ARTINIAN HOPF ALGEBRAS ARE FINITE DIMENSIONAL

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Abstract. We prove that an artinian Hopf algebra over a field is finite dimensional. This answers a question of Bergen.

A classical theorem of Connell [Co] states that if the group algebra $kG$ of a group $G$ over a base field $k$ is left artinian, then $G$ is finite. For a general Hopf algebra, Bergen asked the following question [JR, p. 373]:

If a Hopf algebra $H$ is artinian as an algebra, must $H$ be finite dimensional?

Based on an exercise of Sweedler, we settle the question with an easy proof. Note that the group algebra proof in [Co] is more complicated, because it relies on the characterization of prime group algebras.

Theorem. Let $H$ be a left (or right) noetherian Hopf algebra over a field $k$. If $H$ has a minimal prime of finite codimension, then $H$ is finite dimensional over $k$.

We need the following lemma which does not use the coalgebra structure.

Lemma. Let $R$ be a left noetherian $k$-algebra containing a minimal prime ideal of finite codimension. Then $R$ has a nonzero finite-dimensional right ideal.

Proof. Let $J$ be a minimal prime ideal of $R$ of finite codimension. By [MR] Lemma 6.2.3., the left noetherian ring $R$ has left Krull dimension, which means that the hypothesis of [GK] Lemma 2(i) holds. By [GK] Lemma 2(i), $J$ (and any minimal prime ideal) is a middle annihilator, namely, there are ideals $X$ and $Y$ of $R$ such that $XY \neq 0$ but $XJY = 0$. Pick any $x \in X$ such that the right ideal $xY$ is nonzero. We claim that $xY$ is finite dimensional. Since $R/J$ is finite dimensional, the finitely generated $R/J$-module $Y/JY$ is finite dimensional. Hence there is a finite-dimensional vector space $V \subset Y$ such that $Y = V + JY$. Using $xJY = 0$ we have $xY = xV + xJY = xV$. Therefore $xY$ is finite dimensional, as required.

Proof of the Theorem. An exercise of Sweedler [Sw, p. 108] says that if $H$ contains a nonzero finite-dimensional right (or left) ideal, then $H$ is finite dimensional (see [DNR] Lemma 5.3.1(i), p. 189 for a proof). The assertion follows from the above Lemma and Sweedler’s exercise.

Here is what we aim to show.

Corollary. A left (or right) artinian Hopf algebra over $k$ is finite dimensional.

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Proof. Let \( J \) be the augmentation ideal \( \ker \epsilon \) of a left artinian Hopf algebra \( H \). Then \( J \) is a maximal (and hence prime) ideal of codimension 1. Since \( H \) is left artinian, the prime ideal \( J \) is a minimal prime. By Hopkins’ theorem [MR Corollary 0.1.13], \( H \) is left noetherian. The assertion follows from the Theorem. □

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We remark that in 2003, T.-K. Lee showed us some of his results [Le] using some arguments in [LZ]. Indeed, he proved that: “Let \( R \) be a left artinian algebra with 1 over a field. Then \( R \) has a proper subalgebra of finite codimension if and only if \( R \) has a non-zero finite dimensional right ideal.” We would like to thank T.-K. Lee for showing us his results.

References


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