotent operator admit a moment sequence? In this connection, remember Theorem 3.3.

References


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