OPERATORS THAT ADMIT A MOMENT SEQUENCE, II

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Abstract. As the title indicates, this note is a continuation of a paper by Foias, Jung, Ko and Pearcy, in which it was shown that certain classes of operators on a Hilbert space admit moment sequences. Herein we extend these results.

1. Introduction

In this note \( \mathcal{H} \) will always be a separable, infinite-dimensional, complex Hilbert space, and \( \mathcal{L}(\mathcal{H}) \) will denote the algebra of all bounded linear operators on \( \mathcal{H} \). As usual, \( K = K(\mathcal{H}) \) will denote the ideal of compact operators in \( \mathcal{L}(\mathcal{H}) \), and we write \( \mathbb{N}[\mathbb{N}_0] \) for the set of positive [nonnegative] integers. Following [1] and [7], we say that an operator \( T \) in \( \mathcal{L}(\mathcal{H}) \) admits a moment sequence if there exist nonzero vectors \( x \) and \( y \) in \( \mathcal{H} \) and a (finite, regular) Borel measure \( \mu \) supported on the spectrum \( \sigma(T) \) of \( T \) such that for every complex polynomial \( p \),

\[
\langle p(T)x, y \rangle = \int_{\sigma(T)} p(\lambda) \, d\mu(\lambda). \tag{1.1}
\]

(We use the term measure here in the usual sense of a nonnegative-valued set function.)

The motivation for [7] (and this continuation) is the following nice theorem of Atzmon and Godefroy [1].

Theorem 1.1. Suppose \( X \) is a real separable Banach space and \( T \) is a bounded linear operator on \( X \) that admits a moment sequence (with associated Borel measure \( \mu \) supported on \( \sigma(T) \subset \mathbb{R} \)). Then \( T \) has a nontrivial invariant subspace.

It is obvious that every \( T \) in \( \mathcal{L}(\mathcal{H}) \) that has a nontrivial invariant subspace (n.i.s.) admits a moment sequence (associated with the measure \( \mu \equiv 0 \) on \( \sigma(T) \)), and Theorem 1.1 points in the direction of the possible equivalence of the two concepts. Thus the authors believe that the question of which operators in \( \mathcal{L}(\mathcal{H}) \) can be shown to have a moment sequence is worth further exploration.

The basic tool used in [7] was a rather deep theorem of Foiaş-Pasnicu-Voiculescu [3], together with the theory of quasitriangular operators (cf., e.g., [5]), and its main theorem was the following, where \( (\mathbb{N} + K) \) denotes the set of all operators \( T \)
in \( \mathcal{L}(\mathcal{H}) \) that can be written as a sum \( T = N + K \), where \( N \) is a normal operator and \( K \) is compact.

**Theorem 1.2** ([7]). Every \( T \in (N + K) \) admits a moment sequence.

In this note we first show that Theorem 1.2 has a somewhat shorter proof that can be based on a theorem of Lomonosov [6], and then we modestly enlarge the class of operators known to admit moment sequences.

2. A NEW PROOF

We write, as usual, \( \pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{K} \) for the Calkin map, and \( \sigma_e(T) := \sigma(\pi(T)), \|T\|_e := \|\pi(T)\|. \) The above-mentioned result of Lomonosov is the following.

**Theorem 2.1** ([6]). Let \( \mathcal{A} \) be a proper (i.e., \( \mathcal{A} \neq \mathcal{L}(\mathcal{H}) \)) subalgebra of \( \mathcal{L}(\mathcal{H}) \) that is closed in the weak operator topology (WOT) and contains the identity operator \( 1_{\mathcal{H}} \). Then there exist nonzero vectors \( x \) and \( y \) in \( \mathcal{H} \) such that

1) \( \langle x, y \rangle \geq 0 \), and

2) the linear functional \( \varphi \in \mathcal{A}^* \) defined by \( \varphi(A) = \langle Ax, y \rangle \) satisfies \( \|\varphi(A)\| \leq \langle x, y \rangle \|A\|_e \) for every \( A \in \mathcal{A} \) (and therefore also satisfies \( \varphi(1_{\mathcal{H}}) = \langle x, y \rangle = \|\varphi\| \)).

Following [6], we say that a functional \( \varphi \in \mathcal{A}^* \) with the above properties is a positive, vector functional on \( \mathcal{A} \). Recall that an operator \( T \in \mathcal{L}(\mathcal{H}) \) is called essentially normal if \( T^*T - TT^* \in \mathcal{K} \), i.e., if \( \pi(T) \) is normal. We begin our program with the following new proof of Theorem 1.2.

**New proof of Theorem 1.2.** Let \( \mathcal{A}_T \) be the unital, WOT-closed subalgebra of \( \mathcal{L}(\mathcal{H}) \) generated by \( T \). Since \( \mathcal{A}_T \) is abelian, \( \mathcal{A}_T \neq \mathcal{L}(\mathcal{H}) \). We apply Theorem 2.1 to obtain nonzero vectors \( x, y \) in \( \mathcal{H} \) and the positive vector functional \( \varphi \) on \( \mathcal{A}_T \) satisfying

\[
\varphi(A) = \langle Ax, y \rangle \quad \text{and} \quad \|\varphi(A)\| \leq \langle x, y \rangle \|A\|_e \quad \text{for every} \quad A \in \mathcal{A}_T.
\]

Then, as above,

\[
\varphi(1_{\mathcal{H}}) = \langle x, y \rangle = \|\varphi\|, \quad \text{and we also have} \quad \varphi(\mathcal{A}_T \cap \mathcal{K}) = 0.
\]

Thus, by standard facts about factoring through quotient algebras, there exists \( \tilde{\varphi} \in (\pi(\mathcal{A}_T))^* \) such that \( \varphi = \varphi \circ \pi \) and \( \tilde{\varphi}(\pi(1_{\mathcal{H}})) = \langle x, y \rangle = \|\varphi\| \). In particular, we have, for every (complex) polynomial \( p \),

\[
\varphi(p(T)) = \tilde{\varphi}(\pi(p(T))) = \tilde{\varphi}(p(\pi(T))).
\]

From the hypothesis, we know that \( \pi(T) \) is normal, and writing \( P(\pi(T)) \) for the polynomial algebra generated by \( \pi(T) \), the Hahn-Banach theorem yields the fact that \( \tilde{\varphi}|_{P(\pi(T))} \) has an extension \( \tilde{\varphi}_{\text{ext}} \) to the abelian unital \( C^* \)-algebra \( C^*(\pi(T)) \) generated by \( \pi(T) \) satisfying

\[
\|\tilde{\varphi}_{\text{ext}}\| = \|\tilde{\varphi}_{\text{ext}}(\pi(1_{\mathcal{H}}))\| = \langle x, y \rangle.
\]

In other words, \( \tilde{\varphi}_{\text{ext}} \in (C^*(\pi(T)))^* \), and since this dual space is isometrically isomorphic to the Banach space \( \mathcal{M}(\sigma_e(T)) \) of all complex, regular, Borel measures on \( \sigma(\pi(T)) = \sigma_e(T) \), we obtain, finally, that there exists \( \mu \in \mathcal{M}(\sigma_e(T)) \) such that for every complex polynomial \( p \),

\[
\langle p(T)x, y \rangle = \varphi(p(T)) = \varphi(\pi(p(T))) = \tilde{\varphi}_{\text{ext}}(p(\pi(T))) = \int_{\sigma_e(T)} p(\lambda) \, d\mu(\lambda).
\]

Moreover, since \( \varphi, \tilde{\varphi}, \) and \( \tilde{\varphi}_{\text{ext}} \) are positive linear functionals, the corresponding complex measure \( \mu \) is, in fact, a measure, so the proof is complete. \( \square \)
Using Theorem 1.2, the BDF-theory of essentially normal operators (cf. [3]), and the characterization of quasitriangular operators from [2], as well as the Berger-Shaw theorem [4], we now obtain, as in [7], the following.

**Corollary 2.2** ([7]). Every $T$ in $L(H)$ that is either nonbiquasitriangular, essentially normal, or hyponormal admits a moment sequence.

### 3. Some new results

In this section we enlarge the class of operators in $L(H)$ known to have a moment sequence. We first recall from [9] that an operator $T$ in $L(H)$ is called almost hyponormal if $T^*T - TT^*$ can be written as $P + K$, where $P \geq 0$ and $K \in C_1(H)$, the ideal of trace-class operators in $L(H)$. A little-known theorem from [9] is the following.

**Theorem 3.1** ([9]). Suppose $T \in L(H)$ is almost hyponormal, and let $X$ be any Hilbert-Schmidt operator in $L(H)$ (i.e., $X \in C_2(H)$). Then, if $T^*T - TT^* \notin C_1(H)$, the operator $T + X$ has an n.i.s.

Our first new result partially generalizes Corollary 2.2.

**Theorem 3.2.** Every operator in $L(H)$ of the form $T + X$, where $T$ is almost hyponormal and $X \in C_2(H)$, admits a moment sequence.

**Proof.** If $T^*T - TT^* \notin C_1(H)$, then by Theorem 3.1, $T + X$ has an n.i.s. and thus admits a moment sequence. On the other hand, if $T^*T - TT^* \in C_1(H)$, then $T + X$ is essentially normal, and thus admits a moment sequence via Corollary 2.2.

Our second new result shows that, indeed, the properties of having an n.i.s. and admitting a moment sequence are equivalent for a very special class of operators.

**Theorem 3.3.** Suppose $T \in L(H)$ and $\sigma(T)$ contains at least one isolated point. Then $T$ has an n.i.s. if and only if $T$ admits a moment sequence.

**Proof.** If $\sigma(T)$ is not a singleton, then $\sigma(T)$ is disconnected, and hence $T$ has an n.i.s. and admits a moment sequence. The case remaining is that in which $\sigma(T) = \{\lambda_0\}$ for some $\lambda_0 \in \mathbb{C}$. Note first that if $\lambda_0 = 0$ and $T$ admits a moment sequence, say

$$\langle T^n x, y \rangle = \int_{\lambda_0} \lambda^n d\mu, \quad n \in \mathbb{N}_0,$$

for some nonzero vectors $x$ and $y$ and some (necessarily atomic) measure $\mu$ supported on $\{0\}$, then $\langle T^n x, y \rangle = 0$ for $n \in \mathbb{N}$, and the vector $Tx$ is not cyclic for $T$. Thus

$$\mathcal{M} = \bigvee_{n \in \mathbb{N}} T^n x$$

is an n.i.s. for $T$. Therefore to complete the proof it suffices to show that translation of an arbitrary operator $A$ in $L(H)$ by an arbitrary scalar preserves the property of admitting a moment sequence. But this is immediate from (1.1) and the obvious fact that $\{p(A) : p \in \mathbb{C}[x]\} = \{q(A + \gamma \cdot 1_H) : q \in \mathbb{C}[x]\}$ after making the appropriate change of variables.

We next recall that a corollary of the proof of Theorem 1.2 given in [7] was the following.
Corollary 3.4. Suppose $T \in (N + K)$ and $(0) \neq M$ is an n.i.s. for $T$. Then $T|_M$ admits a moment sequence.

It is worthwhile to consider the structure of such $T|_M$. If $\dim M = n \in \mathbb{N}$, then $T|_M$ is essentially an $n \times n$ complex matrix, and nothing more need be said. On the other hand, if $\dim (H \ominus M) \in \mathbb{N}$, then a matricial calculation and use of the well-known fact that $T \in (N + K)$ if and only if $T$ is essentially normal and biquasitriangular yields the fact that $T|_M \in (N + K)(M)$. Thus we may consider the structure of $T|_M$ when both $M$ and $H \ominus M$ are infinite dimensional. Upon identifying $H \oplus M$ with $\mathcal{M}$, we may suppose that $T \in (N + K)(\mathcal{M} \oplus M)$, and thus may be written matricially as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where the $T_{ij} \in \mathcal{L}(\mathcal{M})$. Let us write $T = N + K$ with $N$ normal and $K \in \mathcal{K}$, and write the corresponding $2 \times 2$ matrices for $N$ and $K$ as

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}, \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},$$

where again the $N_{ij}$ and $K_{ij} \in \mathcal{L}(\mathcal{M})$. Obviously, $K_{ij} \in \mathcal{K}(\mathcal{M})$ for $i, j = 1, 2$ and $N_{21} = -K_{21} \in \mathcal{K}$. Thus $\pi_M(N_{21}) = 0$, $\pi_M(T)$ is normal, and $\pi_M(T_{11})$ is subnormal. Since $T_{11} = T|_M$ has a moment sequence, some essentially subnormal operators have moment sequences. This raises the following interesting problem.

**Problem 3.5.** Does every essentially subnormal or essentially hyponormal operator in $\mathcal{L}(H)$ admit a moment sequence?

Theorem 3.2 above gives a partial answer to this question, and this next proposition gives another.

**Proposition 3.6.** Every $T = S + K \in \mathcal{L}(H)$ with $S$ subnormal and $K \in \mathcal{K}$ has a moment sequence.

**Proof.** Let $N$ be a minimal normal extension of $S$ acting on a Hilbert space $K \supset H$ and let $J = K \oplus 0_{K \ominus H}$ be a (compact) extension of $K$. Then clearly $N + J \in (N + K)(K)$, $(N + J)H \subset H$, and $(N + J)|_H = S + K$. So the result follows from Corollary 3.4.

We close this note with some additional interesting problems in this area.

**Problem 3.7.** Let $T$ be an invertible operator in $\mathcal{L}(H)$ admitting a moment sequence. Does $T^{-1}$ admit a moment sequence?

**Problem 3.8.** Suppose that for $n \in \mathbb{N}$, $T_n \in \mathcal{L}(H)$ admits a moment sequence and $\|T_n - T_0\| \to 0$. Does $T_0$ admit a moment sequence?

**Problem 3.9.** Does every quasinilpotent operator admit a moment sequence? In this connection, remember Theorem 3.3.

**References**


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