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A NEW PROOF AND GENERALIZATIONS OF GEARHART'S THEOREM

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ABSTRACT. Let H be a Hilbert space, let $AP(\mathbf{R}, H)$ be the space of almost periodic functions from $\mathbf{R}$ to H, and let A be a closed densely defined linear operator on H. For a closed subset $\Lambda \subset \mathbf{R}$, let $M(\Lambda)$ be the subspace of $AP(\mathbf{R}, H)$ consisting of functions with spectrum contained in Λ. We prove that the following properties are equivalent: (i) for every function $f \in M(\Lambda)$ there exists a unique mild solution $u \in M(\Lambda)$ of equation $u'(t) = Au(t) + f(t)$; (ii) $i\Lambda \subset \rho(A)$ and $\sup_{\lambda\in\Lambda} \|(i\lambda - A)^{-1}\| < \infty$. The case $\Lambda = \{2\pi k : k = 0, \pm1, \pm2, ...\}$ yields a new proof of the well-known Gearhart's spectral mapping theorem.

1.

Let H be a Hilbert space and let $T(t)$, $t \geq 0$, be a strongly continuous semigroup (C_0-semigroup) of bounded linear operators on H, with the generator A. The following is the well-known Gearhart's spectral mapping theorem. It was proved by Gearhart [2] for contraction semigroups and later independently by Herbst [3], Howland [4] and Prüss [7] for C_0-semigroups (see also [6], p. 95).

Theorem 1. *The following are equivalent:*

(i) $1 \in \rho(T(1))$;

(ii) $2\pi ki \in \rho(A)$ *for every* $k \in \mathbf{Z}$ *and* $\sup_{k\in\mathbf{Z}} \|(2\pi ki - A)^{-1}\| < \infty$;

(iii) *for every* 1*-periodic continuous function* $f : \mathbf{R} \to H$, *there exists a unique* 1*-periodic mild solution of the equation*

$$u'(t) = Au(t) + f(t). \tag{1}$$

In this note, we prove the following generalization of this theorem. Note that the notion of almost periodic functions used in Theorem 2 is in the sense of Hilbert space (see Section 2 for a precise definition).

Theorem 2. *Let A be a closed densely defined linear operator on a Hilbert space H and let Λ be a closed subset of* $\mathbf{R}$. *Then the following are equivalent:*

(i) *For every almost periodic function $f : \mathbf{R} \to H$ such that $\sigma(f) \subset \Lambda$, there exists a unique almost periodic mild solution u of* (1) *such that* $\sigma(u) \subset \Lambda$;

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(ii) $i\Lambda \subset \rho(A)$ *and*

$$\sup_{\lambda\in\Lambda} \|(i\lambda - A)^{-1}\| < \infty. \tag{2}$$

Since f is 1-periodic if and only if $\sigma(f) \subset \{2\pi k : k = 0, \pm 1, \pm 2, ...\}$, the equivalence of (ii) and (iii) in Theorem 1 (which is the main part of the theorem) is a particular case of Theorem 2.

Note that we do not assume that A is a generator of a C_0-semigroup.

2.

Let H be a Hilbert space with the inner product denoted by $(x, y)_H,\ x, y \in H$. Let $AP_b(\mathbf{R}, H)$ be the space of Bohr's almost periodic functions defined on $\mathbf{R}$ with values in H. In $AP_b(\mathbf{R}, H)$ the following limit (mean) exists:

$$\langle f, g\rangle = M\{f, g\} := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (f(t), g(t))_H dt$$

and defines an inner product. Thus, $AP_b(\mathbf{R}, H)$ is a pre-Hilbert space and its completion, denoted by $AP(\mathbf{R}, H)$, is a Hilbert space. Below we will denote the inner product and norm in $AP(\mathbf{R}, H)$ by $\langle\ ,\ \rangle_{AP}$ and $\|\ \|_{AP}$, respectively.

In $AP(\mathbf{R}, H)$, the family of functions $e_{\lambda,x}(t) = e^{i\lambda t}x,\ \lambda \in \mathbf{R}$ and $x \in H$, form a complete system (which are orthogonal for different λ's). If x_α form an orthonormal basis in H, then $e_{\lambda,x_\alpha}(t) = e^{i\lambda t}x_\alpha$ form an orthonormal basis in $AP(\mathbf{R}, H)$.

For each $f \in AP(\mathbf{R}, H)$, the Fourier-Bohr transform is defined by

$$a(\lambda, f) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\lambda t}dt. \tag{3}$$

The set $\sigma(f) := \{\lambda \in \mathbf{R} : a(\lambda, f) \neq 0\}$ is called the *Bohr spectrum* of f. It is well known that $\sigma(f)$ is (at most) countable. The Fourier-Bohr series of f is

$$\sum_{\lambda\in\sigma(f)} a(\lambda, f)e^{i\lambda t},$$

and it converges to f (in the topology of $AP(\mathbf{R}, H)$). Moreover, the following Parseval's equality holds:

$$\|f\|_{AP}^2 = \sum_{\lambda\in\sigma(f)} \|a(\lambda, f)\|_H^2,\ f \in AP(\mathbf{R}, H).$$

In the sequel, if a function f in $AP(\mathbf{R}, H)$ has a Fourier-Bohr series

$$\sum_{\lambda\in\sigma(f)} a(\lambda, f)e^{i\lambda t},$$

then we will write

$$f \sim \sum_{\lambda\in\sigma(f)} a(\lambda, f)e^{i\lambda t}.$$

We will also frequently use the following equality, which is valid for every $u, v \in AP(\mathbf{R}, H)$:

$$\langle u, v\rangle_{AP} = \sum_{\lambda} (a(\lambda, u), a(\lambda, v))_H$$

(the sum is over a countable set of exponents λ). In particular,

$$\langle u, e^{i\lambda t}x\rangle_{AP} = (a(\lambda, u), x)_H, \text{ for all } u \in AP(\mathbf{R}, H),\ x \in H.$$

Note that there is a family of orthogonal projections $P_\lambda,\ \lambda \in \mathbf{R}$, on $AP(\mathbf{R}, H)$ defined by $P(\lambda)f = e^{i\lambda t}a(\lambda, f)$, which satisfies $P_\lambda P_\mu = 0$ if $\lambda \neq \mu$. It is clear that $H_\lambda := P_\lambda H = \{e^{i\lambda t}x : x \in H\}$. The family $H_\lambda, \lambda \in \mathbf{R}$, is pairwise orthogonal and complete in $AP(\mathbf{R}, H)$. For these and other facts about almost periodic functions, we refer the reader to [5].

Consider the translation group $S(t), -\infty < t < \infty$, on $AP(\mathbf{R}, H)$. The operators $S(t)$ are first defined for functions f in $AP_b(\mathbf{R}, H)$ by $(S(t)f)(\cdot) = f(\cdot + t)$, and extended to $AP(\mathbf{R}, H)$ by continuity. It is clear that $S(t)$ is a strongly continuous group of unitary operators. Let $\mathcal{D}$ be the generator of $S(t)$. Then $\mathcal{D}$ is a skew self-adjoint operator on $AP(\mathbf{R}, H)$, i.e. $\mathcal{D}^* = i\mathcal{D}$, and is the closure of the operator of differentiation, with the natural domain.

Let A be a closed, densely defined linear operator on a Hilbert space H. The operator A generates an operator $\mathcal{A}$ on $AP(\mathbf{R}, H)$ in a natural manner. Namely, we define $\mathcal{A}$ on $AP(\mathbf{R}, H)$ by

$$D(\mathcal{A}) := \{f \in AP(\mathbf{R}, H) : a(\lambda, f) \in D(A) \text{ for all } \lambda \in \sigma(f) \\ \text{and } \textstyle\sum_{\lambda\in\sigma(f)} \|Aa(\lambda, f)\|_H^2 < \infty\}$$

and

$$(\mathcal{A}f) \sim \sum_{\lambda\in\sigma(f)} Aa(\lambda, f)e^{i\lambda t}, \text{ for } f \in D(\mathcal{A}).$$

Lemma 3. *$\mathcal{A}$ is a densely defined closed operator and $\sigma(\mathcal{A}) = \sigma(A)$.*

Proof. It is clear that $D(\mathcal{A})$ contains linear combinations of functions of the form $e^{i\lambda t}x$, with $\lambda \in \mathbf{R}$ and $x \in D(A)$. From this it is easily seen that $D(\mathcal{A})$ is dense in $AP(\mathbf{R}, H)$. This implies that $\mathcal{A}^*$ is well defined (and densely defined closed). Moreover, for every $f \in D(\mathcal{A})$ with the Fourier-Bohr series $\sum_{\lambda\in\sigma(f)} a(\lambda, f)e^{i\lambda t}$, and for every $x \in D(A^*)$, we have

$$\langle \mathcal{A}f, e^{i\xi t}x\rangle = (Aa(\xi, f), x)_H = (a(\xi, f), A^*x)_H = \langle f, e^{i\xi t}A^*x\rangle_{AP},$$

which implies that $e^{i\xi t}x \in D(\mathcal{A}^*)$ and $\mathcal{A}^*(e^{i\xi t}x) = e^{i\xi t}A^*x$.

Now assume that $f_n \in D(\mathcal{A}),\ f_n \to f,\ \mathcal{A}f_n \to g$. We must show that $f \in D(\mathcal{A})$ and $\mathcal{A}f = g$. Let

$$f \sim \sum a(\lambda, f)e^{i\lambda t}, \quad f_n \sim \sum a(\lambda, f_n)e^{i\lambda t}, \quad g \sim \sum a(\lambda, g)e^{i\lambda t}.$$

Since $f_n \in D(\mathcal{A})$ and A is closed, we have $a(\lambda, f_n) \in D(A)$ and $Aa(\lambda, f_n) \to a(\lambda, g)$. Moreover, for every $h \in D(\mathcal{A}^*)$ we have

$$\langle f, \mathcal{A}^*h\rangle_{AP} = \lim_{n\to\infty}\langle f_n, \mathcal{A}^*h\rangle_{AP} = \lim_{n\to\infty}\langle \mathcal{A}f_n, h\rangle_{AP} = \langle g, h\rangle_{AP}.$$

In particular, for every $x \in D(A^*)$, we have $h(t) = e^{i\lambda t}x \in D(\mathcal{A}^*)$, $\mathcal{A}^*h = e^{i\lambda t}A^*x$ and

$$\langle f, \mathcal{A}^*h\rangle_{AP} = \langle f, e^{i\lambda t}A^*x\rangle_{AP} = (a(\lambda, f), A^*x)_H = \langle g, e^{i\lambda t}x\rangle_{AP} = (a(\lambda, g), x)_H.$$

This implies that $Aa(\lambda, f) = a(\lambda, g)$, so that $f \in D(\mathcal{A})$ and $\mathcal{A}f = g$. Finally, we show that $\sigma(\mathcal{A}) = \sigma(A)$. If $\lambda \in \rho(A)$, then the operator $\mathcal{B}$ defined by $(\mathcal{B}f)(t) = \sum_{\xi\in\sigma(f)}(\lambda - A)^{-1}a(\xi, f)e^{i\xi t}$ is easily seen to be the bounded inverse of $(\lambda - \mathcal{A})$, hence $\lambda \in \rho(\mathcal{A})$, or $\sigma(\mathcal{A}) \subset \sigma(A)$.

Conversely, if $\lambda \in \rho(\mathcal{A})$, then $\mathcal{A} - \lambda$ has a dense range and satisfies

$$\|(\mathcal{A} - \lambda)f\|_{AP} \geq \delta\|f\|_{AP}$$

for some $\delta > 0$ and all $f \in D(\mathcal{A})$. This implies that $(A-\lambda)$ has a dense range and $\|(A-\lambda)x\|_H \geq \delta\|x\|_H$ for all $x \in D(A)$, so that $\lambda \in \rho(A)$.

Below we denote by $L = \mathcal{D} - \mathcal{A}$ the operator on $AP(\mathbf{R}, H)$ defined by $D(L) = D(\mathcal{D}) \cap D(\mathcal{A})$ and $Lf = \mathcal{D}f - \mathcal{A}f$ for all $f \in D(L)$.

Lemma 4. *The operator $L = \mathcal{D} - \mathcal{A}$ is densely defined and closable.*

Proof. Since $D(\mathcal{D})$ and $D(\mathcal{A})$ contain linear combinations of functions $e^{i\lambda t}x, \lambda \in \mathbf{R}, x \in D(A)$, it follows that L is densely defined. For $v(t) = \sum_{k=1}^{n} e^{\lambda_k t} x_k$ with $\lambda_k \in \mathbf{R}, x_k \in D(A^*)$, let $Kv = \sum_{k=1}^{n}[(i\lambda_k) - A^*]x_k e^{i\lambda_k t}$. It is easily seen that

$$\langle Lf, v\rangle_{AP} = \langle f, Kv\rangle_{AP}$$

for each $f = \sum_{j=1}^{m} y_j e^{i\gamma_j t}, y_j \in D(A)$. Hence, $Kv = L^*v$, so that L^* is densely defined. This implies that L is closable (and its closure is L^{**}).

Below, we denote by $(\mathcal{D} - \mathcal{A})^-$ the closure of $\mathcal{D} - \mathcal{A}$.

For every closed subset $\Lambda \subset \mathbf{R}$, we denote by $M(\Lambda)$ a subspace of $AP(\mathbf{R}, H)$ consisting of functions g such that $\sigma(g) \subset \Lambda$.

Lemma 5. *Let Λ be a closed non-empty subset of $\mathbf{R}$. Then*

(i) *$M(\Lambda)$ is a closed invariant subspace with respect to $S(t), \mathcal{D}, \mathcal{A}$;*

(ii) *$\sigma(\mathcal{D}|M(\Lambda)) = i\Lambda$, $\sigma(\mathcal{A}|M(\Lambda)) = \sigma(A)$;*

(iii) *$\mathcal{D}|M(\Lambda)$ is bounded if (and only if) Λ is compact.*

Proof. (i) It is obvious that $M(\Lambda)$ is linear and invariant with respect to $S(t), \mathcal{D}$ and $\mathcal{A}$. Suppose $g_n \in M(\Lambda)$ and $\|g_n - g\|_{AP} \to 0$ as $n \to \infty$. This implies $\|a(\lambda, g_n) - a(\lambda, g)\|_H \to 0$. Since $\sigma(g_n) \subset \Lambda$, we have $a(\lambda, g_n) = 0$ for all $\lambda \notin \Lambda$, which implies $a(\lambda, g) = 0$ for all $\lambda \notin \Lambda$, or $\sigma(g) \subset \Lambda$. Hence $M(\Lambda)$ is closed.

(ii) If $\lambda \in \Lambda,\ x \in H$, then $h(t) = e^{i\lambda t}x \in D(\mathcal{D}) \cap M(\Lambda)$ and $\mathcal{D}h = i\lambda h$. Hence $i\lambda \in \sigma(\mathcal{D}|_{M(\Lambda)})$, which implies $i\Lambda \subset \sigma(\mathcal{D}|_{M(\Lambda)})$.

Suppose now that $\lambda_0 \notin \Lambda$. Define, for $\lambda_k \in \Lambda,\ x_k \in H$,

$$R(\sum_{k=1}^{n} e^{i\lambda_k t}x_k) = \sum_{k=1}^{n}(i\lambda_k - i\lambda_0)^{-1}e^{i\lambda_k t}x_k.$$

It is clear that

$$\begin{aligned}\|R\sum_{k=1}^{n} e^{i\lambda_k t}x_k\|_{AP}^2 &= \|\sum_{k=1}^{n}(i\lambda_k - i\lambda_0)^{-1}e^{i\lambda_k t}x_k\|_{AP}^2 \\ &= \sum_{k=1}^{n}|(i\lambda_k - i\lambda_0)^{-1}|^2\|x_k\|^2 \\ &\leq (\sup_{\lambda\in\Lambda}|\lambda - \lambda_0|^{-1})^2\|\sum_{k=1}^{n} e^{i\lambda_k t}x_k\|_{AP}^2,\end{aligned}$$

hence R can be extended to a bounded operator on $M(\Lambda)$. It is easily verified that R is the inverse to $(\mathcal{D} - i\lambda_0)|_{M(\Lambda)}$, hence $i\lambda_0 \notin \sigma(\mathcal{D}|_{M(\Lambda)})$. The proof of $\sigma(\mathcal{A}|M(\Lambda)) = \sigma(A)$ is analogous to that of $\sigma(\mathcal{A}) = \sigma(A)$ in Lemma 3.

(iii) The operator $\mathcal{D}|M(\Lambda)$, being skew self-adjoint, is bounded if and only if its spectrum, $i\Lambda$, is compact.

Assume that $f \in AP(\mathbf{R}, H)$. A function $u \in AP(\mathbf{R}, H)$ is called a *mild solution* of (1) if $u \in D((\mathcal{D} - \mathcal{A})^-)$ and $(\mathcal{D} - \mathcal{A})^-u = f$. The space $M(\Lambda)$ is called *regularly* *admissible* (w.r.t. (1)) if for every $f \in M(\Lambda)$, (1) has a unique mild solution u in $M(\Lambda)$.

3.

Let Λ be a closed subset of $\mathbf{R}$. If follows from Lemma 5(i) that $M(\Lambda)$ is invariant under $(\mathcal{D}-\mathcal{A})^-$, so that $(\mathcal{D}-\mathcal{A})^-|M(\Lambda)$ is defined. Assume that $M(\Lambda)$ is regularly admissible. Then we define a linear operator K_Λ on $M(\Lambda)$, called the *solution operator*, by putting $K_\Lambda f = u$, where u is the unique (mild) solution in $M(\Lambda)$ of (1). A standard argument, using the Closed Graph Theorem, shows that K_Λ is a bounded operator on $M(\Lambda)$. Moreover, $(\mathcal{D}-\mathcal{A})^- u = (\mathcal{D}-\mathcal{A})^- K_\Lambda f = f$ (for all f in $M(\Lambda)$). Therefore the operator $(\mathcal{D}-\mathcal{A})^-|M(\Lambda)$ is invertible (with the inverse equal to K_Λ). In particular, for every $\lambda_0 \in \Lambda$ and $y \in H$, there exists a unique $x \in H$ such that $e^{i\lambda_0 t}x$ is the unique mild solution in $M(\Lambda)$ of (1), with $f(t) = e^{i\lambda_0 t}y$, which implies that for every $y \in H$ there exists a unique $x \in H$ such that $(i\lambda_0 - A)x = y$, i.e. $(i\lambda_0 - A)$ is invertible. Thus, $\sigma(A) \cap i\Lambda = \emptyset$. From this the following lemma, which is a version of [8], Theorem 3.1, follows.

Lemma 6 (cf. [8], Theorem 3.1)**.** *Let Λ be a non-empty closed subset of $\mathbf{R}$. Then $M(\Lambda)$ is regularly admissible if and only if $(\mathcal{D}-\mathcal{A})^-|M(\Lambda)$ is invertible. If $M(\Lambda)$ is regularly admissible, then $i\Lambda \cap \sigma(A) = \emptyset$.*

If Λ is compact, then $\mathcal{D}|M(\Lambda)$ is bounded and, therefore, $(\mathcal{D}-\mathcal{A})^-|M(\Lambda)$ is invertible whenever $\sigma(\mathcal{D}|M(\Lambda)) \cap \sigma(\mathcal{A}|M(\Lambda)) = \emptyset$, or $i\Lambda \cap \sigma(A) = \emptyset$ (see [1]). Moreover, K_Λ is given by the following analog of the Krein-Rosenblum integral formula (cf. [8], p. 397, [1]):

$$K_\Lambda = \frac{1}{2\pi i}\int_\Gamma (\lambda - \mathcal{A}_M)^{-1}(\lambda - \mathcal{D}_M)^{-1} d\lambda, \tag{4}$$

where Γ is a Cauchy contour which surrounds $\sigma(\mathcal{D}_M)$ $(= i\Lambda)$ and is separated from $\sigma(A_M)$ $(= \sigma(A))$, and $\mathcal{A}_M$ and $\mathcal{D}_M$ are restrictions of $\mathcal{A}$ and $\mathcal{D}$, respectively, to $M(\Lambda)$. Thus, the following analog of ([8], Theorem 3.3-i) holds.

Lemma 7 ([8], Theorem 3.3-i)**.** *If Λ is compact and $i\Lambda \cap \sigma(A) = \emptyset$, then $M(\Lambda)$ is regularly admissible and the solution operator K_Λ is given by* (4).

Assume that $i\Lambda \cap \sigma(A) = \emptyset$. In general, this condition does not imply that $M(\Lambda)$ is regularly admissible. Let $\Lambda_\alpha, \alpha \in \Omega$, be a family of compact subsets of Λ such that $span\{M(\Lambda_\alpha) : \alpha \in \Omega\}$ is dense in $M(\Lambda)$. Then from $\Lambda \cap \sigma(A) = \emptyset$ it follows that $\Lambda_\alpha \cap \sigma(A) = \emptyset$ for all α. According to Lemma 7, each subspace $M(\Lambda_\alpha)$ is regularly admissible. Let K_{Λ_α} be the solution operator on $M(\Lambda_\alpha)$. From the uniqueness of K_{Λ_α} it follows that if $\Lambda_\alpha \subset \Lambda_\beta$, then $K_{\Lambda_\beta}|M(\Lambda_\alpha) = K_{\Lambda_\alpha}$. Therefore, one can correctly define an operator K_0 with dense domain $D(K_0) = span\{M(\Lambda_\alpha) : \alpha \in \Omega\}$, by putting $K_0|M(\Lambda_\alpha) = K_{\Lambda_\alpha}$. If $M(\Lambda)$ is regularly admissible and K_Λ is the corresponding solution operator, then $K_\Lambda|M(\Lambda_\alpha) = K_{\Lambda_\alpha}$, hence $\sup_{\alpha\in\Omega}\|K_{\Lambda_\alpha}\| = \|K_\Lambda\| < \infty$. Conversely, if $\sup_\alpha \|K_{\Lambda_\alpha}\| = L < \infty$, then $\|K_0\| \le L$ so that K_0 can be extended by continuity to a bounded operator K_Λ on $M(\Lambda)$, which is the inverse of $(\mathcal{D}-\mathcal{A})^-|M(\Lambda)$. Thus, the following statement, which is a version of [8], Theorem 2.2 and Theorem 3.4, holds.

Lemma 8. *Assume that $i\Lambda \cap \sigma(A) = \emptyset$. Let $\Lambda_\alpha, \alpha \in \Omega$, be a family of compact subsets such that $span\{M(\Lambda_\alpha) : \alpha \in \Omega\}$ is dense in $M(\Lambda)$. Then $M(\Lambda)$ is regularly admissible if and only if $\sup_{\alpha\in\Omega}\|K_{\Lambda_\alpha}\| < \infty$.*

4.

Proof of Theorem 2. In light of Lemma 6, we can assume that $\sigma(A) \cap i\Lambda = \emptyset$. Let $\lambda_k \in \Lambda$ and $\Lambda_\alpha = \{\lambda_1, \lambda_2, ..., \lambda_n\}$. By Lemma 7, $M(\Lambda_\alpha)$ is regularly admissible. Let $K_{\Lambda_\alpha} : M(\Lambda_\alpha) \to M(\Lambda_\alpha)$ be the corresponding solution operator. Consider the function $g(t) = \sum_{k=1}^n e^{i\lambda_k t} x_k$, where $x_k,\ k = 1, 2, ..., n$, are arbitrary vectors in H. It is directly verified that $u(t) := \sum_{k=1}^n e^{i\lambda_k t}(i\lambda_k - A)^{-1} x_k$ is a (classical) solution in $M(\Lambda_\alpha)$ of equation $u'(t) = Au(t) + g(t)$, hence $K_{\Lambda_\alpha} g = \sum_{k=1}^n e^{i\lambda_k t}(i\lambda_k - A)^{-1} x_k$. According to Lemma 8, $M(\Lambda)$ is regularly admissible if and only if $\sup_\alpha \|K_{\Lambda_\alpha}\| < \infty$, that is, if and only if there exists $L > 0$ such that

$$\|\sum_{k=1}^n e^{i\lambda_k t}(\lambda_k - A)^{-1} x_k\|_{AP} \le L \|\sum_{j=1}^n e^{i\lambda_j t} x_j\|_{AP} \tag{5}$$

for every $x_j \in H, \lambda_j \in \Lambda, 1 \le j \le n$. By Parseval's equality

$$\|\sum_{k=1}^n e^{i\lambda_k t}(i\lambda_k - A)^{-1} x_k\|_{AP}^2 = \sum_{k=1}^n \|(i\lambda_k - A)^{-1} x_k\|_H^2,$$

$$\|\sum_{j=1}^n e^{i\lambda_j t} x_j\|_{AP}^2 = \sum_{j=1}^n \|x_j\|_H^2.$$

Hence, (5) is equivalent to

$$\sum_{k=1}^n \|(i\lambda_k - A)^{-1} x_k\|_H^2 \le L \sum_{j=1}^n \|x_j\|_H^2, \tag{6}$$

for all $x_1, x_2, ..., x_n$ in H and $\lambda_1, ..., \lambda_n$ in Λ.

Therefore, it remains to show that (2) and (6) are equivalent, which is obvious. □

5.

Let $\Lambda_1 = \{2\pi k : k \in \mathbf{Z}\}$. Then $M(\Lambda_1)$ can be naturally identified with the space $L^2([0,1], H)$.

Corollary 9. *The following are equivalent:*

(i) *For every function $f \in L^2([0,1], H)$, there exists a unique mild solution $u \in L^2([0,1], H)$ of* (1)*;*

(ii) *$i2\pi k \in \rho(A)$ for all $k \in \mathbf{Z}$ and $\sup_{k\in\mathbf{Z}} \|(i2\pi k - A)^{-1}\| < \infty$.*

Now let A be the generator of a C_0-semigroup $T(t)$ on H. Assuming that $M(\Lambda_1)$ is regularly admissible, we show that mild solutions in our definition are mild solutions in the standard sense of the theory of C_0-semigroups (see e.g. [6]), i.e. the following equation holds:

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau) f(\tau) d\tau \ (t \ge s). \tag{7}$$

Proposition 10. *Under the conditions of Corollary 9, for every $f \in L^2([0,1], H)$ the unique mild solution u in $L^2([0,1], H)$ is a continuous 1-periodic function and satisfies* (7).

Proof. Let $f_n = \sum_{k=-n}^{n} e^{i2\pi kt}\hat{f}(k)$ and $u_n = \sum_{k=-n}^{n} e^{i2\pi kt}(i2\pi - A)^{-1}\hat{f}(k)$, where $\hat{f}(k)$ and $\hat{u}(k)$ are Fourier coefficients of f and u, respectively. Using the well-known property

$$A\int_0^t T(s)x ds = T(t)x - x$$

(which is valid for arbitrary semigroups $T(t)$, with generator A, and arbitrary $x \in H$), we obtain

$$(A - i2\pi k)\int_0^t T(s)e^{-i2\pi ks}\hat{f}(k)ds = e^{-i2\pi kt}T(t)\hat{f}(k) - \hat{f}(k),$$

which implies

$$\begin{aligned} &e^{i2\pi kt}(i2\pi k - A)^{-1}\hat{f}(k) \\ &= T(t)[(i2\pi k - A)^{-1}\hat{f}(k)] + \int_0^t T(t-\tau)e^{i2\pi k\tau}\hat{f}(k)d\tau. \end{aligned} \tag{8}$$

From (8) it follows that

$$u_n(t) = T(t)u_n(0) + \int_0^t T(t-\tau)f_n(\tau)d\tau,$$

i.e. u_n is a mild solution of the equation $u'(t) = Au(t) + f_n(t)$ in the traditional sense (of (7)). From the last identity it follows that

$$v_n := [I - T(1)]u_n(0) = \int_0^1 T(1-s)f_n(s)ds,$$

so that $\|v_n - v_m\| \leq \sup_{0\leq t\leq 1}\|T(t)\|\|f_n - f_m\|_{L^2}$, i.e. v_n converges to some $v \in H$. Furthermore

$$\begin{aligned} T(1)u_n(0) &= \int_0^1 T(1-t)T(t)u_n(0) \\ &= \int_0^1 T(1-t)u_n(t)dt - \int_0^1 T(1-t)\int_0^t T(t-\tau)f_n(\tau)d\tau dt, \end{aligned}$$

which also implies that $w_n := T(1)u_n(0)$ is a convergent sequence. Therefore, $u_n(0) = v_n + w_n$ converges to some vector $u_0 \in H$. From

$$\begin{aligned} &\|u_n(t) - u_m(t)\| \\ &\leq \sup_{0\leq t\leq 1}\|T(t)\|\left[\|u_n(0) - u_m(0)\| + \|f_n - f_m\|_{L^2}\right],\ 0 \leq t \leq 1, \end{aligned}$$

it follows that u_n converges to u uniformly on $[0,1]$, so that u is a continuous 1-periodic function. The equality (7) is now immediate.

In conclusion we remark that the presented approach is directly applicable to more general classes of differential, integro-differential and functional-differential equations in a Hilbert space. The details are to be given elsewhere.

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