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COUNTEREXAMPLES TO THE WELL-POSEDNESS OF L^p TRANSMISSION BOUNDARY VALUE PROBLEMS FOR THE LAPLACIAN

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Abstract. In this note we show that the well-posedness range $p \in (1, 2]$ for L^p transmission boundary value problems for the Laplacian in the class of Lipschitz domains established by Escauriaza and Mitrea (2004) is sharp. Our approach relies on Mellin transform techniques for singular integrals naturally associated with the transmission problems and on a careful analysis of the L^p spectra of such singular integrals.

1. Introduction

We consider the transmission boundary value problem for the Laplacian in a domain $\Omega \subset \mathbb{R}^n$ given by

$$(1) \qquad (TBVP) \quad \begin{cases} \Delta u_\pm = 0 \quad \text{in} \quad \Omega_\pm, \\ M(\nabla u_\pm) \in L^p(\partial\Omega), \\ u_+\big|_{\partial\Omega} = u_-\big|_{\partial\Omega}, \\ \partial_\nu u_+ - \gamma \partial_\nu u_- = g \in L^p(\partial\Omega), \end{cases}$$

where $\gamma \in (0, 1)$ is the transmission coefficient, M is the nontangential maximal operator, $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$ (with $\bar{\Omega}$ denoting the closure in $\mathbb{R}^n$). Furthermore, ∂_ν stands for the normal derivative on $\partial\Omega$. Recently, in [2], the authors show that, for any $\gamma \in (0, 1)$, the problem $(TBVP)$ is well posed (uniqueness understood modulo constants) in the class of (special) Lipschitz domains for every $p \in (1, 2]$. In this note we answer the question, posed to us by L. Escauriaza and M. Mitrea, of whether this range is optimal in the class of domains under consideration. Indeed, we prove that

Theorem 1.1. *For any $p > 2$ there exist a special Lipschitz domain Ω and $\gamma \in (0, 1)$ such that $(TBVP)$ is not well posed.*

Our counterexamples, constructed in the simplest geometric context, i.e., when Ω is an infinite sector in $\mathbb{R}^2$ of a sufficiently small aperture $\theta \in (0, 2\pi)$, rely on a careful

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analysis of the L^p spectra of an integral operator which is naturally associated with $(TBVP)$.

2. Preliminaries

Definition 2.1. A domain $\Omega \subset \mathbb{R}^2$ lying above the graph of a Lipschitz function $\phi : \mathbb{R} \to \mathbb{R}$ is called a special Lipschitz domain. That is,

$$\Omega := \{X = (x_1, x_2) \in \mathbb{R}^2 : \ x_2 > \phi(x_1)\}. \tag{2}$$

Throughout the paper we denote by $d\sigma$ the surface measure on $\partial\Omega$, and by ν the outward unit normal vector which exists almost everywhere with respect to $d\sigma$. As before, we set $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^2 \setminus \bar{\Omega}$. Next, for any $P \in \partial\Omega$, we introduce the non-tangential approach regions with vertex at P as

$$\Upsilon^{\pm}(P) := \{X \in \Omega_{\pm} : \ |P - X| < \kappa \operatorname{dist}(X, \partial\Omega)\}, \tag{3}$$

where $\kappa > 1$ is a fixed, sufficiently large constant. The cone-like regions defined in (3) are then used to define non-tangential traces on $\partial\Omega$. Specifically, if $u_{\pm} : \Omega_{\pm} \to \mathbb{R}$ we let

$$u_{\pm}\big|_{\partial\Omega}(P) := \lim_{\substack{X \in \Upsilon^{\pm}(P) \\ X \to P}} u_{\pm}(X), \quad \text{for a.e.} \quad P \in \partial\Omega, \tag{4}$$

and

$$\partial_\nu u_{\pm}(P) := \langle \nu(P), (\nabla u_{\pm})\big|_{\partial\Omega}(P)\rangle, \quad \text{for a.e.} \quad P \in \partial\Omega. \tag{5}$$

Here and elsewhere $\langle \cdot, \cdot \rangle$ stands for the canonical inner product in $\mathbb{R}^2$. Next, we recall the non-tangential maximal operator M acting on functions $u_{\pm} : \Omega_{\pm} \to \mathbb{R}$ which is given at each boundary point $P \in \partial\Omega$ by

$$M(u_{\pm})(P) := \sup\{|u_{\pm}(X)| : \ X \in \Upsilon^{\pm}(P)\}. \tag{6}$$

For each $1 < p < \infty$, the space $L^p(\partial\Omega)$ is the Lebesgue space of p-integrable functions on $\partial\Omega$ with respect to the surface measure $d\sigma$. Also, let

$$\begin{aligned} L^p_1(\partial\Omega) &:= \{f \in L^p(\partial\Omega) : \partial_\tau f \in L^p(\partial\Omega)\}, \\ \dot{L}^p_1(\partial\Omega) &:= \{f \in L^p_{\text{loc}}(\partial\Omega) : \partial_\tau f \in L^p(\partial\Omega)\}/\mathbb{R}, \end{aligned} \tag{7}$$

where ∂_τ is the tangential derivative along $\partial\Omega$ and $L^p_{\text{loc}}(\partial\Omega)$ is the local version of the space $L^p(\partial\Omega)$. If $[g] \in \dot{L}^p_1(\partial\Omega)$ denotes the equivalence class of g, we set

$$\|[g]\|_{\dot{L}^p_1(\partial\Omega)} := \|\partial_\tau g\|_{L^p(\partial\Omega)}. \tag{8}$$

In the remaining part of this section we recall the definitions of the classical harmonic layer potential operators for a Lipschitz domain $\Omega \subset \mathbb{R}^2$. We start with the definition of $\mathcal{S}$, the single layer potential operator associated to the Laplacian, and its boundary version S. Specifically, fix $X_0 \notin \partial\Omega$, and for $f : \partial\Omega \longrightarrow \mathbb{R}$ set

$$\begin{aligned} \mathcal{S}f(X) &:= \tfrac{1}{2\pi} \int_{\partial\Omega} [\log|X - Y| - \log|X_0 - Y|] f(Y)\, d\sigma(Y), \quad X \in \mathbb{R}^2 \setminus \partial\Omega, \\ Sf(X) &:= \tfrac{1}{2\pi} \int_{\partial\Omega} [\log|X - Y| - \log|X_0 - Y|] f(Y)\, d\sigma(Y), \quad X \in \partial\Omega. \end{aligned} \tag{9}$$

We shall also work with K^*, the formal adjoint of the boundary version of the double layer potential operator, given by

$$K^* f(P) := p.v. \frac{1}{\pi} \int_{\partial\Omega} \frac{\langle P - Y, \nu(P) \rangle}{|P - Y|^2} f(Y)\, d\sigma(Y), \qquad P \in \partial\Omega, \tag{10}$$

where *p.v.* indicates that the integral is taken in the principal value sense. Next, let us record the following result which is going to be useful to us in the sequel (cf., [1], [7]).

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a special Lipschitz domain and assume that $1 < p < \infty$.*

(1) *The single layer potential operator $\mathcal{S}$ satisfies*

$$\mathcal{S}f|_{\partial\Omega_+} = \mathcal{S}f|_{\partial\Omega_-} = Sf \quad \text{and} \quad \|\mathcal{N}(\nabla \mathcal{S} f)\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega)}, \tag{11}$$

uniformly for $f \in L^p(\partial\Omega)$.

(2) *Given $f \in L^p(\partial\Omega)$, for almost every $P \in \partial\Omega$ we have*

$$\langle \nu(P), \lim_{\substack{X \in \Upsilon^{\pm}(P) \\ X \to P}} \nabla \mathcal{S} f(X) \rangle = (\mp \tfrac{1}{2} I + K^*) f(P). \tag{12}$$

(3) *The operator K^* acting from $L^p(\partial\Omega)$ into $L^p(\partial\Omega)$ is bounded.*

Finally, if $\mathcal{X}$ is a Banach space and $T : \mathcal{X} \to \mathcal{X}$ is a linear, bounded operator, we denote by $\mathrm{Spec}(T; \mathcal{X})$ the spectrum of the operator T given by

$$\mathrm{Spec}(T; \mathcal{X}) := \{ z \in \mathbb{C} : zI - T \text{ is not invertible on } \mathcal{X} \}, \tag{13}$$

where I denotes the identity.

3. L^p SPECTRUM OF LAYER POTENTIALS

In this section we present several known results regarding spectral properties of the operator K^* which are relevant for our work. For $\theta \in (0, \pi)$, $q \in (1, \infty)$, we introduce

$$R(\theta, q) := \frac{\sin(\frac{\pi - \theta}{q})}{2 \sin(\frac{\pi}{q})}. \tag{14}$$

With this notation, using the Mellin pseudo-differential calculus, the following result can be established (cf., e.g., [5], [6]).

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be an infinite sector of aperture $\theta \in (0, 2\pi)$. Then, for any $1 < p, q < \infty$ such that $1/p + 1/q = 1$, the following holds:*

$$R(\theta, q) \in \mathrm{Spec}(K^*; L^p(\partial\Omega)). \tag{15}$$

Next, consider the assignment

$$(0, 2\pi) \ni \theta \mapsto p_{critic}(\theta) := \begin{cases} \dfrac{2\pi - \theta}{\pi - \theta}, & \text{if} \quad \theta \in (0, \pi), \\[2mm] \dfrac{\theta}{\theta - \pi}, & \text{if} \quad \theta \in (\pi, 2\pi), \\[2mm] \infty, & \text{if} \quad \theta = \pi. \end{cases} \tag{16}$$

Notice that $p_{critic}(\theta) = p_{critic}(2\pi - \theta)$ and $\lim_{\theta \to 0^+} p_{critic}(\theta) = \lim_{\theta \to 2\pi^-} p_{critic}(\theta) = 2$.

We have

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^2$ be an infinite sector of aperture $\theta \in (0, 2\pi)$. Then*

$$\text{(17)} \quad \pm\tfrac{1}{2}I + K^* : L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{are isomorphisms} \quad \forall\, p \in (1,\infty)\setminus\{p_{critic}(\theta)\}$$

and

$$\text{(18)} \quad \partial_\tau S : L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{is an isomorphism} \quad \forall\, p \in (1,\infty) \setminus \{p_{critic}(\theta)\}.$$

Furthermore

(19)

$$L^p(\partial\Omega) \ni f \mapsto [Sf] \in \dot{L}^p_1(\partial\Omega) \quad \textit{is an isomorphism} \quad \forall\, p \in (1,\infty) \setminus \{p_{critic}(\theta)\}.$$

Proof. For (17) see, e.g., [6], [5]. As for (18), we employ the following two-dimensional identity proven in [4]:

$$(\partial_\tau S)^2 = (-\tfrac{1}{2}I + K^*)(\tfrac{1}{2}I + K^*). \tag{20}$$

Then (18) readily follows from this identity and (17). Next, fix $1 < p < \infty$, $p \neq p_{critic}(\theta)$, and make the observation that the following operator (also denoted by ∂_τ),

$$\partial_\tau : \dot{L}^p_1(\partial\Omega) \to L^p(\partial\Omega), \quad \partial_\tau([f]) := \partial_\tau f, \tag{21}$$

is well defined, linear, bounded and invertible. Then (19) follows from (18) and (21). □

4. The main result

In this section we present the proof of the main result of this note. We have

Theorem 4.1. *For each $p > 2$ there exist a special Lipschitz domain Ω and $\gamma = \gamma(p,\Omega) \in (0,1)$ such that the transmission boundary value problem* (1) *fails to be well posed.*

As a preamble to the proof of Theorem 4.1 we present a series of four useful technical lemmas.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^2$ be an infinite sector of aperture $\theta \in (0, 2\pi)$ and recall $p_{critic(\theta)}$ from* (16). *Then for each $p \in (1,\infty) \setminus \{p_{critic(\theta)}\}$ the following implication holds:*

(22)

$$\left.\begin{array}{l} \Delta u = 0 \quad in \quad \Omega \\ M(\nabla u) \in L^p(\partial\Omega) \end{array}\right\} \implies \quad u = \mathcal{S}f + c \ \ in \ \ \Omega, \quad \textit{for some } f \in L^p(\partial\Omega) \textit{ and } c \in \mathbb{R}.$$

Proof. We start by noting that $\big[u\big|_{\partial\Omega}\big] \in \dot{L}^p_1(\partial\Omega)$ as $M(\nabla u) \in L^p(\partial\Omega)$ (where, as before, $[f]$ stands for the equivalence class of f modulo $\mathbb{R}$). Using that $p \neq p_{critic}(\theta)$ and (19) in Theorem 3.2, we obtain that $\big[u\big|_{\partial\Omega}\big] = [Sf]$, for some $f \in L^p(\partial\Omega)$. In particular, $u\big|_{\partial\Omega} = Sf + c$, for some $c \in \mathbb{R}$.

Next consider the function $w := u - \mathcal{S}f - c$ in the domain Ω. Using (11) and the sublinearity of the operator M introduced in (6), the following hold:

$$\begin{cases} \Delta w = 0 \quad \text{in} \quad \Omega, \\ M(\nabla w) \in L^p(\partial\Omega), \\ w\big|_{\partial\Omega} = 0. \end{cases} \tag{23}$$

Employing (2.47) in [2], we further conclude that $(\frac{1}{2}I + K^*)(\partial_\nu w) = 0$ since by (23) we have $\partial_\tau w = 0$. Next, by Theorem 3.2 the operator $\frac{1}{2}I + K^*$ is invertible on $L^p(\partial\Omega)$ since we are assuming $p \neq p_{critic}(\theta)$. In particular, using the fact that $\partial_\nu w \in L^p(\partial\Omega)$ from (23), we obtain $\partial_\nu w = 0$. Since from (23) we also have $\partial_\tau w = 0$, it follows that $(\nabla w)\big|_{\partial\Omega} = 0$.

At this point, let $F : \Omega \to \mathbb{C}$ be given by

$$F(z) := \partial_1 w(x_1, x_2) - i\partial_2 w(x_1, x_2), \quad z := x_1 + ix_2, \quad (x_1, x_2) \in \Omega. \tag{24}$$

We have $\bar{\partial} F = \Delta w = 0$ in Ω, i.e., F is a holomorphic function in Ω, and from (23) it also follows that $M(F) \in L^p(\partial\Omega)$. Then, much as in pp. 125-126 in [3], the following Cauchy integral representation holds:

$$F(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{F(\xi)}{\xi - z}\, d\xi, \quad z \in \Omega. \tag{25}$$

Next, $(\nabla w)\big|_{\partial\Omega} = 0$ gives that $F|_{\partial\Omega} = 0$, and using (25) we obtain $F \equiv 0$ in Ω. This further implies $\nabla w \equiv 0$ in Ω and, therefore, w is constant in Ω. Then (22) readily follows. □

Before stating the next lemma let us introduce one more piece of notation. For each $\gamma \in (0, 1)$ we set

$$\beta(\gamma) := \frac{1}{2} \cdot \frac{1 + \gamma}{(1 - \gamma)}. \tag{26}$$

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^2$ be an infinite sector of aperture $\theta \in (0, 2\pi)$ and $\gamma \in (0, 1)$. Then, for every $p \in (1, \infty) \setminus \{p_{critic}(\theta)\}$ such that the problem* (1) *is well posed, it follows that*

$$\beta(\gamma) \notin \mathrm{Spec}(K^*; L^p(\partial\Omega)). \tag{27}$$

Proof. Fix $\gamma \in (0, 1)$ and $p \in (1, \infty) \setminus \{p_{critic}(\theta)\}$ such that (1) is well posed, and consider $g \in L^p(\partial\Omega)$ such that $(\beta(\gamma)I - K^*)\, g = 0$. Our goal is to show that $g \equiv 0$. To see this, set $v_\pm := \mathcal{S}g$ in $\Omega_\pm$. Employing Theorem 2.2, it is then straightforward to check that the pair (v_+, v_-) solves the homogeneous boundary problem

$$\Delta v_\pm = 0 \quad \text{in} \quad \Omega_\pm, \quad M(\nabla v_\pm) \in L^p(\partial\Omega), \quad v_+|_{\partial\Omega} = v_-|_{\partial\Omega}, \quad \partial_\nu v_+ - \gamma \partial_\nu v_- = 0. \tag{28}$$

Hence, since by assumption the problem (28) is well posed, we obtain that $v_\pm \equiv c$ in $\Omega_\pm$ for some $c \in \mathbb{R}$. In particular, appealing again to the jump relations in Theorem 2.2, we have that

$$g = (\tfrac{1}{2}I + K^*)g - (-\tfrac{1}{2}I + K^*)g = \partial_\nu v_- - \partial_\nu v_+ = 0, \tag{29}$$

which shows that $\beta(\gamma)I - K^*$ is one-to-one on $L^p(\partial\Omega)$.

Next consider $f \in L^p(\partial\Omega)$ and let (u_+, u_-) be the solution of the transmission problem (1) with datum $g := -(1 - \gamma)f \in L^p(\partial\Omega)$. Then $\Delta u_\pm = 0$ in $\Omega_\pm$ and $M(\nabla u_\pm) \in L^p(\partial\Omega)$, and using Lemma 4.2 we obtain $u_\pm = \mathcal{S}h_\pm + c_\pm$ for some $h_\pm \in L^p(\partial\Omega)$ and $c_\pm \in \mathbb{R}$. Since $u_+|_{\partial\Omega} = u_-|_{\partial\Omega}$, by Theorem 2.2, this further implies that $Sh_+ + c_+ = Sh_- + c_-$, and taking the equivalence class (modulo $\mathbb{R}$) of both sides we obtain $[Sh_+] = [Sh_-]$. By (19) in Theorem 3.2, we may then conclude that $h_+ = h_- =: h$. Thus, $u_\pm = \mathcal{S}h + c_\pm$ in $\Omega_\pm$, and Theorem 2.2 allows us to write

$$\begin{aligned}\partial_\nu u_+ - \gamma\partial_\nu u_- &= (-\tfrac{1}{2}I + K^*)h - \gamma(\tfrac{1}{2}I + K^*)h \\ &= (-\tfrac{1+\gamma}{2}I + (1-\gamma)K^*)h = -(1-\gamma)(\beta(\gamma)I - K^*)h. \end{aligned} \tag{30}$$

Also, by assumption, $\partial_\nu u_+ - \gamma\partial_\nu u_- = -(1-\gamma)f$, which in concert with (30) yields $(\beta(\gamma)I - K^*)\,h = f$. Hence, the operator $\beta(\gamma)I - K^* : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is surjective and, ultimately invertible. This proves (27), as desired. □

Recall $R(\theta, q)$ from (14). We have

Lemma 4.4. *For any $p > 2$ there exists $\theta_p \in (0, 2\pi)$ with the following property. If $\theta \in (0, \theta_p)$ and $\Omega \subset \mathbb{R}^2$ is an infinite sector of aperture θ, then*

$$R(\theta, \tfrac{p}{p-1}) \in \mathrm{Spec}(K^*; L^p(\partial\Omega)) \tag{31}$$

and

$$R(\theta, \tfrac{p}{p-1}) > \tfrac{1}{2}. \tag{32}$$

Proof. First, (31) is a direct consequence of (15). As for (32), its proof is elementary. Indeed, for every $q \in (1, \infty)$ we have

$$R(0, q) = \tfrac{1}{2}. \tag{33}$$

Next, differentiating both sides of (14) with respect to the parameter θ yields $\frac{\partial R}{\partial\theta}(0, q) = -\frac{\cos(\pi/q)}{2q\sin(\pi/q)}$. Fix $p > 2$ and let $q := \frac{p}{p-1} \in (1, 2)$. Thus $\frac{\pi}{q} \in \left(\frac{\pi}{2}, \pi\right)$, and consequently $\frac{\partial R}{\partial\theta}(0, \frac{p}{p-1}) > 0$. Invoking the continuity of the function $\frac{\partial R}{\partial\theta}(\theta, \frac{p}{p-1})$ with respect to $\theta \in [0, 2\pi]$, we conclude that there exists $\theta_p \in (0, 2\pi)$ such that

$$\frac{\partial R}{\partial\theta}(\theta, \tfrac{p}{p-1}) > 0 \quad \text{for every} \quad \theta \in (0, \theta_p). \tag{34}$$

To finish checking (32), fix $\theta \in (0, \theta_p)$. Then a simple application of the Mean Value Theorem for the function $R(\cdot, \frac{p}{p-1})$ on the interval $[0, \theta]$, together with (34) and (33), give (32). □

Finally, we have

Lemma 4.5. *For each $p > 2$, there exist $\theta \in (0, 2\pi)$ and $\gamma \in (0, 1)$ such that $p_{critic}(\theta) \neq p$ and, if $\Omega \subset \mathbb{R}^2$ is an infinite sector of aperture θ, then the operator $\beta(\gamma)I - K^*$ is not invertible on $L^p(\partial\Omega)$.*

Proof. Let $p > 2$ be fixed and assume that θ_p is as in Lemma 4.4. Based on (16) it is easy to see that it is always possible to choose $\theta \in (0, \theta_p)$ such that $p_{critic}(\theta) \neq p$. Fix such a θ and set

$$\gamma := \frac{2R(\theta, \frac{p}{p-1}) - 1}{2R(\theta, \frac{p}{p-1}) + 1}. \tag{35}$$

Since $R(\theta, \frac{p}{p-1}) > \frac{1}{2}$ by (32), we see that $\gamma \in (0, 1)$. From (35) we also have

$$1 + \gamma = \frac{4R(\theta, \frac{p}{p-1})}{2R(\theta, \frac{p}{p-1}) + 1} \quad \text{and} \quad 1 - \gamma = \frac{2}{2R(\theta, \frac{p}{p-1}) + 1}. \tag{36}$$

These identities in concert with (26) further imply that $\beta(\gamma) = R(\theta, \frac{p}{p-1})$. Finally, appealing to (31), the operator $\beta(\gamma)I - K^*$ is not invertible on $L^p(\partial\Omega)$. □

We are now ready to present the

Proof of Theorem 4.1. Fix $p > 2$ and let $\theta \in (0, 2\pi)$ and $\gamma \in (0, 1)$ be as in Lemma 4.5. Next, take the special Lipschitz domain $\Omega \subset \mathbb{R}^2$ to be an infinite sector of aperture θ. Then, according to (4.5), we have $\beta(\gamma) \in \mathrm{Spec}(K^*; L^p(\partial\Omega))$. Employing Lemma 4.3, we then obtain that the transmission problem (1) fails to be well posed. □

References

[1] R. R. Coifman, A. McIntosh and Y. Meyer, *L'integrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Annals of Math., 116 (1982), 361–387. MR0672839 (84m:42027)

[2] L. Escauriaza and M. Mitrea, *Transmission problems and spectral theory for singular integral operators on Lipschitz domains*, J. Funct. Anal., 216 (2004), no. 1, 141–171. MR2091359 (2005f:35065)

[3] K. Hoffman, *Banach spaces of analytic functions*, Dover Publications, Inc., New York, 1988. MR1102893 (92d:46066)

[4] D. Mitrea, *Layer potentials and Hodge decompositions in two dimensional Lipschitz domains*, Math. Ann., 322 (2002), no. 1, 75–101. MR1883390 (2003d:42024)

[5] V. Y. Shelepov, *On the index and spectrum of integral operators of potential type along Radon curves*, Math. USSR Sbornik, 70, no. 1 (1991), 175–203. MR1072296 (91i:47070)

[6] R. S. Spencer, *Series Solutions and Spectral Properties of Boundary Integral Equations*, Ph.D. Thesis, University of Minnesota, 1994.

[7] G. C. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, Journal of Functional Analysis, 69 (1984), 572–611. MR0769382 (86e:35038)

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