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ON THE EXTENSIONS OF HOMOGENEOUS POLYNOMIALS

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Abstract. We investigate the problem of the uniqueness of the extension of n-homogeneous polynomials in Banach spaces. We show in particular that in a nonreflexive Banach space X that admits contractive projection of finite rank of at least dimension 2, for every $n \geq 3$ there exists an n-homogeneous polynomial on X that has infinitely many extensions to X^{**}. We also prove that under some geometric conditions imposed on the norm of a complex Banach lattice E, for instance when E satisfies an upper p-estimate with constant one for some $p > 2$, any 2-homogeneous polynomial on E attaining its norm at $x \in E$ with a finite rank band projection P_x, has a unique extension to its bidual E^{**}. We apply these results in a class of Orlicz sequence spaces.

Let $\mathbb{F}$ be either a complex or real field, and let X be a Banach space over $\mathbb{F}$. A bounded multilinear form is an n-linear mapping $L : X^n \to \mathbb{F}$ with a finite norm $\|L\|$ that is defined as

$$\|L\| = \sup\big\{|L(x_1, \ldots, x_n)| : \|x_i\| \leq 1 \text{ for all } i \leq n\big\}.$$

The mapping $P : x \mapsto L(x, x, \ldots, x)$ is called an *n-homogeneous polynomial* on X, and its norm is defined by

$$\|P\| = \sup\big\{|P(x)| : \|x\| \leq 1\big\}.$$

For every n-homogeneous polynomial P there exists [5] a unique symmetric n-linear form $L : X^n \to \mathbb{F}$ such that $P(x) = L(x, \ldots, x)$.

Aron and Berner [1] first studied the problem of norm-preserving extensions of polynomials in Banach spaces. They showed that, in general, a continuous extension of a bounded homogeneous polynomial from a subspace to the entire space does not exist. They showed however that such an extension always exists from the space to its bidual, and Davie and Gamelin [4] later showed that the Aron-Berner extension is a norm-preserving one. Recently the problem of uniqueness of the above extensions have been investigated in [2, 3, 8]. It has been shown in particular that for any real Banach space and a closed proper subspace Y of X, for any $n \geq 2$ there exists an n-homogeneous polynomial on Y which has infinitely many norm-preserving extensions to X. The similar result holds true for $n \geq 3$ in some classes of complex Banach lattices [8]. In the case of a 2-homogeneous polynomial in a complex space X, it was proved that a norm-attaining polynomial in X has

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a unique norm-preserving extension to X^{**}, whenever X is the space c_0 [2] or the Marcinkiewicz sequence space (predual to the Lorentz sequence space) [3, 8].

In this article we continue to study the problems concerning norm-preserving extensions of n-homogeneous polynomials in more general settings. As a consequence we enlarge the class of spaces for which we can find infinite many extensions, as well as the class of spaces that enjoy uniqueness of such extensions. We show in particular that the class of spaces with the unique extension property also includes the spaces with the norms not of supremum type, for instance some Orlicz spaces, contrary to all previously known examples. The proofs use a quite different technique, for example, they do not employ the Maximum Modulus Theorem which was fundamental for proofs in spaces with supremum type norm.

In the first section, we prove that if a Banach space X admits a finite rank contractive projection on a subspace Y, then for every $n \geq 3$ and any proper subspace Z of X that contains Y, there is a norm-attaining n-homogeneous polynomial P on Z that has at least two and hence infinitely many norm-preserving extensions to its bidual X. As corollaries we obtain several results proved in [8], and we provide some new examples as well.

In the second part we study the uniqueness of extension of 2-homogeneous polynomials in complex Banach lattices. Compared to the known results [2, 3, 8], we essentially enlarge the pool of spaces where this rather rare phenomenon of unique extension holds. In particular, it occurs when the spaces are complex Banach lattices satisfying an upper p-estimate with constant one for some $p > 2$. The main result states that if E is a Banach lattice such that for a unit vector $x \in E$ with the property that the band projection P_x is of finite rank, and there exist $p > 2$, $\alpha, \epsilon > 0$ such that

$$\|x + y\| \leq 1 + \alpha \|y\|^p$$

for all $\|y\| \leq \epsilon$ disjoint to x, then every 2-homogeneous polynomial on E attaining its norm at x has a unique norm-preserving extension to its bidual E^{**}. We then apply this result in a class of Orlicz sequence spaces ℓ_M. Under some natural assumptions on M, we show that a norm-attaining 2-homogeneous polynomial on a proper subspace of all order continuous elements h_M of ℓ_M has a unique norm-preserving extension to the entire space ℓ_M, which is the bidual of h_M. The latter result also remains true up to equivalent norm in a larger class of Orlicz spaces.

Now recall some notions in Banach lattices [11]. Let E be a Banach lattice and let Y be a sublattice of E. $Y^\perp$ denotes the set of all elements $x \in E$ which are disjoint from every $y \in Y$. Y is said to be a *projection band* if $E = Y \oplus Y^\perp$. Given a projection band Y of E, a positive projection P from E to Y, which vanishes on $Y^\perp$ is called a *band projection*. Then $0 \leq Px \leq x$ for any $x \geq 0$ and $\|P\| = 1$. Recall that a Banach lattice E is *σ-order complete* if every bounded sequence in E has a least upper bound. Let E be a σ-order complete Banach lattice and x a nonzero element in E. Then the set $Y_x = \{y \in E : |y| = \sup_{n \in N}\big(|nx| \wedge |y|\big)\}$ is a projection band of E and $P_x(y) = \bigvee_{n=1}^{\infty}(nx \wedge y)$ is a band projection from E onto Y_x.

1. Nonunique extensions of polynomials

In this section we discuss several cases when n-homogeneous polynomials, $n \geq 2$, on a subspace Y of a Banach space X cannot have unique norm-preserving extensions to X. The results contained in this section are true for both real and

complex spaces. However, their true value lies in the case of complex spaces in view of Theorem 2.1 in [8].

Proposition 1.1. *Let Y be a subspace of a Banach space X so that there is a contractive projection R from X onto Y. Let n be any natural number. Suppose there are two nonzero linear functionals ϕ_1, ϕ_2 on Y such that $\|\phi_1\| = 1$, and*

$$|\phi_1(y)|^n + |\phi_2(y)|^n \leq \|y\|^n \quad \textit{for all } y \in Y.$$

Then for any $m > n$ and any proper subspace Z of X that contains Y, there is an m-homogeneous polynomial P on Z which has at least two norm-preserving extensions to X.

Proof. Let P be the m-homogeneous polynomial on X defined by

$$P(x) = \big(\phi_1 \circ R(x)\big)^m \quad \text{for all } x \in X.$$

Then

$$\|P\| = \|P|_Z\| = \|P|_Y\| = 1.$$

By the assumptions, ϕ_1, ϕ_2 are linearly independent, $X \neq Z$, and $\|\phi_2\| \leq 1$. So there are $y \in Y$ and $w \in X \setminus Z$ such that $\phi_2(y) \neq 0$ and $R(w) = 0$. By the Hahn-Banach theorem, there is a linear functional ψ on X such that $\|\psi\| = 1$, $\psi|_Z = 0$, and $\psi(w + y) \neq 0$. Define an m-homogeneous polynomial Q by

$$Q(x) = P(x) + \big(\phi_2 \circ R(x)\big)^{m-1}\psi(x).$$

Since $\psi|_Z = 0$, $\phi_2(y) \neq 0$, and $R(w) = 0$, we have

$$\begin{aligned} P|_Z =& Q|_Z, \\ Q(y + w) =& P(y) + \big(\phi_2(y)\big)^{m-1}\psi(y + w) \neq P(y) = P(y + w), \end{aligned}$$

and

$$\begin{aligned} |Q(x)| =& \big|\big(\phi_1 \circ R(x)\big)^m + \big(\phi_2 \circ R(x)\big)^{m-1}\psi(x)\big| \\ \leq& \|x\|^{m-n}\Big(\big|\phi_1 \circ R(x)\big|^n + \big|\phi_2 \circ R(x)\big|^n\Big) \\ \leq& \|x\|^{m-n}\|R(x)\|^n \leq \|x\|^m. \end{aligned}$$

This implies that P and Q are two distinct norm-preserving extensions of $P|_Z$ to X. The proof is complete. □

Note that Theorem 2.2 in [8] is a particular case of the next corollary, when X is a Banach function space with a two-rank conditional expectation contractive projection.

Corollary 1.2. *Let Y be a finite-dimensional subspace of a Banach space X such that $\dim(Y) \geq 2$. Suppose that there is a contractive projection from X to Y. Then for any $m \geq 3$ and any proper subspace Z of X that contains Y, there is an m-homogeneous polynomial P on Z which has at least two norm-preserving extensions.*

Proof. Without loss of generality, we may assume that $Y = \mathbb{F}^n$. Let $\|\cdot\|_2$ be the Euclidean norm on Y and let "$\cdot$" be the inner product associated to the Euclidean norm $\|\cdot\|_2$ on Y. Let S be the set

$$S = \{y \in Y : \|y\| \leq 1\}.$$

Since Y is finite dimensional, there is $u \in S$ such that $\|u\|_2 = \sup\{\|y\|_2 : y \in S\}$. Let v be any unit vector in Y such that $v \cdot u = 0$ and let ϕ_1, ϕ_2 be two linear functionals on Y defined by

$$\begin{aligned}\phi_1(y) =& y \cdot \frac{u}{\|u\|_2^2},\\ \phi_2(y) =& y \cdot \frac{v}{\|u\|_2}.\end{aligned}$$

It is easy to see that $\|\phi_1\| = 1$ (in $(Y, \|\cdot\|)$ norm). Note that for any orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of a Hilbert space W and any $w \in W$,

$$\|w\|_2^2 = \sum_{k=1}^{\infty} |w \cdot e_k|^2.$$

Thus for any $y \in S$,

$$|\phi_1(y)|^2 + |\phi_2(y)|^2 \leq \frac{\|y\|_2^2}{\|u\|_2^2} \leq 1.$$

By Proposition 1.1, for any $m \geq 3$, there is an m-homogeneous polynomial on Z that has at least two norm-preserving extensions to X. □

Corollary 1.3. *Let E be a nonreflexive σ-order complete Banach lattice. Suppose there exist a unit vector $x \in E$, a support functional x^* of x and $y \in E$ such that $|y| \wedge |x| = 0$, $x^*(y) \neq 0$, and the projection P_{x+y} is of finite rank. Then there is a 2-homogeneous polynomial on E that has at least two norm-preserving extensions to E^{**}.*

Proof. Let $Y = P_{x+y}(E)$. Since P_{x+y} has finite rank, $Y = P_{x+y}(E^{**})$. Let ϕ_1, ϕ_2 be linear functionals on Y defined by

$$\begin{aligned}\phi_1(z) =& x^* \circ P_x(z),\\ \phi_2(z) =& x^* \circ (I - P_x)(z).\end{aligned}$$

By the assumptions, $\|\phi_1\| = 1$, ϕ_2 is nonzero. Note that P_x is a positive projection. Thus for any $z \in Y$,

$$\begin{aligned}|\phi_1(z)| + |\phi_2(z)| =& |x^* \circ P_x(z)| + |x^* \circ (I - P_x)(z)|\\ \leq& |x^* \circ P_x|(|z|) + |x^* \circ (I - P_x)|(|z|)\\ \leq& |x^*|(|z|) \leq \|z\|.\end{aligned}$$

By Proposition 1.1, there is a 2-homogeneous polynomial on E that has at least two norm-preserving extensions to E^{**}. The proof is complete. □

We can apply the above results to some particular classes of spaces.

Example 1.4. Let $\{\Psi(n)\}$ be an increasing sequence such that $\Psi(n)/n$ is decreasing, $\Phi(n) > 0$ for all $n \in \mathbb{N}$, and $\lim_n \Psi(n) = \infty$. Recall that the *Marcinkiewicz sequence space* m_Ψ consists of all sequences $x = (x(n))$ such that

$$\|x\|_{m_\Psi} = \sup_{n \geq 1} \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)} < \infty,$$

where $x^* = (x^*(n))$ is the decreasing rearrangement of $(x(n))$. Let m_Ψ^0 be the subspace of m_Ψ consisting of all $x \in m_\Psi$ satisfying

$$\lim_n \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)} = 0.$$

Then the bidual of m_Ψ^0 coincides with m_Ψ and (e_k) is a basis in m_Ψ^0 ([9]). Suppose that $\Psi(k) = \Psi(k+1)$. Then it is easy to check that $\frac{1}{\Psi(k)}\sum_{j=1}^{k+1} e_j^*$ is a support functional of $\sum_{j=1}^{k} e_j$. By Corollary 1.3, there is a 2-homogeneous polynomial P on m_Ψ^0 that has at least two norm-preserving extensions to m_Ψ. We have just proved that if Ψ is not strictly increasing, then there exists a 2-homogeneous polynomial on m_Ψ^0 whose extension to m_Ψ is not unique. This fact was proved in [8] by a different method as part of Theorem 4.4.

Example 1.5. Let $\|\cdot\|$ and $|||\cdot|||$ be equivalent norms defined on ℓ_∞ by

$$\begin{aligned}\|x\| =& |x(1)| + |x(2)| + \sup\{|x(n)| : n \geq 3\},\\ |||x||| =& \sup\{|x(i)| + |x(j)| : i \neq j\}.\end{aligned}$$

Then $(c_0, \|\cdot\|)^{**} = (\ell_\infty, \|\cdot\|)$ and $(c_0, |||\cdot|||)^{**} = (\ell_\infty, |||\cdot|||)$. Let (e_k) be the standard basis of c_0 and (e_k^*) the biorthogonal functionals associated to the basis (e_k). Then $e_1^* + e_2^*$ is a support functional of e_1 in c_0 with respect to both norms $\|\cdot\|$ and $|||\cdot|||$. Thus the assumptions of Corollary 1.3 are satisfied, and there is a 2-homogeneous polynomial P on $(c_0, \|\cdot\|)$ (respectively, $(c_0, |||\cdot|||)$) that has at least two norm-preserving extensions to ℓ_∞. Note that ℓ_∞ with respect to the norm $\|\cdot\|$ has been considered in [8] as Example 3.8.

The next two examples are independent of the results proved above and provide other natural spaces for which there is a 2-homogeneous polynomial with at least two different extensions to their biduals. In fact, the first one suggested using the index $p > 2$ in Theorem 2.1 in the next section, where there are stated conditions on the uniqueness of the extensions of polynomials.

Example 1.6. Let p be a real number such that $1 < p \leq 2$ and let $\|\cdot\|$ be the equivalent norm defined on ℓ_∞ by

$$\|x\| = \sup_{i\neq j}(|x(i)|^p + |x(j)|^p)^{1/p}.$$

Then $(c_0, \|\cdot\|)^{**} = (\ell_\infty, \|\cdot\|)$. Let ϕ be the Banach limit on ℓ_∞ and let P, Q be 2-homogeneous polynomials on ℓ_∞ defined by

$$\begin{aligned}P(x) =& \big(x(1)\big)^2,\\ Q(x) =& P(x) + \big(\phi(x)\big)^2.\end{aligned}$$

It is easy to see that $P|_{c_0} = Q|_{c_0}$ and $\|P\| = \|P|_{c_0}\| = 1$. If x is the constant 1 sequence, then $P(x) = 1$ and $Q(x) = 2$. We claim that $\|Q\| = 1$.

Let x be any unit vector in $(\ell_\infty, \|\cdot\|)$. Then for any $j \geq 2$, $|x(j)| \leq (1-|x(1)|^p)^{1/p}$. So

$$|\phi(x)| \leq \limsup_j |x(j)| \leq (1 - |x(1)|^p)^{1/p}$$

and

$$|Q(x)| \leq |\phi(x)|^2 + |x(1)|^2 \leq |x(1)|^2 + (1 - |x(1)|^p)^{2/p} \leq 1 = \|x\|^2.$$

We have proved that P and Q are two distinct norm-preserving extensions of $P|_{c_0}$ to ℓ_∞.

Example 1.7. Let X be a nonreflexive Banach space and ϕ_1, ϕ_2 two linear functionals on X^{**} such that $\|\phi_1|_X\| = \|\phi_1\| = 1 = \|\phi_2\|$ and $\phi_2|_X = 0$. Let P, Q be the 2-homogeneous polynomials on $(X^{**} \oplus X^{**})_2$ defined by

$$\begin{aligned} P(x_1^{**}, x_2^{**}) =& \phi_1^2(x_1^{**}), \\ Q(x_1^{**}, x_2^{**}) =& \phi_1^2(x_1^{**}) + \phi_2^2(x_2^{**}). \end{aligned}$$

Then $\|P\| = \|Q\| = \|P|_{X\oplus X}\| = 1$. So P and Q are two distinct norm-preserving extensions of $P|_{X\oplus X}$.

2. Unique extensions of polynomials

In this section our main result is Theorem 2.2, which states that under some geometric conditions of the norm of a Banach lattice E, 2-homogeneous polynomials have unique norm-preserving extensions from E to E^{**}. In view of Theorem 2.1 in [8] mentioned before, here we shall assume that all spaces are over the field of complex numbers $\mathbb{C}$. For any complex number $a \in \mathbb{C}$, let $\operatorname{sgn} a = a/|a|$ if $a \neq 0$ and $\operatorname{sgn} a = 0$ if $a = 0$. By $\Re a$ and $\Im a$ we denote the real and imaginary part of a, respectively. We start with a technical result, which is a basis for the next results.

Theorem 2.1. *Let E be a complex σ-order complete Banach lattice and x, y two unit vectors in E such that $|x| \wedge |y| = 0$. Suppose there are $\alpha, \epsilon > 0$ and $p > 2$ such that*

$$\|x + \beta y\| \leq 1 + \alpha|\beta|^p \quad \text{for all } |\beta| \leq \epsilon. \tag{1}$$

If P is a 2-homogeneous polynomial on E that attains its norm at x and if L is the unique symmetric bilinear form associated to P, then

$$L(y, y) = 0 \quad \text{and} \quad L(S(x), y) = 0 \quad \text{for any band projection } S \text{ on } E.$$

Proof. Without loss of generality, we assume that $\|P\| = P(x) = \|x\|^2 = 1$.

Claim 1. We claim that $L(x, y) = 0$ and $L(y, y) = 0$. Replacing y by $\overline{\operatorname{sgn}(L(x, y))}y$, we may assume that $L(x, y) \geq 0$. Then by disjointness of x and y, and by (1), for c close to zero we have

$$0 \leq \frac{1}{c}(\|x + y\|^2 - \|x\|^2) \leq \frac{1}{c}((1 + \alpha|c|^p)^2 - 1).$$

Hence

$$\begin{aligned} 0 =& \lim_{c\downarrow 0} \frac{1}{c}\big(\|x + cy\|^2 - \|x\|^2\big) \\ \geq& \lim_{c\downarrow 0} \frac{1}{c}\big(|L(x + cy, x + cy)| - |L(x, x)|\big) \\ =& \lim_{c\downarrow 0} \frac{1}{c}\big[(P(x) + 2cL(x, y) + c^2\Re(P(y)))^2 + (c^2\Im(P(y))^2)^{1/2} - P(x)\big] \\ =& 2L(x, y). \end{aligned}$$

Thus $L(x, y) = 0$. Now replacing y by $\overline{\big(\operatorname{sgn}(L(y, y))\big)}^{1/2}y$, we may assume that $L(y, y) \geq 0$. Again applying (1) and $L(x, y) = 0$,

$$\begin{aligned} 0 =& \lim_{c\downarrow 0} \frac{1}{c^2}\big(\|x + cy\|^2 - \|x\|^2\big) \\ \geq& \lim_{c\downarrow 0} \frac{1}{c^2}\big(L(x + cy, x + cy) - L(x, x)\big) = L(y, y). \end{aligned}$$

This implies that $L(y,y)=0$, and we have proved Claim 1.

Claim 2. Let S be any band projection on E. We claim that $L(S(x),x)$ is real. Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(\theta)=|P(e^{i\theta}S(x)+(I-S)(x))|$. Then f attains its maximum at 0. Note that $\lim_{\theta\to 0}\frac{e^{i\theta}-1}{\theta}=i$ and $P(x)=1$. Thus

$$\begin{aligned}
0 \le & \lim_{\theta\downarrow 0}\frac{|P(e^{\pm i\theta}S(x)+(I-S)(x))|-|P(x)|}{\theta}\\
= & \lim_{\theta\downarrow 0}\frac{|P((e^{\pm i\theta}-1)S(x)+x)|-1}{\theta}\\
= & \lim_{\theta\downarrow 0}\frac{|P(x)+2(e^{\pm i\theta}-1)L(S(x),x)+(e^{\pm i\theta}-1)^2P(S(x))|-1}{\theta}\\
= & \lim_{\theta\downarrow 0}\frac{|1+2(e^{\pm i\theta}-1)L(S(x),x)|-1}{\theta}\\
\ge & \lim_{\theta\downarrow 0}\frac{\Re((e^{\pm i\theta}-1)L(S(x),x)}{\theta}\\
= & \lim_{\theta\downarrow 0}\frac{(\cos\theta-1)\Re(L(S(x),x))\mp\Im(L(S(x),x))\cdot\sin\theta}{\theta}\\
= & \mp\Im(L(S(x),x),
\end{aligned}$$

and we have proved Claim 2.

Claim 3. Fix a real number θ. We claim that for any $|\beta|\le\epsilon$,

$$|P(S(x)+e^{i\theta}(I-S)(x))|+2|L(S(x)+e^{i\theta}(I-S)(x),\beta y)|\le(1+\alpha|\beta|^p)^2.$$

In fact, for any β with $|\beta|\le\epsilon$, in view of $P(y)=0$ we get

$$\begin{aligned}
&\big|P\big(S(x)+e^{i\theta}(I-S)(x)\big)+2L\big(S(x)+e^{i\theta}(I-S)(x),\beta y\big)\big|\\
&\quad=\big|P\big(S(x)+e^{i\theta}(I-S)(x)\big)+2L\big(S(x)+e^{i\theta}(I-S)(x),\beta y\big)+P(\beta y)\big|\\
&\quad=\big|P\big(S(x)+e^{i\theta}(I-S)(x)+\beta y\big)\big|\le\|S(x)+e^{i\theta}(I-S)(x)+\beta y\|^2\\
&\quad\le\|\,|S(x)|+|(I-S)(x)|+|\beta y|\,\|^2=\|\,|S(x)+x-S(x)|+|\beta y|\,\|^2\\
&\quad=\|x+\beta y\|^2\le(1+\alpha|\beta|^p)^2.
\end{aligned}$$

Since $P(S(x)+e^{i\theta}(I-S)(x))$ is independent of β, we have proved Claim 3.

Now let $h:\mathbb{R}\to\mathbb{R}$ be defined by

$$h(\theta)=|1+2(e^{i\theta}-1)L(x,S(x))+(e^{i\theta}-1)^2P(S(x))|-1.$$

It is clear that $h(0)=0$. Applying Claim 2 we can compute that $h'(0)=0$ and we also see that h is twice differentiable in a neighborhood of zero. So, for $\beta>0$ we obtain

$$\begin{aligned}
&\lim_{\beta\downarrow 0}\frac{\big|1+2(e^{i\beta^{p/2}}-1)L(x,S(x))+(e^{i\beta^{p/2}}-1)^2P(S(x))\big|-1}{\beta^{(p+2)/2}}\\
&\qquad=\lim_{\beta\downarrow 0}\beta^{(p-2)/2}\cdot\frac{h(\beta^{p/2})}{\beta^p}=0.
\end{aligned}$$

Let $\gamma = |L(S(x), y)|$. Then applying Claim 1, for any real number θ,

$$\begin{aligned}
&\big|P\big(S(x) + e^{i\theta}(I-S)(x)\big)\big| + 2\big|L\big(S(x) + e^{i\theta}(I-S)(x), \beta y\big)\big| \\
&\quad = \big|P(x) + 2(e^{-i\theta}-1)L(x, S(x)) + (e^{-i\theta}-1)^2 P(S(x))\big| + 2|\beta|\gamma|1-e^{i\theta}| \\
&\quad = \big|1 + 2(e^{-i\theta}-1)L(x, S(x)) + (e^{-i\theta}-1)^2 P(S(x))\big| + 4|\beta|\gamma\Big|\sin\frac{\theta}{2}\Big|.
\end{aligned}$$

Thus by Claim 3, letting $\theta = -\beta^{p/2}$ where $\beta > 0$, we have

$$\begin{aligned}
0 &= \lim_{\beta\downarrow 0} \frac{(1+\alpha|\beta|^p)^2 - 1}{\beta^{(p+2)/2}} \\
&\geq \lim_{\beta\downarrow 0} \frac{\big|P\big(S(x) + e^{-i\beta^{p/2}}(I-S)(x)\big)\big| + 2\big|L\big(S(x) + e^{-i\beta^{p/2}}(I-S)(x)), \beta y\big)\big| - 1}{\beta^{(p+2)/2}} \\
&= \lim_{\beta\downarrow 0} \frac{\big|1 + 2(e^{i\beta^{p/2}}-1)L(x, S(x)) + (e^{i\beta^{p/2}}-1)^2 P(S(x))\big| + 4\beta\gamma\sin\frac{\beta^{p/2}}{2}}{\beta^{(p+2)/2}} \\
&= \lim_{\beta\downarrow 0} \frac{h(\beta^{p/2})}{\beta^{(p+2)/2}} + \lim_{\beta\downarrow 0} \frac{4\beta\gamma\sin\frac{\beta^{p/2}}{2}}{\beta^{(p+2)/2}} = 2\gamma \geq 0.
\end{aligned}$$

The proof is complete. □

Theorem 2.2. *Let x be a unit vector in a complex σ-order complete Banach lattice E such that P_x is of finite rank. Suppose there are $\alpha, \epsilon > 0$ and $p > 2$ such that for any $y \in E$ with $|y| \wedge |x| = 0$ and $\|y\| \leq \epsilon$,*

$$\|x + y\| \leq \|x\| + \alpha\|y\|^p.$$

If P is any 2-homogeneous polynomial on E that attains its norm at x, then P has a unique norm-preserving extension to its bidual E^{**}.

Proof. Let P be a 2-homogeneous polynomial on E that attains its norm at x and let $\widetilde{P}$ be a norm-preserving extension of P to E^{**}. Then $\widetilde{P}$ also attains its norm at x. Let L be the unique symmetric 2-linear form on E^{**} such that

$$\widetilde{P}(w) = L(w, w) \quad \text{for all } w \in E^{**}.$$

Since P_x is of finite rank, $P_x(w) \in E$ for every $w \in E^{**}$. Thus in view of Theorem 2.1, for every $w \in E^{**}$, $L(P_x(w), w - P_x(w)) = L(w - P_x(w), w - P_x(w)) = 0$. This clearly implies that

$$\widetilde{P}(w) = \widetilde{P}(P_x(w)) = P(P_x(w))$$

for every $w \in E^{**}$, which shows uniqueness of the extension $\widetilde{P}$ and completes the proof. □

Theorem 3.2 in [8] is a particular case of Theorem 2.2. The geometric conditions assumed on the norm are quite strong, and they are naturally satisfied by supremum type of norms. Thus as corollaries we obtained characterizations on the uniqueness of the extensions in c_0 and in Marcinkiewicz space m^0_Ψ. Applying however Theorem 2.2, we are able to cover a wider class of spaces including those satisfying an upper p-estimate with constant one and Orlicz sequence spaces.

Recall that a Banach lattice $(E, \|\cdot\|)$ satisfies an upper p-estimate, $1 < p < \infty$, with constant one whenever $\|x + y\| \leq (\|x\|^p + \|y\|^p)^{1/p}$ for any disjoint elements $x, y \in E$ ([11]). The next corollary is an immediate consequence of Theorem 2.2.

Corollary 2.3. *Let E be a complex σ-order complete Banach lattice which satisfies the upper p-estimate with constant one for some $p > 2$. If x is a unit vector in E such that the projection P_x has finite rank, then any 2-homogeneous polynomial P on E attaining its norm at x has a unique extension to E^{**}.*

We finish the paper with some applications in Orlicz sequence spaces. Recall that a function $M : \mathbb{R}^+ \to \mathbb{R}^+$ is called an Orlicz function if M is continuous, convex, $M(0) = 0$ and M is not identically equal to zero. The Orlicz function M is said to satisfy condition Δ_2 at zero if there exist $u_0, K > 0$ such that $M(u_0) > 0$ and

$$M(2u) \leq KM(u) \quad \text{when } 0 \leq u \leq u_0.$$

By M^* let us denote the conjugate function to M defined as

$$M^*(u) = \sup_{v \geq 0}\{uv - M(v)\}, \quad u > 0.$$

The conjugate function M^* is an extended real-valued Orlicz function such that $M^{**} = M$. Given an Orlicz function M, the Orlicz sequence space ℓ_M and its subspace h_M are subspaces of the set of all complex or real-valued sequences $x = (x_n)$ defined as follows:

$$\ell_M = \Big\{x : \sum_{n=1}^{\infty} M(\lambda|x_n|) < \infty \text{ for some } \lambda > 0\Big\},$$

$$h_M = \Big\{x : \sum_{n=1}^{\infty} M(\lambda|x_n|) < \infty \text{ for all } \lambda > 0\Big\}.$$

The Orlicz space ℓ_M equipped with the norm

$$\|x\|_M = \inf\Big\{\lambda > 0 : \sum_{n=1}^{\infty} M(|x_n/\lambda|) \leq 1\Big\}$$

is a Banach space, and its subspace h_M consisting of all order continuous elements in ℓ_M is closed.

The following facts in sequence Orlicz spaces are well known:

(1) The spaces ℓ_M and h_M coincide if and only if M satisfies the Δ_2-condition at zero, which in turn is equivalent to ℓ_M being separable [10, Proposition 4.a.4].

(2) The dual of h_M is isometrically isomorphic to the Orlicz space $(\ell_{M^*}, \|\cdot\|^0_{M^*})$, where the Orlicz norm $\|\cdot\|^0_M$ is equivalent to the norm $\|\cdot\|_M$. If in addition M^* satisfies the Δ_2-condition at zero, then the bidual of h_M coincides with ℓ_M, since $(\ell_{M^*}, \|\cdot\|^0_{M^*})^* = (\ell_{M^{**}}, \|\cdot\|_{M^{**}})$ and $M^{**} = M$ [10, Proposition 4.b.1].

In the next theorem we show that for some Orlicz functions M and some 2-homogeneous polynomials P on h_M there is a unique extension of P to the entire Orlicz space ℓ_M.

Theorem 2.4. *Let M be an Orlicz function such that for some $p > 2$, $M(u)/u^p$ is increasing on some interval $(0, \epsilon)$ for which $M(\epsilon) > 0$. Assume also that M does not satisfy the Δ_2-condition at zero. Let x be a unit vector in h_M with finite support. If P is a 2-homogeneous polynomial on the complex space h_M attaining its norm at x, then P has a unique norm-preserving extension to its bidual $h_M^{**} = \ell_M$.*

Proof. We assume without loss of generality that $M(1) = 1$. First we shall show that M^* satisfies the Δ_2-condition at zero. By the assumption on M, for every $0 \le u \le \epsilon$ and $0 \le a \le 1$,

$$M(au) \le a^p M(u), \tag{2}$$

where $M(\epsilon) > 0$. Assume that $0 < \epsilon < 1$. It follows that $M(au)/au \le a^{p-1}M(u)/u$ for all $0 < u < \epsilon$, $0 < a < 1$, and so $\lim_{u\to 0^+} M(u)/u = 0$. We also have for any $v > 0$,

$$M^*(2v) = 2\sup_{u>0}\{uv - \frac{1}{2}M(u)\} = 2\sup_{0<u<u_v}\{uv - \frac{1}{2}M(u)\},$$

where

$$u_v = \sup\{u > 0 : \frac{M(u)}{2u} \le v\}.$$

In view of $\lim_{u\to 0^+} M(u)/u = 0$, it is clear that $0 < u_v \le \infty$. Since in addition $M(u)/u$ is increasing on $(0, \infty)$, there exists $\delta > 0$ such that $0 < u < \epsilon$ whenever $M(u)/2u < \delta$. Thus $0 \le u_\delta \le \epsilon$. Hence for any $0 < v < \delta$ we have $u_v \le u_\delta$, and if $0 < u < u_v$, then $0 < u < \epsilon$. Thus in view of (2), for $0 \le v \le \delta$,

$$\begin{aligned} M^*(2v) &= 2\sup_{0<u<u_v}\{uv - \frac{1}{2}M(u)\} \le 2\sup_{0<u<u_v}\{uv - M(u/2^{1/p})\} \\ &\le 2\sup_{u>0}\{uv - M(u/2^{1/p})\} = 2^{1/p+1}M^*(v), \end{aligned}$$

which shows the Δ_2-condition for M^*.

Thus we have that h_M is a proper subspace of ℓ_M and its bidual is ℓ_M. We shall complete the proof by showing that the inequality in Theorem 2.2 is satisfied in h_M. Indeed, let $y \in h_M$ be any element such that $|x| \wedge |y| = 0$ and $0 < \|y\|_M \le \epsilon$. Then $\|y/\epsilon\|_M \le 1$, and so $\sum_{n=1}^\infty M(|y_n|/\epsilon) \le 1$ and $\max_n |y_n| \le \epsilon$ in view of $M(1) = 1$. By inequality (2), for every $b \ge 1$ and $0 \le u \le \epsilon$ such that $bu \le \epsilon$ we have

$$M(bu) \ge b^p M(u).$$

Note that for every $n \in \mathbb{N}$, $|y_n| \le \epsilon$ and $|y_n|/\|y\|_M \le 1$. Setting $b = \epsilon/\|y\|_M$ and $u = |y_n|$ we have $b \ge 1$, $u \le \epsilon$ and $bu = \epsilon|y_n|/\|y\|_M \le \epsilon$. Therefore

$$1 \ge \epsilon\sum_{n=1}^\infty M\Big(\frac{|y_n|}{\|y\|_M}\Big) \ge \sum_{n=1}^\infty M\Big(\epsilon\frac{|y_n|}{\|y\|_M}\Big) \ge \Big(\frac{\epsilon}{\|y\|_M}\Big)^p \sum_{n=1}^\infty M(|y_n|).$$

Thus by the convexity of M and the above inequality,

$$\begin{aligned} \sum_{n=1}^\infty M\Big(\frac{|x_n + y_n|}{1+\epsilon^{-p}\|y\|_M^p}\Big) &= \sum_{n=1}^\infty M\Big(\frac{|x_n|}{1+\epsilon^{-p}\|y\|_M^p}\Big) + \sum_{n=1}^\infty M\Big(\frac{|y_n|}{1+\epsilon^{-p}\|y\|_M^p}\Big) \\ &\le \frac{1}{1+\epsilon^{-p}\|y\|_M^p}\Big(\sum_{n=1}^\infty M(|x_n|) + \sum_{n=1}^\infty M(|y_n|)\Big) \\ &\le \frac{1}{1+\epsilon^{-p}\|y\|_M^p}(1+\epsilon^{-p}\|y\|_M^p) = 1, \end{aligned}$$

and so $\|x + y\|_M \le 1 + \epsilon^{-p}\|y\|_M^p$, which completes the proof. □

The above result remains valid up to renorming for larger classes of Orlicz spaces ℓ_M. We note that the behaviour of the function $M(u)/u^r$ can be described in terms

of indices of M. Recall that the lower Matuszewska-Orlicz index (at zero) of an Orlicz function M is defined as [10]

$$\alpha_M = \sup\Big\{r \geq 1 : \text{ for some } u_0 > 0, \sup_{0<M(u)\leq M(u_0), 0<a<1} \frac{M(au)}{a^r M(u)} < \infty\Big\}.$$

Lemma 2.5. *Let $\alpha_M > r$, $1 < r < \infty$. Then there exists an equivalent renorming $\|\cdot\|_{\overline{M}}$ of $\|\cdot\|_M$ in ℓ_M such that $(\ell_M, \|\cdot\|_{\overline{M}})$ satisfies the upper r-estimate with constant one.*

Proof. Assume without loss of generality that $M(1) = 1$. First observe that by our assumption on M, for all $0 < a, u < 1$ and some $C > 0$,

$$M(au) \leq Ca^r M(u). \tag{3}$$

In fact, in view of definition of α_M the above inequality is satisfied for all $0 < a < 1$ and $0 < u \leq u_0$. However, if $0 < u_0 < 1$, then for all $u_0 \leq u \leq 1$ and $0 < a \leq u_0$ we have for some $D > 0$,

$$\frac{M(au)}{a^r M(u)} \leq \frac{M((a/u_0)u_0)}{a^r M(u_0)} \leq D\Big(\frac{a}{u_0}\Big)^r \frac{M(u_0)}{a^r M(u_0)} = \frac{D}{u_0^r},$$

which shows (3). Define

$$p(u) = \begin{cases} \sup_{0<M(v)\leq M(u)} \frac{M(v)}{v^r}, & \text{if} \quad M(u) > 0, \\ 0, & \text{if} \quad M(u) = 0. \end{cases}$$

Then in view of (3) it is easy to check that p is increasing and

$$\frac{M(u)}{u^r} \leq p(u) \leq C\frac{M(u)}{u^r}$$

for all $0 < u \leq 1$. Now let

$$\overline{M}(u) = \int_0^u p(t)t^{r-1}\,dt, \qquad u > 0.$$

Then $\overline{M}$ is an Orlicz function such that for any $0 < u < 1$,

$$M(\frac{1}{2}u) \leq \int_0^u \frac{M(t)}{t}\,dt \leq \overline{M}(u) \leq C\int_0^u \frac{M(t)}{t}\,dt \leq CM(u).$$

Thus M is equivalent to $\overline{M}$ at zero, and so the norms $\|\cdot\|_M$ and $\|\cdot\|_{\overline{M}}$ are equivalent and $\ell_M = \ell_{\overline{M}}$ [10, Proposition 4.a.5]. By simple checking we also have that for all $0 < a, u < 1$, $\overline{M}(au) \leq a^r\overline{M}(u)$. This implies that $\overline{M}(u)/u^r$ is increasing on the interval $(0, 1)$. Then applying Corollary 3.9 in [6], the space $(\ell_M, \|\cdot\|_{\overline{M}})$ satisfies the upper p-estimate with constant one. □

The last result is a consequence of Lemma 2.5 and Corollary 2.3.

Theorem 2.6. *Let M be an Orlicz function such that $\alpha_M > p$ for $2 < p < \infty$, and M does not satisfy the Δ_2-condition at zero. Then there exists a norm $\|\cdot\|_{\overline{M}}$ in ℓ_M equivalent to $\|\cdot\|_M$ such that Theorem 2.4 is satisfied in ℓ_M and h_M equipped with the norm $\|\cdot\|_{\overline{M}}$.*

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