A NEW EXPPLICIT EXPRESSION OF THE CONTOU-CARRÈRE SYMBOL

FERNANDO PABLOS ROMO

(Communicated by Wen-Ching Winnie Li)

Abstract. The aim of this note is to offer a new explicit expression of the Contou-Carrère symbol that depends only on a product of a finite number of terms. As an application, we obtain an explicit formula for a Witt Residue.

1. Introduction

In 1994 C. Contou-Carrère defined a natural transformation greatly generalizing the tame symbol. In the case of an artinian local base ring $A$ with maximal ideal $m$, the natural transformation takes the following form. Let $f,g \in A((t))^\times$ be given, where $t$ is a variable. (Here and below $R^\times$ denotes the multiplicative group of a ring $R$ with unit.) It is possible in exactly one way to write

$$
\begin{align*}
&f = a_0 \cdot t^{w(f)} \cdot \prod_{i=1}^\infty (1 - a_i t^i) \cdot \prod_{i=1}^\infty (1 - a_{-i} t^{-i}), \\
g = b_0 \cdot t^{w(g)} \cdot \prod_{i=1}^\infty (1 - b_i t^i) \cdot \prod_{i=1}^\infty (1 - b_{-i} t^{-i}),
\end{align*}
$$

with $w(f), w(g) \in \mathbb{Z}$, $a_i, b_i \in A$ for $i > 0$, $a_0, b_0 \in A^\times$, $a_{-i}, b_{-i} \in m$ for $i > 0$, and $a_{-i} = b_{-i} = 0$ for $i \gg 0$. By definition, the value of the Contou-Carrère symbol is

$$
\langle f, g \rangle_A := (-1)^{w(f)w(g)} a_0^{w(g)} \prod_{i=1}^\infty \prod_{j=1}^\infty (1 - a_i^{j/(i,j)} b_j^{j/(i,j)})^{(i,j)} \\
b_0^{w(f)} \prod_{i=1}^\infty \prod_{j=1}^\infty (1 - a_{-i}^{j/(i,j)} b_{-j}^{j/(i,j)})^{(i,j)} \in A^\times.
$$

The definition makes sense because only finitely many of the terms appearing in the infinite products differ from 1. The symbol $\langle \cdot, \cdot \rangle_A$ is clearly antisymmetric and, although it is not immediately obvious from the definition, also bimultiplicative.

G. W. Anderson and the author have interpreted the Contou-Carrère symbol $\langle f, g \rangle_A$—up to signs—as a commutator of liftings of $f$ and $g$ to a certain central extension of a group containing $A((t))^\times$, and they have exploited the commutator interpretation to prove, in the style of Tate, a reciprocity law for the Contou-Carrère symbol on a nonsingular complete curve defined over an algebraically closed field $k$, $A$ being an artinian local $k$-algebra.

Received by the editors February 6, 2006 and, in revised form, February 23, 2006 and March 1, 2006.

2000 Mathematics Subject Classification. Primary 19F15, 19C20.

Key words and phrases. Contou-Carrère symbol, explicit expression, artinian local ring, Witt Residue.

This work was partially supported by the DGESYC research contract no. BFM2003-00078 and Castilla y León regional government contract SA071/04.

©2007 American Mathematical Society

Reverts to public domain 28 years from publication.
The author has also obtained a similar result for an algebraic curve over a perfect field \cite{10}, and A. Beilinson, S. Bloch and H. Esnault \cite{3} have defined the Contou-Carrère symbol as the commutator pairing in a Heisenberg super extension. Moreover, recently, this symbol has played an important role in the work of M. Kapranov and E. Vasserot \cite{6}, and M. Asakura \cite{2} has shown that the Contou-Carrère symbol coincides with the boundary map \( \delta: K_{i+1}(A((t))) \to K_i(A) \) described by K. Kato in \cite{7}.

In this context, the goal of this paper is to offer a new explicit expression of the Contou-Carrère symbol that depends only on a product of a finite number of terms (Proposition 3.4). As an application, we study a Witt Residue and we also obtain an explicit expression for it.

2. Preliminaries

2.1. Contou-Carrère symbol. Using the theory of groupoids, we can construct a central extension of groups

\[
1 \to A^\times \to \widehat{A((t))^\times} \xrightarrow{\pi} A((t))^\times \to 1,
\]

and we have a commutator map

\[
\{\cdot, \cdot\}_{A((t))}^A: A((t))^\times \times A((t))^\times \to A^\times.
\]

That is, if \( \tau \) and \( \sigma \) are two elements of \( A((t))^\times \) and \( \bar{\tau}, \bar{\sigma} \in \widehat{A((t))^\times} \) are elements such that \( \pi(\bar{\tau}) = \tau \) and \( \pi(\bar{\sigma}) = \sigma \), then the commutator map is

\[
\{\tau, \sigma\}_{A((t))}^A = \bar{\tau} \cdot \bar{\sigma} \cdot (\bar{\tau}^{-1} \cdot \bar{\sigma}^{-1}) \in A^\times.
\]

With the notations of the preceding section, the Contou-Carrère symbol \cite{4} is

\[
\langle f, g \rangle_A = (-1)^{w(f)w(g)} \cdot \{f, g\}_{A((t))}^A.
\]

For details about the central extension (2.1) and the commutator \( \{\cdot, \cdot\}_{A((t))}^A \) readers are referred to \cite{4}.

For arbitrary elements \( f, g, h \in A((t))^\times \), the following relations hold:

- \( \langle f, g \cdot h \rangle_A = \langle f, g \rangle_A \cdot \langle f, h \rangle_A \).
- \( \langle g, f \rangle_A = \langle f, g \rangle_A^{-1} \).
- \( \langle f, -f \rangle_A = 1 \).
- Given \( \varphi \in A((t))^\times \) with positive winding number \( n \), one has that

\[
\langle f, g \circ \varphi \rangle_A = \langle N_{\varphi}[f], g \rangle_A,
\]

where \( N_{\varphi}: A((t))^\times \to A((t))^\times \) denotes the corresponding norm mapping: viewing \( A((t)) \) via the homomorphism \( h \mapsto h \circ \varphi \) as a free \( A((t)) \)-module of rank \( n \) (\cite{9}, Proposition 3.6).

Remark 2.1. With the notation of the previous section, we should note that the original expression of the Contou-Carrère symbol \cite{4} is

\[
\langle f, g \rangle_A = (-1)^{w(f)w(g)} \frac{a_0^{w(g)}}{b_0^{w(f)}} \cdot \exp\left( \sum_{i>0} \frac{\delta_i(f) \cdot \delta_{-i}(g)/i}{\delta_{-i}(f) \cdot \delta_i(g)/i} \right),
\]

where \( \delta_m(f) = \text{Res}(t^m \frac{df}{f}) \).
Let $\xi_1, \xi_2, \ldots; \eta_1, \eta_2, \ldots$ now be indeterminates. Recall from \cite{5} that we can define the sequences $\bar{x} = (\bar{x}_i)$ and $\bar{y} = (\bar{y}_j)$ by the equations

\[
\prod_{i}(1 - \xi_i t) = 1 + \bar{x}_1 t + \bar{x}_2 t^2 + \ldots, \\
\prod_{i}(1 - \eta_i t) = 1 + \bar{y}_1 t + \bar{y}_2 t^2 + \ldots.
\]

Hence, from these definitions we can construct a polynomial sequence $P_1, P_2, \ldots$ satisfying the relations

\[
(2.4) \quad \prod_{i,j}(1 - \xi_i \eta_j t) = 1 + P_1 t + P_2 t^2 + \ldots.
\]

Bearing in mind the fundamental theorem of symmetric functions, $P_j$ can be written as $P_j(\bar{x}_1, \ldots, \bar{x}_j; \bar{y}_1, \ldots, \bar{y}_j)$.

Thus, if $A$ is a commutative ring, we can consider on the multiplicative group

\[
\bigwedge(A) = \{1 + a_1 t + a_2 t^2 + \ldots, \ a_i \in A \} \subseteq A[[t]]
\]

a second operation by means of the formula

\[
(1 + a_1 t + a_2 t^2 + \ldots) \ast (1 + b_1 t + b_2 t^2 + \ldots) = 1 + P_1(a, b) t + P_2(a, b) t^2 + \ldots,
\]

such that $(\bigwedge(A), \ast, \cdot)$ is a ring.

Moreover, $\mathcal{W}(A)$ being the ring of Witt vectors with coefficients in $A$, we can define a map

\[
E_A: \mathcal{W}(A) \longrightarrow \bigwedge(A), \quad (a_1, a_2, \ldots) \longrightarrow \prod_{i \geq 1}(1 - a_i t^i),
\]

which is an isomorphism of rings because $E_A(a + b)E_A(a) \cdot E_A(b)$ and $E_A(a \cdot b) = E_A(a) \ast E_A(b)$.

Since $\mathcal{W}(A)$ is a commutative ring and with unit element (\cite{5}, page 117), we can denote by $\mathcal{W}_+(A)$ the abelian group induced by the ring structure.

Furthermore, we can define the abelian group

\[
\mathcal{W}_+(A) = \left\{(b_1, b_2, \ldots) \in \mathcal{W}_+(A) \text{ with } b_i \text{ nilpotent for all } i \right\}.
\]

If the artinian local ring $A$ is a $k$-algebra with $\text{ch}(k) = 0$, bearing in mind Proposition 2.4 (\cite{10}, p. 44) and denoting $E_A(\cdot) = E_A(\cdot, t)$, we have that

\[
\exp\left(\sum_{i > 0}[\delta_i(f) \cdot \delta_{-i}(g)/i!]\right) = E_A(a \cdot b', 1),
\]

where $a = (a_1, a_2, \ldots) \in \mathcal{W}_+(A)$ and $b' = (b_{-1}, b_{-2}, \ldots) \in \mathcal{W}_+(A)$, with the notation of \cite{11}.

**Remark 2.2.** If $A$ is again an artinian local $k$-algebra with $\text{ch}(k) = 0$, let us now consider again $f, g \in A((t))^\times$ and let us assume that there exist decompositions

\[
f = \exp\left(\sum_{i > 0}[\delta_i(f)/i!]\right) = a_0 \cdot t^{\nu(f)} \cdot \prod_{i \geq 1}(1 - \bar{a}_i t) \cdot \prod_{i \geq 1}(1 - \bar{a}_{-i} t^{-1}),
\]

\[
g = \exp\left(\sum_{i > 0}[\delta_i(g)/i!]\right) = b_0 \cdot t^{\nu(g)} \cdot \prod_{j \geq 1}(1 - \bar{b}_j t) \cdot \prod_{j \geq 1}(1 - \bar{b}_{-j} t^{-1}),
\]

with $\bar{a}_i, \bar{b}_j \in A$ and $\bar{a}_{-i}, \bar{b}_{-j} \in m$. 
It follows from the above considerations, in particular from equality \((2.4)\), and from the results of \([8]\) (formulas (4.1) on p. 62 and (4.3) on p. 63), that
\[
\exp\left(\sum_{i>0}[\delta_i(f) \cdot \delta_{-i}(g)] \right) = \left(\sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}(\bar{a}) p_{\lambda}(\bar{b}')\right)^{-1},
\]
\[
\exp\left(\sum_{i>0}[\delta_i(f) \cdot \delta_{-i}(g)] \right) = \left(\sum_{\lambda} s_{\lambda}(\bar{a}) s_{\lambda}(\bar{b}')\right)^{-1},
\]
summed over all partitions \(\lambda = (\lambda_1, \lambda_2, \ldots)\), with:
- \(\bar{a} = (\bar{a}_1, \bar{a}_2, \ldots)\);
- \(\bar{b}' = (\bar{b}_{-1}, \bar{b}_{-2}, \ldots)\);
- \(z_{\lambda} = \prod_{i \geq 1} i^{m_i} m_i!\), where \(m_i = m_i(\lambda)\) is the number of parts of \(\lambda\) equal to \(i\);
- \(p_{\lambda} = p_{\lambda_1} \cdot p_{\lambda_2} \cdots\), where \(p_{\lambda_1}(x)\) is the \(\lambda_1\)-th power sum \(\sum x_1^{\lambda_1}\);
- \(s_{\lambda}\) is the Schur function associated with the partition \(\lambda\).

Therefore, setting \(\bar{b} = (\bar{b}_1, \bar{b}_2, \ldots)\) and \(\bar{a}' = (\bar{a}_{-1}, \bar{a}_{-2}, \ldots)\), equivalent expressions to \((2.4)\) are:
\[
(1) \quad (f, g)_A = (-1)^w(f) w(g) \frac{\bar{a}^{w(\bar{a})} \sum z_{\lambda}^{-1} p_{\lambda}(\bar{b}) p_{\lambda}(\bar{b}')}{b_0^1 \sum z_{\lambda}^{-1} p_{\lambda}(\bar{a}) p_{\lambda}(\bar{a}')};
\]
\[
(2) \quad (f, g)_A = (-1)^w(f) w(g) \frac{\bar{a}^{w(\bar{a})} \sum s_{\lambda}(\bar{b}) s_{\lambda}(\bar{a}')}{b_0^1 \sum s_{\lambda}(\bar{a}) s_{\lambda}(\bar{a}')}.
\]

2.2. Witt Residue symbol. Similar to \([1]\), let
\[
\{\epsilon\} \prod_{i=1}^{\infty} \{x_i, y_i\}_{i=1}^{\infty}
\]
be a family of independent variables. Let us write
\[
\prod_{i=1}^{\infty} ((1 - x_i \epsilon^i)(1 - y_i \epsilon^i)) = \prod_{i=1}^{\infty} (1 - A_i \epsilon^i),
\]
\[
\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \left(1 - x_i^{(i,j)} y_j^{(i,j)} \epsilon^{(i,j)}\right)^{(i,j)} \prod_{i=1}^{\infty} (1 - M_i \epsilon^i),
\]
thereby defining families of polynomials
\[
\{A_n, M_n \in \mathbb{Z} \mid \{x_i, y_i\}_{i=1}^{n}\}_{n=1}^{\infty}.
\]
For any commutative ring \(B\) with unit and finite subset \(\Delta\) of the set of positive integers closed under passage to divisors, let \(\mathbb{W}_\Delta(B)\) denote the set of vectors with entries in \(B\) indexed by \(\Delta\). It can be shown that the \(A\)’s and \(M\)’s define addition and multiplication laws with respect to which \(\mathbb{W}_\Delta(B)\) becomes a commutative ring with unit, functorially in commutative rings \(B\) with unit.

Let us fix a positive integer \(N\). If \(\mathbb{W}_{\leq N}(B)\) is the ring of Witt vectors associated with \(\Delta = \{1, 2, \ldots, N\}\), one has that the map
\[
x = (x_i)_{i=1}^{N} \mapsto \prod_{i=1}^{N} (1 - x_i \epsilon^i) \mod \epsilon^{N+1}
\]
identifies the additive group underlying \(\mathbb{W}_{\leq N}(B)\) with the group of units in the ring \(B[\epsilon]/(\epsilon^{N+1})\) congruent to 1 modulo \((\epsilon)\) functorially in commutative rings \(B\) with unit.

If \(F\) is a field, and \(A := F[\epsilon]/(\epsilon^{N+1})\), we can define a pairing
\[
\text{Res}_{\leq N}^\mathbb{W}(\cdot, \cdot) : F((t))^\otimes \times \mathbb{W}_{\leq N}(F((t))) \to \mathbb{W}_{\leq N}(F)
\]
by the rule
\[ \left\langle f, \prod_{i=1}^{N} (1 - x_i e^t) \right\rangle_{A((t))}^{A((t))} = \prod_{i=1}^{N} \left( 1 - e^t \left[ \text{Res}^{w}_{\leq N}(f, x) \right] \right) \mod (e^{N+1}), \]
where \( \langle \cdot, \cdot \rangle_{A((t))}^{A((t))} \) is the Contou-Carrère symbol.

The pairing \( \text{Res}^{w}_{\leq N} \) is essentially the pairing introduced in Witt’s paper [12].

Hence, when \( N = 1 \) we have a map
\[ \text{Res}^{w}((t)) \times F((t)) \rightarrow F, \]
where
\[ \langle f, 1 - \epsilon g \rangle_{A((t))} = 1 - \epsilon [\text{Res}^{w}(f, g)] \mod (\epsilon^2), \]
for all \( f \in F((t))^{\times} \) and \( g \in F((t)) \).

One has that:
- \( \text{Res}^{w}(f \cdot f', g) = \text{Res}^{w}(f, g) + \text{Res}^{w}(f', g) \).
- \( \text{Res}^{w}(f, g + g') = \text{Res}^{w}(f, g) + \text{Res}^{w}(f, g') \).

3. **NEW EXPLICIT EXPRESSION OF THE CONTOU-CARRÈRE SYMBOL**

Given an element \( f \in A((t))^{\times} \), it is possible in exactly one way to write
\[ f = a_0 \cdot t^{w(f)} \prod_{i=1}^{\infty} (1 - a_i t^{-i}) \cdot \prod_{i=1}^{\infty} (1 - a_i t^{i}), \]
where
\[ w(f) \in \mathbb{Z}, \]
\[ w(f) = \begin{cases} a_i = 0 & \text{if } i < 0, \\ a_i \in m & \text{if } i < 0, \\ a_i \in A^{m} & \text{if } i = 0, \\ a_i \in A & \text{if } i > 0. \end{cases} \]

The integer number \( w(f) \) is the winding number of \( f \), and the set \( \{a_i\}_{i=-\infty}^{\infty} \) is the family of Witt parameters of \( f \).

Thus, we can consider the morphism of groups
\[ \phi : A((t))^{\times} \rightarrow A((t))^{\times}, \]
\[ f \mapsto a_0. \]

**Remark 3.1 (Characterization of the Contou-Carrère symbol).** It follows from the properties described in Subsection [2.1] that the commutator map \( \{\cdot, \cdot\}_{A[[t]]}^{A[[t]]} \) is the only bimultiplicative map
\[ \{\cdot, \cdot\}_{A[[t]]}^{A[[t]]} : A((t))^{\times} \times A((t))^{\times} \rightarrow A^{\times} \]
that satisfies the conditions:
- \( \{\cdot, \cdot\}_{A[[t]]}^{A[[t]]} = 1. \)
- \( \{f, g\}_{A[[t]]}^{A[[t]]} = \phi(N_g[f]) \) for all \( g \in A((t))^{\times} \) with \( w(g) > 0. \)

Hence, the Contou-Carrère symbol is the only map
\[ \langle \cdot, \cdot \rangle_{A} : A((t))^{\times} \times A((t))^{\times} \rightarrow A^{\times} \]
that satisfies the properties:
- \( \langle \cdot, \cdot \rangle_{A} \) is bimultiplicative.
- \( \langle \cdot, \cdot \rangle_{A} = 1. \)
• \((f, g)_A = (-1)^{w(f)w(g)} \cdot \phi(N_{g}[f])\) for all \(g \in A(t)\) with \(w(g) > 0\).

As far as we know a characterization relating the Contou-Carrere symbol with a norm map is not stated explicitly in the literature.

Let us now consider a series \(s(t) \in A[[t]]^\times\) and an element \(a \in m\). For each positive integer \(n\), setting

\[
s(t) = s_0^n(t^n) + t \cdot s_1^n(t^n) + \cdots + t^{n-1} \cdot s_{n-1}^n(t^n),
\]

we can construct a matrix \(C_{s(t)}^{n,a} \in \text{Gl}(n, A)\) where the coefficients are

\[
[C_{s(t)}^{n,a}]_{ij} = \begin{cases} 
    s_i^n(a) & \text{if } i \geq j, \\
    a \cdot s_{i-j}(a) & \text{if } i < j.
\end{cases}
\]

That is, the expression of the matrix is

\[
C_{s(t)}^{n,a} = \begin{pmatrix} 
    s_0^n(a) & a \cdot s_1^n(a) & \cdots & a \cdot s_{n-1}^n(a) \\
    s_1^n(a) & s_0^n(a) & \cdots & a \cdot s_{n-2}^n(a) \\
    \vdots & \vdots & \ddots & \vdots \\
    s_{n-2}^n(a) & s_{n-3}^n(a) & \cdots & s_0^n(a) \\
    s_{n-1}^n(a) & s_{n-2}^n(a) & \cdots & s_1^n(a)
\end{pmatrix}.
\]

Note that in \((3.1)\) the series \(s_0^n(t) \in A[[t]]^\times\). Moreover, for all \(s(t) \in A[[t]]^\times\) and \(a \in m\), one has that

\[
\det C_{s(t)}^{n,a} = \phi(s(t)) = s(a).
\]

We shall now give an explicit expression of the Contou-Carrere symbol by using the determinants of the matrices \(C_{s(t)}^{n,a}\).

**Lemma 3.2.** If \(n > 0\), \(a \in m\) and \(s(t) \in A[[t]]^\times\), one has that

\[
(s(t), t^n - a)_A = \det C_{s(t)}^{n,a}.
\]

**Proof.** Viewing \(A((t))\) via the homomorphism \(h \mapsto h \circ (t^n - a)\) as a free \(A((t))\)-module of rank \(n\) with basis \(\{1, \ldots, t^{n-1}\}\), from the relations

\[
s_i^n(t^n) = s_i^n(a) + (t^n - a) \cdot s_i^n(t^n - a),
\]

\[
t^n \cdot s_i^n(t^n) = a \cdot s_i^n(a) + (t^n - a) \cdot s_i^n(t^n - a),
\]

we have that the matrix of the homothety \(h_{s(t)}(t)\) is obtained from the equality

\[
h_{s(t)}(1) = s_0^n(t^n) + t \cdot s_1^n(t^n) + \cdots + t^{n-1} \cdot s_{n-1}^n(t^n)
\]

\[
= ([s_0^n(a) + t \cdot s_0^n(t)] \circ (t^n - a)) + \cdots + t^{n-1} \cdot [s_{n-1}^n(a) + t \cdot s_{n-1}^n(t)] \circ (t^n - a)
\]

\[
\equiv (s_0^n(a) + t \cdot s_0^n(t), \ldots, s_{n-1}^n(a) + t \cdot s_{n-1}^n(t))
\]

and from the expressions

\[
h_{s(t)}(t^i) = t^i \cdot s_0^n(t^n) + \cdots + t^{n-1}
\]

\[
\cdot s_{n-1}^n(t^n) + t^n + s_{n-1}^n(t^n) + \cdots + t^{n-1} \cdot t^n \cdot s_{n-1}^n(t^n)
\]

\[
= t^i \cdot ([s_0^n(a) + t \cdot s_0^n(t)] \circ (t^n - a)) + \cdots + t^{n-1}
\]

\[
\cdot [s_{n-1}^n(a) + t \cdot s_{n-1}^n(t)] \circ (t^n - a)
\]

\[
\equiv (a \cdot s_{n-i}^n(a) + t \cdot s_{n-i}^n(t), \ldots, s_{n-1}^n(a) + t \cdot s_{n-1}^n(t))
\]

when \(i \geq 1\).
Thus, bearing in mind the definition of $C_{s(t)}^{n,a}$, since $N_{N - a}[s(t)] \in A[[t]]^\times$ and
$(s(t), t^n - a)_A = \phi(N_{N - a}[s(t)])$,
the claim is deduced.

A direct consequence of Lemma 3.2 is the following result that appeared in [2].

**Corollary 3.3.** For every $s(t) \in A[[t]]^\times$ and $a \in m$, one has that
$(s(t), t - a)_A = s(a)$.

Given two elements $f, g \in A((t))^\times$, let us now write

$$f = t^{-N} \cdot s(t) \prod_{i=1}^k (t^i - a_{-i}), \quad g = t^{-M} \cdot s'(t) \prod_{j=1}^h (t^j - b_{-j}),$$

with $N, M \in \mathbb{Z}^+$; $a_{-i}, b_{-j} \in m$; $s(t), s'(t) \in A[[t]]^\times$; $w(f) = \frac{k(k+1)}{2} - N$ and $w(g) = \frac{b_0(k+1)}{2} - M$.

**Proposition 3.4 (Explicit expression of the Contou-Carr`ere symbol).** With the previous notation, if $\phi(f) = a_0$ and $\phi(g) = b_0$, the Contou-Carr`ere symbol is

$$(f, g)_A = (-1)^{w(f)w(g)} \frac{b_0^N \prod_{j=1}^h \det C^{t,j}_{s(t)}}{a_0^M \prod_{i=1}^k \det C^{t,i-1}_{s(t)}}.$$

**Proof.** The expression follows from Lemma 3.2 bearing in mind that

- $(t^{-N}, t^{-M})_A = (-1)^{N,M}$;
- $(t^{-N}, t^{-j} - b_{-j})_A = (-1)^{j,N}$;
- $(t^i - a_{-i}, t^{-M})_A = (-1)^{i,M}$;
- $(t^{-N}, s'(t))_A = b_0^M$;
- $(s(t), t^{-M})_A = a_0^N$;
- $(t^j - b_{-j})_A = (-1)^j$.

**Corollary 3.5.** Given two elements $f, g \in A((t))^\times$, such that

$$f = t^{-N} \cdot a_0 \cdot \bar{s}(t) \prod_{i=1}^k (t^i - a_{-i}), \quad g = t^{-M} \cdot b_0 \cdot \bar{s}'(t) \prod_{j=1}^h (t^j - b_{-j}),$$

with $N, M \in \mathbb{Z}^+$; $a_0, b_0 \in A^\times$; $a_{-i}, b_{-j} \in m$; $\bar{s}(t), \bar{s}'(t) \in A[[t]]^\times$; $w(f) = \frac{k(k+1)}{2} - N$ and $w(g) = \frac{b_0(k+1)}{2} - M$, the Contou-Carr`ere symbol is

$$(f, g)_A = (-1)^{w(f)w(g)} \frac{a_0^{w(f)} \prod_{j=1}^h \det C^{t,j}_{\bar{s}(t)}}{b_0^{w(g)} \prod_{i=1}^k \det C^{t,i-1}_{\bar{s}'(t)}}.$$

**Proof.** Since $(a_0, t^j - b_{-j})_A = a_j^0$ and $(t^i - a_{-i}, b_0)_A = b_0^j$, this formula follows immediately from the explicit expression given in Proposition 3.4.

Finally, as an application of this explicit formula of the Contou-Carr`ere symbol, we shall study the map $\operatorname{Res}^W(\cdot, \cdot) : F((t))^\times \times F((t)) \to F$ defined in Subsection 2.2 by the rule

$$(f, 1 - \epsilon g)_A((t))^\times \equiv 1 - \epsilon[\operatorname{Res}^W(f, g)] \mod (\epsilon^2).$$
Note that if \( g = \sum_{j \geq -h} b_j t^j \), then
\[
1 - \epsilon g = t^{h+1} \cdot (1 - \epsilon b_0) \cdot s'(t) \cdot \prod_{j=1}^{h} (t^j - \epsilon b_j),
\]
where \( s'(t) \in A[[t]]^\times \) with \( A = F[x]/(x^2) \).
Thus, if \( f = t^w(f) \cdot a_0 \cdot s(t) \), we have that
\[
\langle f, 1 - \epsilon g \rangle_{A((t))} = \prod_{j=1}^{h} \det C^{j,eb_j}_{s(t)} = \prod_{j=1}^{h} \det C^{j,eb_j}_{s(t)} / [1 - \epsilon w(f) b_0].
\]
Accordingly, if \( \text{Tr} A \) denotes the trace of a square matrix \( A \), bearing in mind that
\[
C^{j,eb_j}_{s(t)} = A^{j}_{s(t)} \cdot (\text{Id} + \epsilon \cdot b_{-j} \cdot [A^{j}_{s(t)}]^{-1} \cdot B^{j}_{s(t)}),
\]
with
\[
A^{j}_{s(t)} = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
a_1 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
a_{j-2} & \cdots & a_1 & 1 & 0 \\
a_{j-1} & \cdots & a_2 & a_1 & 1
\end{pmatrix}
\]
and
\[
B^{j}_{s(t)} = \begin{pmatrix}
a_j & a_{j-1} & \cdots & \cdots & a_1 \\
a_{j+1} & a_j & a_{j-1} & \cdots & a_2 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
a_{2j-2} & \cdots & a_{j+1} & a_j & a_{j-1} \\
a_{2j-1} & \cdots & a_{j+2} & a_{j+1} & a_j
\end{pmatrix},
\]
where \( s(t) = 1 + \sum_{i \geq 1} a_i \cdot t^i \), we have that
\[
\det C^{j,eb_j}_{s(t)} = 1 + \epsilon \cdot b_{-j} \cdot \text{Tr}([A^{j}_{s(t)}]^{-1} \cdot B^{j}_{s(t)}).
\]
Hence, since
\[
([A^{j}_{s(t)}]^{-1})_{\alpha\beta} = \begin{cases}
1 & \text{if } \alpha = \beta, \\
0 & \text{if } \alpha < \beta, \\
\sum_{i_1 + \cdots + i_t = \alpha - \beta} (-1)^t a_{i_1} \cdots a_{i_t} & \text{if } \alpha > \beta,
\end{cases}
\]
then
\[
\text{Tr}([A^{j}_{s(t)}]^{-1} \cdot B^{j}_{s(t)}) = j \cdot a_j + \sum_{r=1}^{j-1} \left[ \sum_{i_1 + \cdots + i_t = r} (-1)^t a_{i_1} \cdots a_{i_t} \right] a_{j-r},
\]
and the explicit expression of the Witt Residue is:
\[
\text{Res}^W(f, g) = -w(f) \cdot b_0 - \sum_{j=1}^{h} b_{-j} \cdot (j \cdot a_j + \sum_{r=1}^{j-1} \sum_{i_1 + \cdots + i_t = r} (-1)^t a_{i_1} \cdots a_{i_t} \cdot a_{j-r})
\]
for \( f = t^w(f) \cdot a_0 \cdot [1 + \sum_{i \geq 1} a_i \cdot t^i] \) and \( g = \sum_{j \geq -h} b_j t^j \).
Remark 3.6 (Residue Theorem). Let $X$ be a nonsingular and irreducible curve over the field $F$ and let $\Sigma_X$ be its function field. Fixing a closed point $x \in X$, we have an immersion of rings

$$i_x: \Sigma_X \hookrightarrow (\hat{O}_x)_0 \simeq F((t)),$$

such that, by restriction of $\text{Res}^W (\cdot, \cdot)$, we can consider the map

$$\text{Res}^W_x (\cdot, \cdot): \Sigma_X \times \Sigma_X \to F,$$

which is the Witt Residue associated with the closed point $x$.

When $X$ is a complete curve, a direct consequence of the reciprocity law for the Contou-Carrèrè symbol [1] is the following Residue Theorem:

$$\sum_{x \in X} \text{Res}^W_x (f, g) = 0$$

for every $f \in \Sigma_X^*$ and $g \in \Sigma_X$.

**ACKNOWLEDGMENT**

The author thanks the anonymous referee for her/his constructive comments aimed at improving the preliminary version of this paper.

**REFERENCES**


Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, España

E-mail address: fpablos@usal.es