ON FUNDAMENTAL GROUPS OF COMPACT HAUSDORFF SPACES

JAMES E. KEESLING AND YULI B. RUDYAK

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Abstract. We discuss which groups can be realized as the fundamental groups of compact Hausdorff spaces. In particular, we prove that the claim "every group can be realized as the fundamental group of a compact Hausdorff space" is consistent with the Zermelo–Fraenkel–Choice set theory.

In this note we prove that every group \( \pi \) can be realized as the fundamental group of a compact Hausdorff space, if we work in the pattern of the set theory without inaccessible cardinals. (See e.g. [2] concerning inaccessible cardinals.) Note that, because of a theorem of Kuratowski [2, Proposition 1.2], the absence of inaccessible cardinals is consistent with the Zermelo–Fraenkel–Choice set theory.

The idea is the following. Take a CW-space \( X \) with \( \pi_1(X) = \pi \) and let \( \beta X \) be the Stone–Čech compactification of \( X \), [4]. Then \( X \) is a path-connected component of \( \beta X \), and therefore \( \pi_1(\beta X, \ast) = \pi \) for all \( \ast \in X \).

This result should be contrasted with the result of Saharon Shelah [5] that for path connected, locally path connected compact metric spaces \( X \), \( \pi_1(X) \) is either a finitely generated group or has cardinality \( 2^{\aleph_0} \).

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All spaces are assumed to be Hausdorff, all maps and functions are assumed to be continuous. We denote by \( I \) the unit segment \([0, 1]\). Let \( \mathbb{N}^* \) denote the one-point compactification of the natural numbers \( \mathbb{N} \) with \( \ast \) being the point at infinity.

A dated, but useful, compendium of information on the Stone–Čech compactification can be found in [6]. See also [1]. The Stone–Čech compactification \( \beta X \) of the completely regular space \( X \) can be characterized as a topological embedding \( X \subset \beta X \) with \( \beta X \) compact and such that \( X \) is dense in \( \beta X \) and such that every function \( f : X \to I \) can be extended to a function \( \hat{f} : \beta X \to I \). There are several standard constructions of the Stone–Čech compactification. One construction uses the maximal ideals in the ring of bounded functions on \( X \), \( C^*(X) \), with the hull-kernel topology [1, Chapter 7].

Let \( \nu X \) denote the Hewitt realcompactification of \( X \), [1, Chapter 8]. Recall that \( X \subset \nu X \subset \beta X \) and is characterized by the property that every continuous function
X → ℜ (not necessarily bounded) can be extended to vX. A space X is called
realcompact if vX = X. Note that a realcompact space is not necessarily compact.

1. **Theorem.** If X is a paracompact space of non-measurable cardinality, then
vX = X.

**Proof.** Katetov [3] proved that a paracompact space X is realcompact iff each of its
closed discrete subspaces is realcompact. On the other hand, a discrete subspace is
realcompact iff it has non-measurable cardinality, [1, Chapter 12]. □

2. **Theorem.** Each non-discrete, closed subset in βX \ vX contains a copy of βN,
and so its cardinality is at least 2^c.

**Proof.** See [1, Theorem 9.11]. □

3. **Theorem.** If X is a path connected paracompact space of non-measurable car-
dinality, then X is a path component of βX.

**Proof.** Suppose that there exists a path α : I → βX with α(0) ∈ X and α(1) ∈ βX \ X. Then, by Theorems 1 and 2, α(I) ∩ (βX \ X) is a discrete set. So, we may
assume without loss of generality that α([0, 1)) is an infinite set; otherwise α(1) ∈ X.

Now let {t_n}_{n=1} be a sequence of distinct points in α([0, 1)) converging to α(1).
Define \( f(t_n) = n \mod 2 : \{t_n\}_{n=1}^{∞} \rightarrow [0, 1] \). Since X is paracompact and therefore
normal, let F : X → [0, 1] be any extension of f to all of X using Tietze’s Extension
Theorem. Then let \( \hat{F} : \beta X \rightarrow [0, 1] \) be the Stone-Čech extension of F to βX. Then
\( \hat{F}|\{t_n\}_{n=1}^{∞} = f \) has an extension to \( \{t_n\}_{n=1}^{∞} \cup \alpha(1) \), which is clearly a contradiction. □

4. **Corollary.** Every group of non-measurable cardinality is the fundamental group
of a compact space.

**Proof.** Let \( \pi \) be a given group. Let X be a connected CW-space having \( \pi_1(X) = \pi \).
We can assume that X has non-measurable cardinality. Since X is paracompact,
we conclude that \( \pi_1(\beta X, x_0) = \pi \) for any \( x_0 \in X \) in view of Theorem 3 □

5. **Remark.** Similar arguments show that, given a sequence \( \{A_n\}_{n=1}^{∞} \) of groups
of non-measurable cardinality, with \( A_n \) abelian for \( n ≥ 2 \), there exists a compact space
X with \( \pi_n(X) = A_n \) for all \( n ≥ 1 \).

6. **Remark.** Let us recall that every measurable cardinal is inaccessible, [2].

7. **Remark.** Of course, there exist groups of measurable cardinality that can be
realized as the fundamental groups of compact spaces. For example, the product
of any family of circles is compact, and so \( \mathbb{Z}^m \) with any cardinal \( m \) can be realized.

8. **Remark.** In the first version of the paper (that we put in an electronic archive)
we asked whether there exists a path connected compact space with prescribed
fundamental group of non-measurable cardinality. Recently Adam Przeździecki
notified us that the answer is affirmative. Also, he announced an example of a
group of measurable cardinality that is not the fundamental group of any compact
space.
References


Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, Florida 32611-8105
E-mail address: jek@math.ufl.edu

Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, Florida 32611-8105
E-mail address: rudyak@math.ufl.edu