PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 7, July 2007, Pages 2065–2072
S 0002-9939(07)08691-1
Article electronically published on February 2, 2007

A NEW PROOF AND GENERALIZATIONS OF GEARHART'S THEOREM

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(Communicated by Carmen C. Chicone)

ABSTRACT. Let H be a Hilbert space, let $AP(\mathbf{R}, H)$ be the space of almost periodic functions from $\mathbf{R}$ to H, and let A be a closed densely defined linear operator on H. For a closed subset $\Lambda \subset \mathbf{R}$, let $M(\Lambda)$ be the subspace of $AP(\mathbf{R}, H)$ consisting of functions with spectrum contained in Λ. We prove that the following properties are equivalent: (i) for every function $f \in M(\Lambda)$ there exists a unique mild solution $u \in M(\Lambda)$ of equation $u'(t) = Au(t) + f(t)$; (ii) $i\Lambda \subset \rho(A)$ and $\sup_{\lambda\in\Lambda} \|(i\lambda - A)^{-1}\| < \infty$. The case $\Lambda = \{2\pi k : k = 0, \pm1, \pm2, ...\}$ yields a new proof of the well-known Gearhart's spectral mapping theorem.

1.

Let H be a Hilbert space and let $T(t)$, $t \geq 0$, be a strongly continuous semigroup (C_0-semigroup) of bounded linear operators on H, with the generator A. The following is the well-known Gearhart's spectral mapping theorem. It was proved by Gearhart [2] for contraction semigroups and later independently by Herbst [3], Howland [4] and Prüss [7] for C_0-semigroups (see also [6], p. 95).

Theorem 1. *The following are equivalent:*

(i) $1 \in \rho(T(1))$*;*

(ii) $2\pi ki \in \rho(A)$ *for every* $k \in \mathbf{Z}$ *and* $\sup_{k\in\mathbf{Z}} \|(2\pi ki - A)^{-1}\| < \infty$*;*

(iii) *for every* 1*-periodic continuous function* $f : \mathbf{R} \to H$*, there exists a unique* 1*-periodic mild solution of the equation*

$$u'(t) = Au(t) + f(t). \tag{1}$$

In this note, we prove the following generalization of this theorem. Note that the notion of almost periodic functions used in Theorem 2 is in the sense of Hilbert space (see Section 2 for a precise definition).

Theorem 2. *Let* A *be a closed densely defined linear operator on a Hilbert space* H *and let* Λ *be a closed subset of* $\mathbf{R}$*. Then the following are equivalent:*

(i) *For every almost periodic function* $f : \mathbf{R} \to H$ *such that* $\sigma(f) \subset \Lambda$*, there exists a unique almost periodic mild solution* u *of* (1) *such that* $\sigma(u) \subset \Lambda$*;*

Received by the editors December 29, 2005 and, in revised form, March 2, 2006.
2000 *Mathematics Subject Classification.* Primary 47D06, 35B40.
Key words and phrases. C_0-semigroup, almost periodic, admissible subspace, spectral mapping theorem.

(ii) $i\Lambda \subset \rho(A)$ *and*

$$\sup_{\lambda\in\Lambda} \|(i\lambda - A)^{-1}\| < \infty. \tag{2}$$

Since f is 1-periodic if and only if $\sigma(f) \subset \{2\pi k : k = 0, \pm 1, \pm 2, ...\}$, the equivalence of (ii) and (iii) in Theorem 1 (which is the main part of the theorem) is a particular case of Theorem 2.

Note that we do not assume that A is a generator of a C_0-semigroup.

2.

Let H be a Hilbert space with the inner product denoted by $(x, y)_H,\ x, y \in H$. Let $AP_b(\mathbf{R}, H)$ be the space of Bohr's almost periodic functions defined on $\mathbf{R}$ with values in H. In $AP_b(\mathbf{R}, H)$ the following limit (mean) exists:

$$\langle f, g\rangle = M\{f, g\} := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (f(t), g(t))_H dt$$

and defines an inner product. Thus, $AP_b(\mathbf{R}, H)$ is a pre-Hilbert space and its completion, denoted by $AP(\mathbf{R}, H)$, is a Hilbert space. Below we will denote the inner product and norm in $AP(\mathbf{R}, H)$ by $\langle\ ,\ \rangle_{AP}$ and $\|\ \|_{AP}$, respectively.

In $AP(\mathbf{R}, H)$, the family of functions $e_{\lambda,x}(t) = e^{i\lambda t}x$, $\lambda \in \mathbf{R}$ and $x \in H$, form a complete system (which are orthogonal for different λ's). If x_α form an orthonormal basis in H, then $e_{\lambda,x_\alpha}(t) = e^{i\lambda t}x_\alpha$ form an orthonormal basis in $AP(\mathbf{R}, H)$.

For each $f \in AP(\mathbf{R}, H)$, the Fourier-Bohr transform is defined by

$$a(\lambda, f) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\lambda t}dt. \tag{3}$$

The set $\sigma(f) := \{\lambda \in \mathbf{R} : a(\lambda, f) \neq 0\}$ is called the *Bohr spectrum* of f. It is well known that $\sigma(f)$ is (at most) countable. The Fourier-Bohr series of f is

$$\sum_{\lambda\in\sigma(f)} a(\lambda, f)e^{i\lambda t},$$

and it converges to f (in the topology of $AP(\mathbf{R}, H)$). Moreover, the following Parseval's equality holds:

$$\|f\|^2_{AP} = \sum_{\lambda\in\sigma(f)} \|a(\lambda, f)\|^2_H,\ f \in AP(\mathbf{R}, H).$$

In the sequel, if a function f in $AP(\mathbf{R}, H)$ has a Fourier-Bohr series

$$\sum_{\lambda\in\sigma(f)} a(\lambda, f)e^{i\lambda t},$$

then we will write

$$f \sim \sum_{\lambda\in\sigma(f)} a(\lambda, f)e^{i\lambda t}.$$

We will also frequently use the following equality, which is valid for every $u, v \in AP(\mathbf{R}, H)$:

$$\langle u, v\rangle_{AP} = \sum_{\lambda} (a(\lambda, u), a(\lambda, v))_H$$

(the sum is over a countable set of exponents λ). In particular,

$$\langle u, e^{i\lambda t}x\rangle_{AP} = (a(\lambda, u), x)_H, \text{ for all } u \in AP(\mathbf{R}, H),\ x \in H.$$

Note that there is a family of orthogonal projections $P_\lambda,\ \lambda \in \mathbf{R}$, on $AP(\mathbf{R}, H)$ defined by $P(\lambda)f = e^{i\lambda t}a(\lambda, f)$, which satisfies $P_\lambda P_\mu = 0$ if $\lambda \neq \mu$. It is clear that $H_\lambda := P_\lambda H = \{e^{i\lambda t}x : x \in H\}$. The family $H_\lambda, \lambda \in \mathbf{R}$, is pairwise orthogonal and complete in $AP(\mathbf{R}, H)$. For these and other facts about almost periodic functions, we refer the reader to [5].

Consider the translation group $S(t), -\infty < t < \infty$, on $AP(\mathbf{R}, H)$. The operators $S(t)$ are first defined for functions f in $AP_b(\mathbf{R}, H)$ by $(S(t)f)(\cdot) = f(\cdot + t)$, and extended to $AP(\mathbf{R}, H)$ by continuity. It is clear that $S(t)$ is a strongly continuous group of unitary operators. Let $\mathcal{D}$ be the generator of $S(t)$. Then $\mathcal{D}$ is a skew self-adjoint operator on $AP(\mathbf{R}, H)$, i.e. $\mathcal{D}^* = i\mathcal{D}$, and is the closure of the operator of differentiation, with the natural domain.

Let A be a closed, densely defined linear operator on a Hilbert space H. The operator A generates an operator $\mathcal{A}$ on $AP(\mathbf{R}, H)$ in a natural manner. Namely, we define $\mathcal{A}$ on $AP(\mathbf{R}, H)$ by

$$\begin{aligned} D(\mathcal{A}) := \{f \in AP(\mathbf{R}, H) : a(\lambda, f) \in D(A) \text{ for all } \lambda \in \sigma(f) \\ \text{and } \textstyle\sum_{\lambda \in \sigma(f)} \|Aa(\lambda, f)\|_H^2 < \infty\} \end{aligned}$$

and

$$(\mathcal{A}f) \sim \sum_{\lambda \in \sigma(f)} Aa(\lambda, f)e^{i\lambda t}, \text{ for } f \in D(\mathcal{A}).$$

Lemma 3. *$\mathcal{A}$ is a densely defined closed operator and $\sigma(\mathcal{A}) = \sigma(A)$.*

Proof. It is clear that $D(\mathcal{A})$ contains linear combinations of functions of the form $e^{i\lambda t}x$, with $\lambda \in \mathbf{R}$ and $x \in D(A)$. From this it is easily seen that $D(\mathcal{A})$ is dense in $AP(\mathbf{R}, H)$. This implies that $\mathcal{A}^*$ is well defined (and densely defined closed). Moreover, for every $f \in D(\mathcal{A})$ with the Fourier-Bohr series $\sum_{\lambda \in \sigma(f)} a(\lambda, f)e^{i\lambda t}$, and for every $x \in D(A^*)$, we have

$$\langle \mathcal{A}f, e^{i\xi t}x\rangle = (Aa(\xi, f), x)_H = (a(\xi, f), A^*x)_H = \langle f, e^{i\xi t}A^*x\rangle_{AP},$$

which implies that $e^{i\xi t}x \in D(\mathcal{A}^*)$ and $\mathcal{A}^*(e^{i\xi t}x) = e^{i\xi t}A^*x$.

Now assume that $f_n \in D(\mathcal{A}),\ f_n \to f,\ \mathcal{A}f_n \to g$. We must show that $f \in D(\mathcal{A})$ and $\mathcal{A}f = g$. Let

$$f \sim \sum a(\lambda, f)e^{i\lambda t}, \quad f_n \sim \sum a(\lambda, f_n)e^{i\lambda t}, \quad g \sim \sum a(\lambda, g)e^{i\lambda t}.$$

Since $f_n \in D(\mathcal{A})$ and A is closed, we have $a(\lambda, f_n) \in D(A)$ and $Aa(\lambda, f_n) \to a(\lambda, g)$. Moreover, for every $h \in D(\mathcal{A}^*)$ we have

$$\langle f, \mathcal{A}^*h\rangle_{AP} = \lim_{n\to\infty} \langle f_n, \mathcal{A}^*h\rangle_{AP} = \lim_{n\to\infty} \langle \mathcal{A}f_n, h\rangle_{AP} = \langle g, h\rangle_{AP}.$$

In particular, for every $x \in D(A^*)$, we have $h(t) = e^{i\lambda t}x \in D(\mathcal{A}^*)$, $\mathcal{A}^*h = e^{i\lambda t}A^*x$ and

$$\langle f, \mathcal{A}^*h\rangle_{AP} = \langle f, e^{i\lambda t}A^*x\rangle_{AP} = (a(\lambda, f), A^*x)_H = \langle g, e^{i\lambda t}x\rangle_{AP} = (a(\lambda, g), x)_H.$$

This implies that $Aa(\lambda, f) = a(\lambda, g)$, so that $f \in D(\mathcal{A})$ and $\mathcal{A}f = g$. Finally, we show that $\sigma(\mathcal{A}) = \sigma(A)$. If $\lambda \in \rho(A)$, then the operator $\mathcal{B}$ defined by $(\mathcal{B}f)(t) = \sum_{\xi \in \sigma(f)} (\lambda - A)^{-1}a(\xi, f)e^{i\xi t}$ is easily seen to be the bounded inverse of $(\lambda - \mathcal{A})$, hence $\lambda \in \rho(\mathcal{A})$, or $\sigma(\mathcal{A}) \subset \sigma(A)$.

Conversely, if $\lambda \in \rho(\mathcal{A})$, then $\mathcal{A} - \lambda$ has a dense range and satisfies

$$\|(\mathcal{A} - \lambda)f\|_{AP} \geq \delta \|f\|_{AP}$$

for some $\delta > 0$ and all $f \in D(\mathcal{A})$. This implies that $(A-\lambda)$ has a dense range and $\|(A-\lambda)x\|_H \geq \delta\|x\|_H$ for all $x \in D(A)$, so that $\lambda \in \rho(A)$.

Below we denote by $L = \mathcal{D} - \mathcal{A}$ the operator on $AP(\mathbf{R}, H)$ defined by $D(L) = D(\mathcal{D}) \cap D(\mathcal{A})$ and $Lf = \mathcal{D}f - \mathcal{A}f$ for all $f \in D(L)$.

Lemma 4. *The operator $L = \mathcal{D} - \mathcal{A}$ is densely defined and closable.*

Proof. Since $D(\mathcal{D})$ and $D(\mathcal{A})$ contain linear combinations of functions $e^{i\lambda t}x, \lambda \in \mathbf{R}, x \in D(A)$, it follows that L is densely defined. For $v(t) = \sum_{k=1}^n e^{\lambda_k t}x_k$ with $\lambda_k \in \mathbf{R}, x_k \in D(A^*)$, let $Kv = \sum_{k=1}^n [(i\lambda_k) - A^*]x_k e^{i\lambda_k t}$. It is easily seen that

$$\langle Lf, v\rangle_{AP} = \langle f, Kv\rangle_{AP}$$

for each $f = \sum_{j=1}^m y_j e^{i\gamma_j t}, y_j \in D(A)$. Hence, $Kv = L^*v$, so that L^* is densely defined. This implies that L is closable (and its closure is L^{**}).

Below, we denote by $(\mathcal{D} - \mathcal{A})^-$ the closure of $\mathcal{D} - \mathcal{A}$.

For every closed subset $\Lambda \subset \mathbf{R}$, we denote by $M(\Lambda)$ a subspace of $AP(\mathbf{R}, H)$ consisting of functions g such that $\sigma(g) \subset \Lambda$.

Lemma 5. *Let Λ be a closed non-empty subset of $\mathbf{R}$. Then*
(i) *$M(\Lambda)$ is a closed invariant subspace with respect to $S(t), \mathcal{D}, \mathcal{A}$;*
(ii) *$\sigma(\mathcal{D}|M(\Lambda)) = i\Lambda$, $\sigma(\mathcal{A}|M(\Lambda)) = \sigma(A)$;*
(iii) *$\mathcal{D}|M(\Lambda)$ is bounded if (and only if) Λ is compact.*

Proof. (i) It is obvious that $M(\Lambda)$ is linear and invariant with respect to $S(t), \mathcal{D}$ and $\mathcal{A}$. Suppose $g_n \in M(\Lambda)$ and $\|g_n - g\|_{AP} \to 0$ as $n \to \infty$. This implies $\|a(\lambda, g_n) - a(\lambda, g)\|_H \to 0$. Since $\sigma(g_n) \subset \Lambda$, we have $a(\lambda, g_n) = 0$ for all $\lambda \notin \Lambda$, which implies $a(\lambda, g) = 0$ for all $\lambda \notin \Lambda$, or $\sigma(g) \subset \Lambda$. Hence $M(\Lambda)$ is closed.

(ii) If $\lambda \in \Lambda,\ x \in H$, then $h(t) = e^{i\lambda t}x \in D(\mathcal{D}) \cap M(\Lambda)$ and $\mathcal{D}h = i\lambda h$. Hence $i\lambda \in \sigma(\mathcal{D}|_{M(\Lambda)})$, which implies $i\Lambda \subset \sigma(\mathcal{D}|_{M(\Lambda)})$.

Suppose now that $\lambda_0 \notin \Lambda$. Define, for $\lambda_k \in \Lambda,\ x_k \in H$,

$$R(\sum_{k=1}^n e^{i\lambda_k t}x_k) = \sum_{k=1}^n (i\lambda_k - i\lambda_0)^{-1} e^{i\lambda_k t}x_k.$$

It is clear that

$$\begin{aligned}\|R\sum_{k=1}^n e^{i\lambda_k t}x_k\|_{AP}^2 &= \|\sum_{k=1}^n (i\lambda_k - i\lambda_0)^{-1}e^{i\lambda_k t}x_k\|_{AP}^2 \\ &= \sum_{k=1}^n |(i\lambda_k - i\lambda_0)^{-1}|^2 \|x_k\|^2 \\ &\leq (\sup_{\lambda\in\Lambda} |\lambda - \lambda_0|^{-1})^2 \|\sum_{k=1}^n e^{i\lambda_k t}x_k\|_{AP}^2,\end{aligned}$$

hence R can be extended to a bounded operator on $M(\Lambda)$. It is easily verified that R is the inverse to $(\mathcal{D} - i\lambda_0)|_{M(\Lambda)}$, hence $i\lambda_0 \notin \sigma(\mathcal{D}|_{M(\Lambda)})$. The proof of $\sigma(\mathcal{A}|M(\Lambda)) = \sigma(A)$ is analogous to that of $\sigma(\mathcal{A}) = \sigma(A)$ in Lemma 3.

(iii) The operator $\mathcal{D}|M(\Lambda)$, being skew self-adjoint, is bounded if and only if its spectrum, $i\Lambda$, is compact.

Assume that $f \in AP(\mathbf{R}, H)$. A function $u \in AP(\mathbf{R}, H)$ is called a *mild solution* of (1) if $u \in D((\mathcal{D} - \mathcal{A})^-)$ and $(\mathcal{D} - \mathcal{A})^- u = f$. The space $M(\Lambda)$ is called *regularly* *admissible* (w.r.t. (1)) if for every $f \in M(\Lambda)$, (1) has a unique mild solution u in $M(\Lambda)$.

3.

Let Λ be a closed subset of $\mathbf{R}$. If follows from Lemma 5(i) that $M(\Lambda)$ is invariant under $(\mathcal{D}-\mathcal{A})^{-}$, so that $(\mathcal{D}-\mathcal{A})^{-}|M(\Lambda)$ is defined. Assume that $M(\Lambda)$ is regularly admissible. Then we define a linear operator K_Λ on $M(\Lambda)$, called the *solution operator*, by putting $K_\Lambda f = u$, where u is the unique (mild) solution in $M(\Lambda)$ of (1). A standard argument, using the Closed Graph Theorem, shows that K_Λ is a bounded operator on $M(\Lambda)$. Moreover, $(\mathcal{D}-\mathcal{A})^{-}u = (\mathcal{D}-\mathcal{A})^{-}K_\Lambda f = f$ (for all f in $M(\Lambda)$). Therefore the operator $(\mathcal{D}-\mathcal{A})^{-}|M(\Lambda)$ is invertible (with the inverse equal to K_Λ). In particular, for every $\lambda_0 \in \Lambda$ and $y \in H$, there exists a unique $x \in H$ such that $e^{i\lambda_0 t}x$ is the unique mild solution in $M(\Lambda)$ of (1), with $f(t) = e^{i\lambda_0 t}y$, which implies that for every $y \in H$ there exists a unique $x \in H$ such that $(i\lambda_0 - A)x = y$, i.e. $(i\lambda_0 - A)$ is invertible. Thus, $\sigma(A) \cap i\Lambda = \emptyset$. From this the following lemma, which is a version of [8], Theorem 3.1, follows.

Lemma 6 (cf. [8], Theorem 3.1)**.** *Let Λ be a non-empty closed subset of $\mathbf{R}$. Then $M(\Lambda)$ is regularly admissible if and only if $(\mathcal{D}-\mathcal{A})^{-}|M(\Lambda)$ is invertible. If $M(\Lambda)$ is regularly admissible, then $i\Lambda \cap \sigma(A) = \emptyset$.*

If Λ is compact, then $\mathcal{D}|M(\Lambda)$ is bounded and, therefore, $(\mathcal{D}-\mathcal{A})^{-}|M(\Lambda)$ is invertible whenever $\sigma(\mathcal{D}|M(\Lambda)) \cap \sigma(\mathcal{A}|M(\Lambda)) = \emptyset$, or $i\Lambda \cap \sigma(A) = \emptyset$ (see [1]). Moreover, K_Λ is given by the following analog of the Krein-Rosenblum integral formula (cf. [8], p. 397, [1]):

$$K_\Lambda = \frac{1}{2\pi i}\int_\Gamma (\lambda - \mathcal{A}_M)^{-1}(\lambda - \mathcal{D}_M)^{-1}d\lambda, \tag{4}$$

where Γ is a Cauchy contour which surrounds $\sigma(\mathcal{D}_M)$ $(= i\Lambda)$ and is separated from $\sigma(A_M)$ $(= \sigma(A))$, and $\mathcal{A}_M$ and $\mathcal{D}_M$ are restrictions of $\mathcal{A}$ and $\mathcal{D}$, respectively, to $M(\Lambda)$. Thus, the following analog of ([8], Theorem 3.3-i) holds.

Lemma 7 ([8], Theorem 3.3-i)**.** *If Λ is compact and $i\Lambda \cap \sigma(A) = \emptyset$, then $M(\Lambda)$ is regularly admissible and the solution operator K_Λ is given by* (4)*.*

Assume that $i\Lambda \cap \sigma(A) = \emptyset$. In general, this condition does not imply that $M(\Lambda)$ is regularly admissible. Let $\Lambda_\alpha, \alpha \in \Omega$, be a family of compact subsets of Λ such that $span\{M(\Lambda_\alpha) : \alpha \in \Omega\}$ is dense in $M(\Lambda)$. Then from $\Lambda \cap \sigma(A) = \emptyset$ it follows that $\Lambda_\alpha \cap \sigma(A) = \emptyset$ for all α. According to Lemma 7, each subspace $M(\Lambda_\alpha)$ is regularly admissible. Let K_{Λ_α} be the solution operator on $M(\Lambda_\alpha)$. From the uniqueness of K_{Λ_α} it follows that if $\Lambda_\alpha \subset \Lambda_\beta$, then $K_{\Lambda_\beta}|M(\Lambda_\alpha) = K_{\Lambda_\alpha}$. Therefore, one can correctly define an operator K_0 with dense domain $D(K_0) = span\{M(\Lambda_\alpha) : \alpha \in \Omega\}$, by putting $K_0|M(\Lambda_\alpha) = K_{\Lambda_\alpha}$. If $M(\Lambda)$ is regularly admissible and K_Λ is the corresponding solution operator, then $K_\Lambda|M(\Lambda_\alpha) = K_{\Lambda_\alpha}$, hence $\sup_{\alpha\in\Omega}\|K_{\Lambda_\alpha}\| = \|K_\Lambda\| < \infty$. Conversely, if $\sup_\alpha \|K_{\Lambda_\alpha}\| = L < \infty$, then $\|K_0\| \le L$ so that K_0 can be extended by continuity to a bounded operator K_Λ on $M(\Lambda)$, which is the inverse of $(\mathcal{D}-\mathcal{A})^{-}|M(\Lambda)$. Thus, the following statement, which is a version of [8], Theorem 2.2 and Theorem 3.4, holds.

Lemma 8. *Assume that $i\Lambda \cap \sigma(A) = \emptyset$. Let $\Lambda_\alpha, \alpha \in \Omega$, be a family of compact subsets such that $span\{M(\Lambda_\alpha) : \alpha \in \Omega\}$ is dense in $M(\Lambda)$. Then $M(\Lambda)$ is regularly admissible if and only if $\sup_{\alpha\in\Omega}\|K_{\Lambda_\alpha}\| < \infty$.*

4.

Proof of Theorem 2. In light of Lemma 6, we can assume that $\sigma(A) \cap i\Lambda = \emptyset$. Let $\lambda_k \in \Lambda$ and $\Lambda_\alpha = \{\lambda_1, \lambda_2, ..., \lambda_n\}$. By Lemma 7, $M(\Lambda_\alpha)$ is regularly admissible. Let $K_{\Lambda_\alpha} : M(\Lambda_\alpha) \to M(\Lambda_\alpha)$ be the corresponding solution operator. Consider the function $g(t) = \sum_{k=1}^n e^{i\lambda_k t} x_k$, where $x_k,\ k = 1, 2, ..., n$, are arbitrary vectors in H. It is directly verified that $u(t) := \sum_{k=1}^n e^{i\lambda_k t}(i\lambda_k - A)^{-1} x_k$ is a (classical) solution in $M(\Lambda_\alpha)$ of equation $u'(t) = Au(t) + g(t)$, hence $K_{\Lambda_\alpha} g = \sum_{k=1}^n e^{i\lambda_k t}(i\lambda_k - A)^{-1} x_k$. According to Lemma 8, $M(\Lambda)$ is regularly admissible if and only if $\sup_\alpha \|K_{\Lambda_\alpha}\| < \infty$, that is, if and only if there exists $L > 0$ such that

$$\|\sum_{k=1}^n e^{i\lambda_k t}(\lambda_k - A)^{-1} x_k\|_{AP} \le L \|\sum_{j=1}^n e^{i\lambda_j t} x_j\|_{AP} \tag{5}$$

for every $x_j \in H, \lambda_j \in \Lambda, 1 \le j \le n$. By Parseval's equality

$$\|\sum_{k=1}^n e^{i\lambda_k t}(i\lambda_k - A)^{-1} x_k\|_{AP}^2 = \sum_{k=1}^n \|(i\lambda_k - A)^{-1} x_k\|_H^2,$$

$$\|\sum_{j=1}^n e^{i\lambda_j t} x_j\|_{AP}^2 = \sum_{j=1}^n \|x_j\|_H^2.$$

Hence, (5) is equivalent to

$$\sum_{k=1}^n \|(i\lambda_k - A)^{-1} x_k\|_H^2 \le L \sum_{j=1}^n \|x_j\|_H^2, \tag{6}$$

for all $x_1, x_2, ..., x_n$ in H and $\lambda_1, ..., \lambda_n$ in Λ.

Therefore, it remains to show that (2) and (6) are equivalent, which is obvious. □

5.

Let $\Lambda_1 = \{2\pi k : k \in \mathbf{Z}\}$. Then $M(\Lambda_1)$ can be naturally identified with the space $L^2([0,1], H)$.

Corollary 9. *The following are equivalent:*

(i) *For every function $f \in L^2([0,1], H)$, there exists a unique mild solution $u \in L^2([0,1], H)$ of* (1)*;*

(ii) *$i2\pi k \in \rho(A)$ for all $k \in \mathbf{Z}$ and $\sup_{k \in \mathbf{Z}} \|(i2\pi k - A)^{-1}\| < \infty$.*

Now let A be the generator of a C_0-semigroup $T(t)$ on H. Assuming that $M(\Lambda_1)$ is regularly admissible, we show that mild solutions in our definition are mild solutions in the standard sense of the theory of C_0-semigroups (see e.g. [6]), i.e. the following equation holds:

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau) f(\tau) d\tau \ (t \ge s). \tag{7}$$

Proposition 10. *Under the conditions of Corollary* 9*, for every $f \in L^2([0,1], H)$ the unique mild solution u in $L^2([0,1], H)$ is a continuous* 1*-periodic function and satisfies* (7)*.*

Proof. Let $f_n = \sum_{k=-n}^{n} e^{i2\pi kt}\hat{f}(k)$ and $u_n = \sum_{k=-n}^{n} e^{i2\pi kt}(i2\pi - A)^{-1}\hat{f}(k)$, where $\hat{f}(k)$ and $\hat{u}(k)$ are Fourier coefficients of f and u, respectively. Using the well-known property

$$A\int_0^t T(s)xds = T(t)x - x$$

(which is valid for arbitrary semigroups $T(t)$, with generator A, and arbitrary $x \in H$), we obtain

$$(A - i2\pi k)\int_0^t T(s)e^{-i2\pi ks}\hat{f}(k)ds = e^{-i2\pi kt}T(t)\hat{f}(k) - \hat{f}(k),$$

which implies

$$\begin{aligned} &e^{i2\pi kt}(i2\pi k - A)^{-1}\hat{f}(k) \\ &= T(t)[(i2\pi k - A)^{-1}\hat{f}(k)] + \int_0^t T(t-\tau)e^{i2\pi k\tau}\hat{f}(k)d\tau. \end{aligned} \tag{8}$$

From (8) it follows that

$$u_n(t) = T(t)u_n(0) + \int_0^t T(t-\tau)f_n(\tau)d\tau,$$

i.e. u_n is a mild solution of the equation $u'(t) = Au(t) + f_n(t)$ in the traditional sense (of (7)). From the last identity it follows that

$$v_n := [I - T(1)]u_n(0) = \int_0^1 T(1-s)f_n(s)ds,$$

so that $\|v_n - v_m\| \le \sup_{0\le t\le 1}\|T(t)\|\|f_n - f_m\|_{L^2}$, i.e. v_n converges to some $v \in H$. Furthermore

$$\begin{aligned} T(1)u_n(0) &= \int_0^1 T(1-t)T(t)u_n(0) \\ &= \int_0^1 T(1-t)u_n(t)dt - \int_0^1 T(1-t)\int_0^t T(t-\tau)f_n(\tau)d\tau dt, \end{aligned}$$

which also implies that $w_n := T(1)u_n(0)$ is a convergent sequence. Therefore, $u_n(0) = v_n + w_n$ converges to some vector $u_0 \in H$. From

$$\begin{aligned} &\|u_n(t) - u_m(t)\| \\ &\le \sup_{0\le t\le 1}\|T(t)\|\left[\|u_n(0) - u_m(0)\| + \|f_n - f_m\|_{L^2}\right],\ 0 \le t \le 1, \end{aligned}$$

it follows that u_n converges to u uniformly on $[0,1]$, so that u is a continuous 1-periodic function. The equality (7) is now immediate.

In conclusion we remark that the presented approach is directly applicable to more general classes of differential, integro-differential and functional-differential equations in a Hilbert space. The details are to be given elsewhere.

Acknowledgment

The author would like to thank the anonymous referee for pointing out some inaccuracies in the original version of the paper.

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