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TRANSVERSALS FOR STRONGLY ALMOST DISJOINT FAMILIES

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Abstract. For a family of sets A, and a set X, X is said to be a transversal of A if $X \subseteq \bigcup A$ and $|a \cap X| = 1$ for each $a \in A$. X is said to be a Bernstein set for A if $\emptyset \neq a \cap X \neq a$ for each $a \in A$. Erdős and Hajnal first studied when an almost disjoint family admits a set such as a transversal or Bernstein set. In this note we introduce the following notion: a family of sets A is said to admit a σ-transversal if A can be written as $A = \bigcup\{A_n : n \in \omega\}$ such that each A_n admits a transversal. We study the question of when an almost disjoint family admits a σ-transversal and related questions.

1. Introduction

Given a family of sets A, we can say that a set X splits the sets in A if for every $a \in A$, $X \cap a$ is nonempty but small. It is natural to consider notions of smallness that include cardinality, small in measure, topologically small and others. For example, a Bernstein set $X \subseteq \mathbb{R}$ is a set that splits the family of perfect subsets of $\mathbb{R}$, where small means "is not equal to". In [2], the question of which almost disjoint families can be split was considered.

Most of our notation and terminology are standard and can be found in [6]. A family $A \subseteq [\lambda]^\kappa$ is *almost disjoint* if pairwise intersections are of cardinality $< \kappa$. A family of sets is said to be *strongly almost disjoint* if pairwise intersections are finite, and *r-almost disjoint* if pairwise intersections have cardinality less than r. A family of sets A is said to be point-finite (point-countable) if $\{a \in A : x \in a\}$ is finite (countable).

The following theorems were proven in [2].

Theorem 1. *Assume GCH. For every strongly almost disjoint family A consisting of sets of size $\geq \aleph_1$, if $|A| \leq \aleph_\omega$, then A has the Bernstein property B. I.e., there is X such that $\emptyset \neq a \cap X \neq a$ for each $a \in A$.*

Theorem 2. *For each $r < \omega$ and each r-almost disjoint family A of countable sets, if $|A| = \aleph_n$, then there is X such that $0 \neq |a \cap X| < (r-1)(n+1)+2$ for each $a \in A$.*

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In the case where the set X meets each element of a family A in exactly one point, we will refer to X as a *transversal* of A.

The paper [3] is a natural sequel to the paper of Erdös and Hajnal. It answers a number of questions from [2] and considers conditions when almost disjoint families $A \subseteq [\lambda]^\kappa$ are essentially disjoint: removing a set of size κ from each member produces a disjoint family. Clearly an essentially disjoint family can be split where small means of size less than κ. (In a further sequel [4] certain results of [2] are shown to be sharp.)

One can compare the above theorems to the following theorem of Sierpiński [10].

Theorem 3. *$\omega_1 \times \omega_1$ can be covered by countably many functions and their inverses.*

The rows and columns form a 2-almost disjoint family, and each function is a transversal of the columns and each inverse is a transversal of the rows. The content of the theorem is that there are countably many of these partial transversals that cover $\omega_1 \times \omega_1$.

In [5] the question of whether certain strongly almost disjoint families of sets may admit a countable family of partial transversals covering the underlying set was central to the analysis of scattered compact spaces of finite Cantor-Bendixson height (e.g., the one-point compactification of $\omega_1 \times \omega_1$). Motivated by those preliminary results we now study more systematically when a strongly almost disjoint family admits a countable family of partial transversals (possible covering the underlying set).

Following the notation of [2], we introduce the following notation.

Definition 4. Suppose that m, p, r, s are cardinals. $M(m,p,r) \to \sigma - B(s)$ means that for every r-almost disjoint family A of size m consisting of infinite sets of size $\leq p$, there are sets $\{X_n : n < \omega\}$ and $\{A_n : n < \omega\}$ such that

(1) $A = \bigcup\{A_n : n < \omega\}$, and
(2) $0 \neq |X_n \cap a| < s$ for each $a \in A_n$,
(3) $X_n \subseteq \bigcup A_n$.

Also, $M(m,p,r) \to \sigma - B$ means (1), (3), and instead of (2) we require

(2′) $\emptyset \neq X_n \cap a \neq a$ for each $a \in A_n$.

In the case that $s = 2$, the family of sets $\{X_n : n \in \omega\}$ satisfying (1)–(3) will be called a *σ-transversal of A*. A σ-transversal that covers the underlying set will be called *a covering σ-transversal*. Even if a family A admits a transversal, one cannot expect to obtain a covering σ-transversal unless for every $a \in A$, all but countably many points of a are covered by $A \setminus \{a\}$. For example, any disjoint family of uncountable sets obviously admits a transversal, but there can be no covering σ-transversal.

In general, one cannot expect to prove $M(m,p,r) \to \sigma - B(s)$ for $s \leq \omega$ unless $r \leq \omega$. Indeed, if $A = \{a_{\beta,\alpha} : \beta < \alpha < \omega_1\}$ is an almost disjoint family of uncountable subsets of ω_1 such that $(\beta, \alpha] \subseteq a_{\beta,\alpha} \subseteq (\beta, \omega_1)$ for each $\beta < \alpha$, then A witnesses that $M(\omega_1, \omega_1, \omega_1) \not\to \sigma - B(\aleph_0)$; indeed, if $\{X_n : n \in \omega\}$ is a family of subsets of ω_1. Let β be such that $X_n \subseteq \beta$ whenever X_n is countable. Then pick $\alpha > \beta$ so that $X_n \cap \alpha$ is infinite whenever X_n is uncountable. Then $a_{\beta,\alpha}$ is either disjoint from X_n or has infinite intersection with X_n for every $n \in \omega$.

Thus, we may only expect to obtain positive results about σ-transversals for strongly almost disjoint families or almost disjoint families with additional properties.

Note also that a strongly almost disjoint family may not admit a transversal, even if it is countable and 2-almost disjoint. Indeed, if A is a maximal 2-almost disjoint family in $[\omega]^{\aleph_0}$, then by maximality, it does not admit a transversal. Even a 2-almost disjoint and point-finite family may not admit a transversal: the collection of rows and columns of $\omega \times \omega_1$ has no transversal.

2. σ-TRANSVERSALS OF STRONGLY ALMOST DISJOINT FAMILIES

Any family A of subsets of ω admits a σ-transversal for trivial reasons: the familes $X_n = \{n\}$ and $A_n = \{a \in A : n \in a\}$ form a σ-transversal of A.

almost disjoint family of subsets of ω consisting of infinite X_n's. However, any almost disjoint family A of subsets of ω admits a $\sigma - B(\aleph_0)$ family consisting of infinite sets: to see this find $\{a_n : n \in \omega\} \subseteq A$ such that

For strongly almost disjoint families of subsets of ω_1 the situation is more complicated. In [5] it was proven, assuming MA_{ω_1}, that certain strongly almost disjoint families admit nice families of partial transversals. In this note, we refine that result to obtain:

Theorem 5. *Assume MA_{ω_1}. Then any strongly almost disjoint family A such that $|A| = \omega_1$ consisting of infinite sets of size $\leq \aleph_1$ admits a σ-transversal. Moreover, under the additional assumption that each $a \in A$ is covered by $A \setminus \{a\}$, then A admits a covering σ-transversal.*

Proof. The proof is similar to the result from [5] mentioned above. For completeness sake we give a full proof. First note that we may assume that A is a family of subsets of ω_1. We need the following lemma about strongly almost disjoint families:

Lemma 6. *Suppose that $\{a_\alpha : \alpha < \omega_1\}$ is a strongly almost disjoint family of sets. Suppose also that $\{p_\alpha : \alpha < \omega_1\}$ is a sequence of pairwise disjoint finite sets. Then there are $\alpha < \beta$ such that $p_\alpha \cap a_\beta = \emptyset$ and $p_\beta \cap a_\alpha = \emptyset$.*

Proof. By going to a subsequence, we may assume that there is $n \in \omega$ such that $|p_\alpha| = n$ for all α. Let M be a countable elementary submodel of a suitably large $H(\theta)$ containing everything relevant and let $\gamma = M \cap \omega_1$.

Claim 7. There are $\{\alpha_i : i < n+1\} \subseteq \gamma$ and a $\beta > \gamma$ such that

$$(\bigcup_{i<n+1} p_{\alpha_i}) \cap a_\beta = \emptyset.$$

Proof. If not, then for each $\beta > \gamma$ there are at most n α's below γ such that $p_\alpha \cap a_\beta = \emptyset$. Thus, there is a $\alpha_\beta < \gamma$ such that $p_\alpha \cap a_\beta \neq \emptyset$ for each $\alpha \in M \setminus \alpha_\beta$. Choose $\{\beta_i : i < n+1\} \subseteq \omega_1 \setminus \gamma$. Choose an index $\alpha \in M$ above $\{\alpha_{\beta_i} : i < n+1\}$ such that p_α is disjoint from the following finite set:

$$\bigcup_{0 \leq i < j < n+1} (a_{\beta_i} \cap a_{\beta_j}) \cap M.$$

Then for $i < n+1$ we have that $p_\alpha \cap a_{\beta_i} \neq \emptyset$ and the family $\{p_\alpha \cap a_{\beta_i} : i < n+1\}$ is pairwise disjoint. This contradicts that $|p_\alpha| = n$. □

To complete the proof of the main lemma, note that by elementarity $a_{\alpha_i} \cap a_{\alpha_j} \subseteq M$ for each $i \neq j$ and $p_\beta \cap M = \emptyset$ since $\beta \notin M$ and the p_α's are pairwise disjoint. By the pigeon-hole argument just presented in the proof of Claim 7, we may conclude that $p_\beta \cap a_{\alpha_i} = \emptyset$ for some $i < n+1$. □

Returning now to the proof of the theorem we need to find sets A_n and X_n satisfying (1)–(3) of Definition 4 and, assuming that each $a \in A$ is covered by the rest of A, also

(4) $\bigcup\{X_n : n \in \omega\} = \bigcup A$.

Let $(M_\alpha : \alpha < \omega_1)$ be a continuous $\in$-chain of countable elementary submodels of a suitably large $H(\theta)$ containing everything relevant. For each $a \in A$, let α be minimal such that $a \in M_\alpha$. Let $a' = a \cap M_\alpha$. (To obtain (1)–(3) the elementary submodels are not needed, and it suffices to take any $a' \in [a]^{\aleph_0}$. Elementarity will be used to obtain (4).)

Now define the poset $\mathbb{P}$ to be the set of all pairs $p = (x_p, F_p)$, where $x_p \in [\omega_1]^{<\omega}$ and $F_p \in [A]^{<\omega}$ with the property that $|x_p \cap a'| = |x_p \cap a| = 1$ for all $a \in F_p$. We define $p \leq q$ if $x_q \subseteq x_p$ and $F_q \subseteq F_p$.

Claim 8. $\mathbb{P}$ has the ccc.

Proof. Given an uncountable subset $\{(y_\alpha, G_\alpha) : \alpha \in \omega_1\}$, we may assume that the y_α's form a Δ-system with root r. Let $x_\alpha = y_\alpha \setminus r$. We may also assume that the G_α's form a Δ-system with root R. Let $F_\alpha = G_\alpha \setminus R$. By going to a subsequence we may assume that $x_\alpha \cap \bigcup\{a' : a \in R\} = \emptyset$ for each α. Thus, for each $a \in R$, $y_\alpha \cap a' = r \cap a'$. Thus, if we let $a_\alpha = \bigcup F_\alpha$ it follows that (y_α, G_α) is compatible with some (y_β, G_β) if and only if $x_\alpha \cap a_\beta = \emptyset = x_\beta \cap a_\alpha$. The existence of such a pair is given by Lemma 6.

To finish the proof, let $\mathbb{P}^\omega$ denote the finite support product of countably many copies of $\mathbb{P}$. Consider the following subsets of $\mathbb{P}^\omega$:

$D_a = \{p \in \mathbb{P}^\omega : \exists n\ a \in F_{p(n)}\}$.

It is easy to show that D_a is dense for each $a \in A$. By MA_{ω_1} we may let $\mathcal{G}$ be a $\{D_a : a \in A\}$-generic filter. Let $X_n = \bigcup\{x_{p(n)} : p \in \mathcal{G}\}$ and $A_n = \bigcup\{F_{p(n)} : p \in \mathcal{G}\}$. Then $\bigcup_n A_n = A$ by genericity of $\mathcal{G}$, and the rest of (1)–(3) easily follows from the definition of the partial order.

In order to obtain (4) assume now that each a is covered by $A \setminus \{a\}$. It suffices to show that for each $\alpha \in \bigcup A$ the following set is dense:

$E_\alpha = \{p \in \mathbb{P}^\omega : \exists n\ \alpha \in x_{p(n)}\}$.

Note that E_α is dense if $\alpha \in a'$ for some $a \in A$. To see this, consider any $a \in A$ and any $\alpha \in a$. Let β be such that $\alpha \in M_{\beta+1} \setminus M_\beta$. If $a \notin M_\beta$, it follows that $\alpha \in a'$. So assume that $a \in M_\beta$. Since a is covered by $A \setminus \{a\}$, there is a $b \in A$ such that $b \neq a$ and $\alpha \in b$. Now it suffices to observe that $b \notin M_\beta$. For if $b \in M_\beta$, then, since $a \cap b$ is finite, by elementarity it would follow that $a \cap b \subseteq M_\beta$, which is impossible. $\square$

Remarks. (1) Note that the natural poset for producing a partial transversal is not necessarily ccc: the natural poset being the collection of pairs (x, F) where x is a finite subset of $\bigcup A$ and F is a finite subset of A with the property that $|x \cap a| = 1$ for each $a \in F$ with the ordering of reverse inclusion on both coordinates. If A contains an uncountable element a, then this poset is not ccc. Indeed, the collection of pairs $(\{\beta\}, \{a\})$, where $\beta \in a$, is an uncountable antichain.

(2) The only place where the elementary submodels played a role in the proof is in order to get the σ-transversal to be covering. Otherwise it would have sufficed to choose $a' \in [a]^{\aleph_0}$ arbitrary. Thus, we obtain as a corollary to the proof of Theorem 5 that MA_κ implies $M(\kappa, \kappa, \omega) \to \sigma - B(2)$. The restriction on $\kappa = \omega_1$ to get the covering σ-transversal is necessary (see Theorem 14 below).

Now we show that the assumption of MA in Theorem 5 is necessary.

Theorem 9. *Assume $\diamondsuit$. Then $M(\omega_1, \omega, \omega) \not\to \sigma - B(\aleph_0)$.*

Proof. Let $\{D_\alpha : \alpha < \omega_1\}$ be a $\diamondsuit$-sequence for subsets of $\omega_1 \times \omega$. So each D_α is a subset of $\alpha \times \omega$ and for any countable family $\{X_n : n \in \omega\}$ of subsets of ω_1, there are stationary many α for which $\bigcup\{(X_n \cap \alpha) \times \{n\} : n \in \omega\} = D_\alpha$. For each n let $D_\alpha(n) = \{\beta < \alpha : (\beta, n) \in D_\alpha\}$. For each limit $\alpha \in \omega_1$ choose an ω-sequence a_α cofinal in α such that $a_\alpha \cap D_\alpha(n)$ is infinite whenever $D_\alpha(n)$ is cofinal in α. Moreover, if there is a β such that $D_\alpha(n) \subseteq \beta$ whenever $D_\alpha(n)$ is bounded in α, then take β_α to be the least such β, and choose a_α such that $a_\alpha \cap \beta_\alpha = \emptyset$. Then $A = \{a_\alpha : \alpha \in \omega_1\}$ is an almost disjoint family in $[\omega_1]^\omega$.

Suppose that A admits families $\{X_n : n \in \omega\}$, $\{A_n : n \in \omega\}$ such that for each $a_\alpha \in A_n$, we have that $a_\alpha \cap X_n$ is finite and nonempty. Some of the X_n's may be countable, and we may let β be an upper bound for all those countable X_n's. If X_n is uncountable, let E_n be the club set of limit α's for which X_n is cofinal in α. We may fix $\alpha > \beta$ in $\bigcap E_n$ such that $X_n \cap \alpha = D_\alpha(n)$ for every n. It follows that $D_\alpha(n)$ is cofinal in α whenever X_n is uncountable; otherwise $D_\alpha(n)$ is bounded below β. Fix n such that $a_\alpha \in A_n$. Then, if X_n is countable, by choice of a_α we have that $a_\alpha \cap X_n$ is empty. So X_n must be uncountable. Thus, $X_n \cap \alpha$ is cofinal in α, and by the choice of a_α, we have that $a_\alpha \cap X_n$ is infinite. Contradiction. $\square$

While some extra set-theoretic assumption is needed to obtain σ-transversals of strongly almost disjoint families of subsets of ω_1, if we make some additional almost-disjointness restrictions on the families, we can obtain ZFC results. Namely, for any $r < \omega$, any r-almost disjoint family admits a σ-transversal. As a corollary to the proof we will also obtain the fact that any point-countable almost disjoint family admits a σ-transversal (Corollary 11 below).

Theorem 10. *$M(\kappa, \eta, r) \to \sigma - B(2)$ for every infinite κ, η and every $r < \omega$.*

Proof. We prove by induction on $|A|$ that something stronger than A admits a σ-transversal. Namely:

$(IH)_\kappa$ For every r-almost disjoint family B of cardinality κ, if $B = \bigcup B_n$, then there is $\{X_{nk} : n, k \in \omega\}$ and $\{B_{nk} : n, k < \omega\}$ such that
  (a) $\{X_{nk} : n, k \in \omega\}$, $\{B_{nk} : n, k < \omega\}$ is a σ-transversal of B,
  (b) $B_{nk} \subseteq B_n$ for every n and $k \in \omega$, and
  (c) $\{X_{nk} : n, k \in \omega\}$ is a point-finite family.

We will refer to such a family of sets as a point-finite σ-transversal refining $\{B_n : n \in \omega\}$.

It is easy to see that $(IH)_{\aleph_0}$ holds.

So assume that A is r-almost disjoint, $A = \bigcup_n A_n$, $|A| = \kappa$ and $(IH)_\lambda$ holds for all $\lambda < \kappa$. First note that there is a set S of cardinality κ such that $\{a \setminus S : a \in A\}$ is a disjoint family. Thus, we may assume that $A \subseteq P(\kappa)$. Moreover, it suffices to find the appropriate point-finite σ-transversal for the subsets $A_0 = \{a \in A : |a| = \kappa\}$ and $A_1 = \{a \in A : |a| < \kappa\}$ (putting them together gives the required point-finite σ-transversal for A). Therefore we may assume that either $|a| = \kappa$ for every $a \in A$ or $|a| < \kappa$ for every $a \in A$.

Case 1: κ is a successor λ^+. Let $\{M_\alpha : \alpha < \kappa\}$ be a continuous $\in$-chain of elementary submodels of a suitably large $H(\theta)$ each of cardinality λ such that M_0 contains everything relevant and such that $A \subseteq \bigcup\{M_\alpha : \alpha < \kappa\}$.

Subcase 1: $A \subseteq [\kappa]^{<\kappa}$. Thus, it follow that

(d) for each $\alpha \in \kappa$ and $a \in A \cap M_\alpha$, $a \subseteq M_\alpha$.

Also, since A is r-almost disjoint, we have that

(e) for each $\alpha \in \kappa$ and $a \in A \setminus M_\alpha$ $|a \cap M_\alpha| < \omega$.

Indeed this intersection must be of cardinality less than r. This follows by elementarity: For if $x \in [a \cap M_\alpha]^r$, then x is an element of M_α and the set of $a \in A$ such that $x \subseteq a$ is of cardinality κ. Contradiction.

We construct, by recursion on $\alpha < \kappa$, sets $X^\alpha_{nk} \subseteq \kappa \cap M_\alpha$ and $A^\alpha_{nk} \subseteq A \cap M_\alpha$ for each $n, k < \omega$ so that

(f) $\{X^\alpha_{nk} : n, k \in \omega\}$, $\{A^\alpha_{nk} : n, k < \omega\}$ is a σ-transversal of $A \cap M_\alpha$,
(g) $A^\alpha_{nk} \subseteq A_n \cap M_\alpha$ for every n and $k \in \omega$, and
(h) $\{X^\alpha_{nk} : n, k \in \omega\}$ is a point-finite family.
(i) If $\beta < \alpha$, then $X^\alpha_{nk} \cap M_\beta = X^\beta_{nk}$ and $A^\alpha_{nk} \cap M_\beta = A^\beta_{nk}$ for all $n, k \in \omega$.

First note that if such a sequence is constructed, then $X_{nk} = \bigcup_{\alpha<\kappa} X^\alpha_{nk}$, and $A_{nk} = \bigcup_{\alpha<\kappa} A^\alpha_{nk}$ defines the required point-finite σ-transversal refining $\{A_n : n \in \omega\}$.

If α is a limit we define $X^\alpha_{nk} = \bigcup_{\beta<\alpha} X^\beta_{nk}$ and $A_{nk} = \bigcup_{\beta<\alpha} A^\beta_{nk}$. Then by (i) all the inductive hypotheses are preserved.

So suppose that $\alpha < \kappa$ is given and X^α_{nk} and A^α_{nk} have been constructed satisfying (f), (g), (h) and (i). For each finite $x \subseteq \omega$ let

$$A^{\alpha+1}_{x,n} = \{a \in A_n \cap (M_{\alpha+1} \setminus M_\alpha) : \ x = \{k : a \cap X^\alpha_{nk} \neq \emptyset\}\}.$$

By (e) and (h) we have that

(j) $A_n \cap (M_{\alpha+1} \setminus M_\alpha) = \bigcup\{A^{\alpha+1}_{x,n} : x \in [\omega]^{<\omega}\}$.

By our inductive hypotheses we may find sets $Y^{\alpha+1}_{x,n,i}$ and $A^{\alpha+1}_{x,n,i}$ forming a point-finite σ-transversal refining $\{A^{\alpha+1}_{x,n} : n \in \omega, x \in [\omega]^{<\omega}\}$. For each i, x, n choose $k_{i,x,n}$ such that

(k) $k_{i,x,n} \notin x$,
(l) $k_{i,x,n} \neq k_{j,y,n}$ whenever $(i, x) \neq (j, y)$.

Now, let

$$A^{\alpha+1}_{nk_{i,x,n}} = A^\alpha_{nk_{i,x,n}} \cup A^{\alpha+1}_{x,n,i}$$

and

$$X^{\alpha+1}_{nk_{i,x,n}} = X^\alpha_{nk_{i,x,n}} \cup Y^{\alpha+1}_{x,n,i}$$

for each $i, n \in \omega$ and each finite $x \subseteq \omega$. If k is not of the form $k_{i,x,n}$, let

$$A^{\alpha+1}_{nk} = A^\alpha_{nk}$$

and

$$X^{\alpha+1}_{nk} = X^\alpha_{nk}.$$

It is straightforward to check that the inductive hypotheses (f), (g), (h) and (i) are preserved: (h) follows from the way the family of $X^{\alpha+1}_{nk}$'s are constructed from two point-finite families. (g) and (i) are directly from construction. To see (f), fix $a \in A \cap M_{\alpha+1}$. If $a \in M_\alpha$, then there is n, k such that $a \in A^{\alpha+1}_{nk}$. And since $|a \cap X^\alpha_{nk}| = 1$ and $a \subseteq M_\alpha$ and $X^{\alpha+1}_{nk} \cap M_\alpha = X^\alpha_{nk}$, it follows that $|a \cap X^{\alpha+1}_{nk}| = 1$. In the case that $a \in M_{\alpha+1} \setminus M_\alpha$, first note by (j) that there is an x such that $a \in A^{\alpha+1}_{x,n}$. Also, by choice of the $A^{\alpha+1}_{x,n,i}$'s, there is an i such that $a \in A^{\alpha+1}_{x,n,i}$. Thus,

$a \in A^{\alpha+1}_{nk_{i,x,n}}$. Moreover, $|a \cap Y^{\alpha+1}_{x,n,i}| = 1$ and $a \cap X^{\alpha}_{nk_{i,x,n}} = \emptyset$ by choice of $k_{n,x,i}$. Thus, $|a \cap X^{\alpha+1}_{nk_{i,x,n}}| = 1$, as required.

This completes the construction in the case that $A \subseteq [\kappa]^{<\kappa}$.

Subcase 2: $A \subseteq [\kappa]^{\kappa}$. The proof in this case is almost identical. One builds by recursion on $\alpha < \kappa$ the same families $\{X^{\alpha}_{nk} : n, k \in \omega\}$ and $\{A^{\alpha}_{nk} : n, k \in \omega\}$. At stage α of the construction, it suffices to note that $a' = a \setminus \bigcup(A \cap M_\alpha)$ has cardinality κ for each $a \in M_{\alpha+1} \setminus M_\alpha$. Therefore, working with the sets

$$A^{\alpha+1}_{x,n} = \{a' : a \in A_n \cap (M_{\alpha+1} \setminus M_\alpha) \text{ and } x = \{k : a \cap X^{\alpha}_{nk} \neq \emptyset\}\},$$

the rest of the proof is identical to the previous case.

This completes the case that κ is a successor. □

Limit case $cf(\kappa) = \omega$. This case is almost trivial. Fix an $\in$-chain $\{M_m : m < \omega\}$ of elementary submodels of a suitably large $H(\theta)$ each of size $< \kappa$ such that M_0 contains everything relevant and $A \subseteq \bigcup\{M_m : m < \omega\}$. For each m fix A^m_{kn}'s and X^m_{kn}'s forming the required point-finite σ-transversal of $\{A_n \cap M_m \setminus M_{m-1} : n \in \omega\}$ so that each $X^m_{nk} \subseteq M_m \setminus M_{m-1}$. Then $\{X^m_{nk} : m, k, n < \omega\}$ is point-finite and thus the required σ-transversal of $\{A_n : n \in \omega\}$.

Limit case $cf(\kappa)$ is uncountable. Let $\{M_\alpha : \alpha < cf(\kappa)\}$ be an $\in$-chain of elementary submodels of a suitably large $H(\theta)$ each of cardinality $< \kappa$ such that M_0 contains everything relevant, for each α $|M_\alpha| < |M_{\alpha+1}|$, and $A \subseteq \bigcup_{\alpha < cf(\kappa)} M_\alpha$. Let $A_0 = \{a \in A :$ if $a \in M_\alpha$, then $|a| \leq |M_\alpha|\}$ and let $A_1 = A \setminus A_0$. As in the successor case, we may assume that either $A = A_0$ or $A = A_1$. The proof is now almost identical to the successor case. In the case that $A = A_0$, we have that the key property (d) holds, and we proceed as in Subcase 1. In the case that $A = A_1$, we proceed as in Subcase 2.

From the proof of Theorem 10 we obtain the following corollary.

Corollary 11. *Assume that A is a point-countable strongly-almost disjoint family consisting of sets of size $\kappa \geq \omega$. Then A admits a σ-transversal.*

Proof. Following the proof of Theorem 10 the key fact is clause (e). However, in the case that A is point-countable, we even have that $a \cap M_\alpha = \emptyset$ if $a \notin M_\alpha$. □

3. σ-TRANSVERSALS COVERING THE UNDERLYING SET

Towards generalizing Sierpiński's Theorem 3, we have the following.

Theorem 12. *Suppose that $A \subseteq [\omega_1]^{\aleph_1}$ is a 2-almost disjoint, point-countable family of size $\aleph_1$ with the property that for each $a \in A$, $A \setminus \{a\}$ covers a. Then there is a σ-transversal $\{X_n : n \in \omega\}$, $\{A_n : n \in \omega\}$ such that $\{X_n : n \in \omega\}$ covers $\bigcup A$.*

Proof. Let $\{M_\alpha : \alpha < \omega_1\}$ be a continuous $\in$-sequence of countable elementary submodels of a suitably large $H(\theta)$ such that $A \in M_0$. Let $A \cap M_0 = \{a^0_n : n \in \omega\}$. For each n let $X^0_n = a^0_n \cap M_0$ and let $A^0_n = \{a \in M_0 \cap A : |a \cap a^0_n| = 1$ and $a \notin A^0_k$ for any $k < n\}$. By our assumptions and by elementarity we have

(a) $\bigcup\{X^0_n : n \in \omega\} = (\bigcup A) \cap M_0$,
(b) for each $\xi \in X^0_n$ there is $a \in A^0_n$ such that $a \cap X^0_n = \{\xi\}$.

Suppose X^β_n and A^β_n have been constructed for each $\beta < \alpha$ and each $n \in \omega$ such that the following inductive hypotheses are satisfied:

(c) $\bigcup\{X_n^\beta : n \in \omega\} = \bigcup A \cap M_\beta$,
(d) for each $a \in A_n^\beta$, $|a \cap X_n^\beta| = 1$,
(e) for each $\xi \in X_n^\beta$ there is $a \in A_n^\beta$ such that $a \cap X_n^\beta = \{\xi\}$,
(f) for $\beta < \gamma < \alpha$, $X_n^\beta \subseteq X_n^\gamma$ and $A_n^\beta \subseteq A_n^\gamma$.
(g) $A \cap M_\beta = \bigcup_n A_n^\beta$.
(h) $A_n^\beta \cap A_m^\beta = \emptyset$ for all $m \notin n$.

If α is a limit, we let $X_n^\alpha = \bigcup_{\beta<\alpha} X_n^\beta$ and $A_n^\alpha = \bigcup_{\beta<\alpha} A_n^\beta$. The inductive hypotheses are preserved.

If $\alpha = \beta + 1$, enumerate $M_\alpha \cap A$ as $\{a_n^\alpha : n \in \omega\}$ such that if $a_n^\alpha \in M_\beta \cap A_k^\beta$, then $n \neq k$. For each n let

$$(a_n^\alpha)' = \left[a_n^\alpha \setminus \bigcup(A \cap M_\beta \setminus \{a_n^\alpha\})\right] \cap M_\alpha \subseteq M_\alpha \setminus M_\beta$$

and let

$$X_n^\alpha = X_n^\beta \cup (a_n^\alpha)'.$$

Finally, let

$$A_n^\alpha = A_n^\beta \cup \{a \in A \cap M_\alpha \setminus M_\beta : |a \cap X_n^\alpha| = 1 \text{ and } a \notin A_k^\alpha \text{ for any } k < n\}.$$

It remains to verify the inductive hypotheses:

(c) is easy but the only nontrivial case: if $\xi \in \bigcup A \cap M_\alpha \setminus M_\beta$, fix $a_n^\alpha \in A$ such that $\xi \in a_n^\alpha$. If $a_n^\alpha \in M_\beta$, then since A is strongly almost disjoint, by elementarity, a_n^α is the only element of M_β containing ξ. Thus, $\xi \in (a_n^\alpha)'$. If $a_n^\alpha \notin M_\beta$ and $\xi \notin (a_n^\alpha)'$, then there must be $a \in A \cap M_\beta$ such that $\xi \in a$. But this a is enumerated as some a_m^α and as in the previous case we would have $\xi \in (a_m^\alpha)'$.

(d), (f), (g), and (h) follow by construction.

(e) follows from point-countable and the assumption that each a is covered by $A \setminus \{a\}$: to see this, fix $\xi \in X_n^\alpha$. We may assume that $\xi \in (a_n^\alpha)'$ and that $\xi \notin M_\beta$. By assumption there is $a \in A \setminus \{a_n^\alpha\}$ such that $\xi \in a$. By elementarity this a may be taken from M_α, and by definition of $(a_n^\alpha)'$ we have that $a \notin M_\beta$. By point countable, we know that $a \cap M_\beta = \emptyset$. Thus, $a \cap X_n^\beta = \emptyset$, and since A is 2-almost disjoint we have that $|a \cap (a_n^\alpha)'| = 1$. Thus, $|a \cap X_n^\alpha| = 1$, as required. □

Corollary 13 (Sierpiński)**.** *$\omega_1 \times \omega_1$ can be covered by countably many functions and their inverses.*

Proof. Let A be the collection of rows and columns of $\omega_1 \times \omega_1$. Then A satisfies the hypotheses of Theorem 12. Let $\{X_n, A_n : n \in \omega\}$ be the σ-transversal that covers $\omega_1 \times \omega_1$. For each n let $f_n : \omega_1 \to \omega_1$ be defined so that if $\{\alpha\} \times \omega_1 \in A_n$ and if $(\alpha, \gamma) \in X_n$, then $f_n(\alpha) = \gamma$. Also, let $g_n : \omega_1 \to \omega_1$ be such that if $\omega_1 \times \{\alpha\} \in A_n$ and $(\gamma, \alpha) \in X_n$, then $g_n(\alpha) = \gamma$. Since $\{X_n : n \in \omega\}$ covers $\omega_1 \times \omega_1$ it follows that the functions $\{f_n : n \in \omega\}$ and the inverses $\{g_n^{-1} : n \in \omega\}$ cover $\omega_1 \times \omega_1$ as well. □

Theorem 14 (Kuratowski)**.** *The family of rows and columns of $\omega_2 \times \omega_2$ does not admit a covering σ-transversal. Moreover, no $\sigma - B(\aleph_1)$ family can be covering.*

Proof. Recall that $\omega_2 \times \omega_2$ cannot be covered by countably many functions and inverses of functions (see [7]). If $\{X_n : n \in \omega\}$ were a $\sigma - B(\aleph_1)$ family, wlog we can assume that for each n, X_n either intersects every column in a countable set, or intersects every row in a countable set. So, each X_n can either be covered by a countable family of functions or a countable family of inverses of functions. Contradiction □

Example. Consider the Σ-product of ω_1 many copies of ω_1 with the discrete topology. For each $f \in X$ and each $\alpha \in \omega_1$, let

$$A_{f\alpha} = \{g \in X : g(\alpha) \neq 0 \text{ and } g(\beta) = f(\beta) \text{ for } \beta \neq \alpha\}.$$

It is easy to see that the $A_{f\alpha}$'s form a point-countable 2-almost disjoint family. Thus, by Theorem 12 for any family Y of $\aleph_1$-many f's in X, there is a covering σ-transversal of the family $\{A_{f\alpha} : f \in Y,\ \alpha \in \omega_1\}$. In particular, if CH is assumed, then the whole space X admits a covering σ-transversal.

4. Open problems

It follows from Kuratowski's example that the assumption that $|A| \leq \omega_1$ in Theorem 12 (as well as Theorem 5) is necessary. However, we do not know if the other assumptions are also necessary.

Question 1. Given a point-countable strongly almost disjoint family $A \subseteq [\omega_1]^{\leq \aleph_1}$ of size $\aleph_1$ such that for each $a \in A$, $A \setminus \{a\}$ covers a, is there a σ-transversal $\{X_n : n \in \omega\}$ covering $\bigcup A$?

Question 2. Given a 2-almost disjoint family $A \subseteq [\omega_1]^{\leq \aleph_1}$ of size $\aleph_1$ such that for each $a \in A$, $A \setminus \{a\}$ covers a, is there a σ-transversal $\{X_n : n \in \omega\}$ covering $\bigcup A$?

Another basic question left open is what can be said about strongly almost disjoint families of size $2^{\aleph_0}$. Theorem 9 implies that $M(2^{\aleph_0}, \omega, \omega) \not\to \sigma - B(\aleph_0)$ is consistent.

Question 3. Is it consistent that $M(2^{\aleph_0}, \omega, \omega) \to \sigma - B(\aleph_0)$?

The existence of σ-Bernstein sets seems significantly weaker than the existence of Bernstein sets. Indeed, for κ any cardinal, any countable point-separating family of subsets of κ is a σ-Bernstein set for $\{X \in P(\kappa) : |X| > 1\}$. Thus,

Proposition 15. *For any X of cardinality at most $2^{\aleph_0}$, the collection of nonsingleton subsets of X admits a σ-Bernstein set. Thus, $M(2^{\aleph_0}, \omega, \omega) \to \sigma - B$.*

There are families of sets with no σ-Bernstein set. An interesting example comes from a measurable cardinal.

Example 16. Let κ be a measurable cardinal. If u is a countably complete ultrafilter on κ, then u does not admit a σ-Bernstein set.

Proof. Indeed if u is such an ultrafilter, and $\{X_n : n \in \omega\}$ is any family of subsets of κ, then for each n either X_n or its complement is in u. By countable completeness, there is an $X \in u$ such that $X \subseteq X_n$ or $X \cap X_n = \emptyset$ for every n. I.e., u has no σ-Bernstein set. □

However, large cardinals are not needed to get an example.

Proposition 17. *If $cf(\kappa) > 2^{\aleph_0}$, then $[\kappa]^\kappa$ has no σ-Bernstein set. If in addition $2^\kappa = \kappa^+$, then there is an almost disjoint family $A \subseteq [\kappa]^\kappa$ with no σ-Bernstein set.*

Proof. Consider a countable collection $\mathcal{X} = (X_n : n \in \omega)$ of subsets of κ. Let $B \in [\kappa]^\kappa$. For each $\alpha \in B$, let $a_\alpha = \{n : \alpha \in X_n\}$. Since $cf(\kappa) > 2^{\aleph_0}$, there is $a \subseteq \omega$ and $A \in [B]^\kappa$ such that, for each $\alpha \in A$, $\alpha \in X_n$ if and only if $n \in a$. Therefore A is either contained in or disjoint from each X_n. Thus, $[\kappa]^\kappa$ has no σ-Bernstein set. Moreover, if we assume $2^\kappa = \kappa^+$, we may first enumerate all countable collections

of subsets of κ as $\{\mathcal{X}_\alpha : \alpha \in \kappa^+\}$. Then let $\{B_\alpha : \alpha \in \kappa^+\} \subseteq [\kappa]^\kappa$ be an almost disjoint family. By the above argument, we may find $A_\alpha \in [B_\alpha]^\kappa$ such that A_α is not split by any member of $\mathcal{X}_\alpha$. Thus, $\{A_\alpha : \alpha \in \kappa^+\}$ has no σ-Bernstein set. $\square$

We can contrast this with the result from [3] that if S is a stationary subset of some λ and $\diamondsuit(S)$ holds, then for any family $\{A_\alpha : \alpha \in S\}$ such that each $A_\alpha \subseteq \alpha$ is infinite, there are $B_\alpha \subseteq A_\alpha$ such that the family $\{B_\alpha : \alpha \in S\}$ has no Bernstein set. Moreover, it was shown consistent that there is a strongly almost disjoint family $A \subseteq [\omega_{\omega+1}]^{\aleph_1}$ with no Bernstein set. However, it is possible that every strongly almost disjoint family admits a σ-Bernstein set.

Question 4. Does $M(\kappa, \lambda, \omega) \to \sigma - B$ for all κ and λ?[1]

Given a topological space X, a Bernstein set for X is a $B \subseteq X$ that intersects each copy of the Cantor set in X but does not contain any copies of the Cantor set. It was proven by Weiss [11] and Bregman, Šapirovskij and Šostak [1] that it is consistent that every topological space admits a Bernstein set. Whether any extra assumptions are needed to prove this was open until Shelah's recent construction [9]. It would be of interest to see if one can prove under weaker set-theoretic assumptions (or even ZFC) that every topological space admits a σ-Bernstein set.

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[1] Neil Rogers has shown that $\diamondsuit(\omega_2)$ implies that there is an almost disjoint family $A \subseteq [\omega_2]^{\aleph_0}$ which does not admit a σ-Bernstein set [8]. So the answer to Question 4 is consistently no for $\kappa = \aleph_2$ and $\lambda = \aleph_0$.