PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 8, August 2007, Pages 2519–2526
S 0002-9939(07)08745-X
Article electronically published on March 2, 2007

INNER SEQUENCE BASED INVARIANT SUBSPACES IN $H^2(D^2)$

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(Communicated by Joseph A. Ball)

ABSTRACT. A closed subspace $H^2(D^2)$ is said to be invariant if it is invariant under the Toeplitz operators T_z and T_w. Invariant subspaces of $H^2(D^2)$ are well-known to be very complicated. So discovering some good examples of invariant subspaces will be beneficial to the general study. This paper studies a type of invariant subspace constructed through a sequence of inner functions. It will be shown that this type of invariant subspace has direct connections with the Jordan operator. Related calculations also give rise to a simple upper bound for $\sum_j 1-|\lambda_j|$, where $\{\lambda_j\}$ are zeros of a Blaschke product.

1. INTRODUCTION

In $H^2(D^2)$ with coordinates z and w, multiplications by z and w (denoted by T_z and T_w, respectively) are shift operators with infinite multiplicity. A subspace M is said to be $z(\textit{or } w)-\textit{invariant}$ if M is invariant under T_z (or T_w, respectively), and M is said to be invariant if it is invariant under both T_z and T_w. It is well-known that in general invariant subspaces of $H^2(D^2)$ can be very complex (cf. [Ru]), and their study demands new ideas and techniques (cf. [DP], [DM], [Ya1]), as well as good understanding of some examples. This paper studies a type of invariant subpace that is constructed through an *inner sequence*. A sequence of inner functions $\{q_j(z) : \ 0 \leq j \leq m\}$, where m may be infinite, is called an inner sequence if $q_{j+1} \mid q_j$ for each j. We will see that this type of invariant subspace has a simple structure, and it has direct connections with the Jordan operator.

The classical Hardy space $H^2(D)$ in the variable z and that in the variable w are different subspaces in $H^2(D^2)$, and we denote them by $H^2(z)$ and $H^2(w)$, respectively. For every $g \in H^2(w)$, we define an operator $\pi_g : \ H^2(D^2) \longrightarrow H^2(z)$ by

$$\pi_g(f)(z) = \int_T f(z,w)\overline{g}(w)dm(w), \quad f \in H^2(D^2),$$

where T is the unit circle and $dm(w)$ is the normalized Lebesgue measure on T. It is easy to check that π_g is bounded. In fact, one verifies that

$$\pi_g^* h = g(w)h(z), \quad h \in H^2(z),$$

Received by the editors November 4, 2005 and, in revised form, April 6, 2006.
2000 *Mathematics Subject Classification.* Primary 47A13; Secondary 46E20.
Key words and phrases. Core operator, Hardy space over the bidisk, Jordan operator, Blaschke product.

and hence $\|\pi_g\| = \|g\|$. We now look at a few facts regarding π_g. If M is z-invariant, then for every $g \in H^2(w)$

$$z\pi_g(M) = \pi_g(zM) \subset \pi_g(M).$$

If M is w-invariant, then

$$\pi_{w^i}(M) = \pi_{w^{i+1}}(wM) \subset \pi_{w^{i+1}}(M)$$

for every integer $i \geq 0$. So in the case when M is invariant, the closures $\overline{\pi_{w^i}(M)}$, $i = 0, 1, 2, ...$, is an increasing sequence of invariant subspaces of $H^2(z)$. By Beurling's theorem, this gives rise to a sequence of inner functions $\{q_i(z) : \ i \geq 0\}$ with $q_{j+1} \mid q_j$, i.e., an inner sequence, such that $\overline{\pi_{w^i}(M)} = q_i H^2(z)$. It then follows easily that

$$M \subset \bigoplus_{j=0}^{\infty} q_i H^2(z) w^j.$$

One observes that $\bigoplus_{j=0}^{\infty} q_i H^2(z) w^j$ is clearly z-invariant. Moreover, since $q_{j+1} \mid q_j$,

$$wq_j H^2(z) w^j = q_j H^2(z) w^{j+1} \subset q_{j+1} H^2(z) w^{j+1},$$

and this shows that $\bigoplus_{j=0}^{\infty} q_i H^2(z) w^j$ is also w-invariant.

Invariant subspace of the form $\bigoplus_{j=0}^{\infty} q_i H^2(z) w^j$, where $\{q_i(z) : \ i \geq 0\}$ is an inner sequence, shall be said to be inner-sequence-based in this paper, and it is the primary subject of this paper. The above observations indicate that every invariant subspace has a smallest inner-sequence-based invariant subspaces containing it (i.e., an inner-sequence-based envelope). Of course, this envelope is non-trivial only when q_0 is not a constant, or equivalently, $\overline{\pi_1(M)}$ is not dense in $H^2(z)$, since if q_0 is a constant, the inner sequence is a sequence of non-zero scalars and hence

$$\bigoplus_{j=0}^{\infty} q_i H^2(z) w^j = H^2(D^2).$$

Let M be z-invariant and $N = H^2(D^2) \ominus M$, and we denote the compression of T_z to N by S_z. It is well-known that S_z serves as a model for the so-called $C_{\cdot\, 0}$ class contractions, namely the class of contractions A with $(A^*)^n$ converging strongly to 0. Clearly, the $C_{\cdot\, 0}$ class is very large, for instance, every strict contraction is in it. If M is invariant (i.e., also invariant for T_w), then S_z on N is much less general, but it still represents a good class of interesting operators. For example, the Bergman shift and unilateral shifts with any multiplicity are all unitarily equivalent to S_z for M an invariant subspace. For convenience, we let $\mathcal{S}$ denote the collection of operators that are unitarily equivalent to S_z or S_w on $N = H^2(D^2) \ominus M$ for some invariant subspace M; here S_w is the compression of T_w to N. In Section 2, we will show that every Jordan operator is in $\mathcal{S}$. In Section 3, we show that Rudin's invariant subspace (cf. [Ru]), formerly believed to be pathological, is in fact inner-sequence-based. The *core operator* is an important associate of invariant subspaces of $H^2(D^2)$ (cf. [GY]). In Section 4 we show how one can compute the core operator explicitly for the case of an inner-sequence-base invariant subspace. Calculations in Section 4 lead to an interesting upper bound for $\sum_j 1 - |\lambda_j|$, where $\{\lambda_j\}$ are zeros of a Blaschke product. We will address this point in Section 5.

2. A note on Jordan operators

Let $H^2(D)$ be the Hardy space over the unit disk. Multiplication by coordinate function z on $H^2(D)$ is the unilateral shift, and its invariant subspace is of the form $\theta H^2(D)$, where θ is an inner function. The compression $S(\theta)$ of the unilateral shift to the quotient space $N_\theta := H^2(D) \ominus \theta H^2(D)$ is called a Jordan block. To be precise, $S(\theta)f = P_{N_\theta} zf, \quad f \in N_\theta$, where P_{N_θ} is the projection from $H^2(D)$ onto N_θ. For an inner sequence $\{q_j(z) : \ 0 \le j \le m\}$, the direct sum $\bigoplus_{j=0}^m S(q_j)$ is often called a Jordan operator. A celebrated result in the 70's is that every C_0 class operator on a separable Hilbert space is quasisimilar to a Jordan operator (cf. [Be]).

Jordan operators are directly connected with inner-sequence-based invariant subspaces. For an inner sequence $\{q_j(z) : \ j \ge 0\}$, and $M = \bigoplus_{j=0}^\infty q_j H^2(z) w^j$, we observe that:

$$N = (\bigoplus_{j=0}^\infty H^2(z)w^j) \ominus (\bigoplus_{j=0}^\infty q_j H^2(z)w^j) = \bigoplus_{j=0}^\infty (H^2(z) \ominus q_j H^2(z))w^j.$$

Let P_j denote the orthogonal projection from $H^2(D^2)$ onto $(H^2(z)\ominus q_j H^2(z))w^j$. Clearly, $P_N = \bigoplus_{j=0}^\infty P_j$. For every $f = \sum_{j=0}^\infty f_j(z)w^j \in N$, where $f_j \in H^2(z) \ominus q_j H^2(z)$,

$$\begin{aligned} S_z f &= (\bigoplus_{j=0}^\infty P_j)(\sum_{i=0}^\infty z f_i(z) w^j) \\ &= \sum_{i=0}^\infty (\bigoplus_{j=0}^\infty P_j z f_i(z) w^i) \\ &= \sum_{i=0}^\infty P_i z f_i(z) w^i \\ &= \sum_{i=0}^\infty (S(q_i) f_i) w^i. \end{aligned}$$

So with respect to the decomposition $N = \bigoplus_{j=0}^\infty (H^2(z) \ominus q_j H^2(z))w^j$,

$$S_z = \bigoplus_{j=0}^\infty S(q_j), \tag{2-1}$$

and we obtain the following fact.

Corollary 2.1. *Every Jordan operator is in $\mathcal{S}$.*

It is not hard to see that if q_j is a scalar starting from $j = k$, then the direct sum above is a finite sum of k terms.

It is also worth noting that (2-1) gives a necessary condition for an invariant subspace to be of inner-sequence-based type. For instance, (2-1) implies that $\sigma(S_z) \cap D$, being the zeros of $S(q_0)$, is discrete. One can easily come up with invariant subspaces for which $\sigma(S_z) \cap D$ is not discrete. For example, if we let $M = [z - w]$ be the closure of the ideal $(z - w)$ in the polynomial ring $C[z, w]$, then S_z on $H^2(D^2) \ominus M$ is equivalent to the Bergman shift, and hence $\sigma(S_z) = \bar{D}$. This indicates that $[z - w]$ is not inner sequence based. It is not difficult to check that the envelope of $[z - w]$ is $zH^2(z) \oplus \bigoplus_{j=1}^\infty H^2(z)w^j$.

3. Rudin's invariant subspace

In [Ru], Rudin constructed an invariant subspace of infinite rank as follows. Let M be the Hardy invariant subspace consisting of all functions in $H^2(D^2)$ which have a zero of order greater than or equal to n at $(\alpha_n, 0) = (1 - n^{-3}, 0)$ for any positive integer n. For quite a long time this invariant subspace was viewed as somewhat pathological. Here, we will show that it is in fact an inner-sequence-based invariant subspace. It is interesting because this means that Rudin's invariant subspace, despite the fact that it has infinite rank, has a very simple structure.

For simplicity, we denote $H^2(D^2)$ by H^2 and set $b_n(z) = (z - \alpha_n)/(1 - \alpha_n z)$. Then we have $M = \bigcap_{n\geq 1} M_n$, where

$$M_n = b_n^n(z)H^2 \vee b_n^{n-1}(z)wH^2 \vee \cdots \vee w^n H^2,$$

that is, M_n is the invariant subspace consisting of all functions in H^2 which have a zero of order greater than or equal to n at $(\alpha_n, 0)$. Further, we have

$$\begin{aligned} M_n &= b_n^n(z)H^2(z) \oplus b_n^{n-1}(z)wH^2(z) \oplus \cdots \oplus w^n H^2 \\ &= \sum_{k=0}^{n-1} \oplus b_n^{n-k}(z)w^k H^2(z) \oplus w^n H^2. \end{aligned}$$

We define a family of inner functions inductively as follows:

$$\begin{cases} q_0(z) = \prod_{n=1}^{\infty} b_n^n(z), \\ q_j(z) = q_{j-1}(z)/\prod_{n=j}^{\infty} b_n(z) \quad (j \geq 1). \end{cases}$$

Then $q_j(z)$ is divisible by $q_{j+1}(z)$ for every $j \geq 0$, and we have

$$M = \bigcap_{n\geq 1} M_n = \bigoplus_{j=0}^{\infty} q_j(z)H^2(z)w^j.$$

The fact that Rudin's example has infinite rank prompts the following conjecture.

Conjecture. *If $\{q_j(z)\}$ is an inner sequence such that q_{j+1} is a proper factor of q_j for each $j \geq 0$, then $\bigoplus_{j=0}^{\infty} q_j H^2(z)w^j$ has infinite rank.*

4. The core operator

For a invariant subspace M, we let $(R_z,\ R_w)$ denote the restriction of $(T_z,\ T_w)$ to M. So it is clear that $(R_z,\ R_w)$ is a pair of commuting isometries. The core operator C for M is defined as

$$C = I - R_z R_z^* - R_w R_w^* + R_z R_w R_z^* R_w^*.$$

A parallel associate for $(S_z,\ S_w)$ is

$$\Delta_S := I - S_z^* S_z - S_w^* S_w + S_z^* S_w^* S_z S_w.$$

Both C and Δ_S are useful tools in the study of invariant subspaces. We refer the readers to [GY] and [Ya2] for details. In this section, we will see that both C and Δ_S can be written as a direct sum with respect to the decomposition $M = \bigoplus_{j=0}^{\infty} q_j(z)H^2(z)w^j$. For simplicity we let $q_j' = \frac{q_{j-1}}{q_j},\ j = 1,\ 2,\ 3,\ ....$

Let $g = q_j f_j w^j$ be any function in $q_j H^2(z) w^j$, where $j \geq 1$. Then

$$\begin{aligned} R_w^*(q_j f_j w^j) &= [(I - P_{j-1})(q_j f_j)] w^{j-1} \\ &= \left(\sum_{i=0}^{\infty} \langle q_j f_j,\ q_{j-1} z^i \rangle q_{j-1} z^i \right) w^{j-1} \end{aligned}$$

and

$$\begin{aligned} (I - R_z R_z^*) g &= \sum_{i=0}^{\infty} \langle g,\ q_i w^i \rangle q_i w^i \\ &= \langle q_j f_j w^j,\ q_j w^j \rangle q_j w^j \\ &= f_j(0) q_j w^j. \end{aligned}$$

It follows that

$$\begin{aligned} Cg &= [I - R_z R_z^* - R_w (I - R_z R_z^*) R_w^*] g \\ &= f_j(0) q_j w^j - \langle q_j f_j,\ q_{j-1} \rangle q_{j-1} w^j \\ &= (\langle f_j,\ 1 \rangle 1 - \langle f_j,\ q_j' \rangle q_j') q_j w^j. \end{aligned}$$

This shows that $q_j H^2(z) w^j$ is invariant for C and on $q_j H^2(z) w^j$

$$C \cong 1 \otimes 1 - q_j' \otimes q_j'.$$

Moreover, since R_w^* is 0 on $q_0 H^2(z)$, it is easy to verify that $C \cong 1 \otimes 1$ on $q_0 H^2(z)$. Summarizing these observations we have

Corollary 4.1. *With respect to the decomposition $M = \bigoplus_{j=0}^{\infty} q_j H^2(z) w^j$,*

$$C \cong 1 \otimes 1 \bigoplus_{j=1}^{\infty} (1 \otimes 1 - q_j' \otimes q_j').$$

Now we take a look at Δ_S. Let f_j be any function in $H^2(z) \ominus q_j H^2(z)$, where $j \geq 1$. Then as indicated before, $S_z(f_j w^j) = (S(q_j) f_j) w^j$, and $S_w(f_j w^j) = (P_{j+1} f_j) w^j$. Writing

$$\Delta_S = (I - S_z^* S_z) - S_w^* (I - S_z^* S_z) S_w$$

and $I_j - S^*(q_j) S(q_j)$ as D_j, where I_j stands for the identity map on $H^2(z) \ominus q_j H^2(z)$, we compute that

$$\begin{aligned} \Delta_S(f_j w^j) &= w^j (I_j - S^*(q_j) S(q_j)) f_j - S_w^* \left(w^{j+1} (I_{j+1} - S^*(q_{j+1}) S(q_{j+1})) (P_{j+1} f_j) \right) \\ &= w^j D_j f_j - w^j D_{j+1} P_{j+1} f_j \\ &= w^j D_j f_j - w^j D_{j+1} (P_j - (P_j - P_{j+1})) f_j. \end{aligned}$$

One verifies that $D_{j+1}(P_j - P_{j+1})) = 0$, and hence

$$\Delta_S(f_j w^j) = (D_j - D_{j+1}) f_j w^j, \quad j \geq 0.$$

We summarize these observations in the following corollary.

Corollary 4.2. *With respect to the decomposition $N = \bigoplus_{j=0}^{\infty} (H^2(z) \ominus q_j H^2(z)) w^j$,*

$$\Delta_S \cong \bigoplus_{j=0}^{\infty} (D_j - D_{j+1}).$$

It is well-known that D_j is of rank 1 when q_j is non-trivial, and it can be explicitly expressed in terms of q_j. We will return to this point in Section 5.

Lemma 4.3. *Let f and g be non-zero functions in $H^2(z)$ and $A = f \otimes f - g \otimes g$. Then*

$$trA^2 = \|f\|^4 + \|g\|^4 - 2|\langle f,\ g\rangle|^2.$$

Proof. It is a simple calculation. Clearly, A is selfadjoint. Since

$$Af = \|f\|^2 f - \langle f,\ g\rangle g, \quad Ag = \langle g,\ f\rangle f - \|g\|^2 g,$$

with respect to the basis $\{f,\ g\}$, A has the matrix form

$$\begin{pmatrix} \|f\|^2 & \langle g,\ f\rangle \\ -\langle f,\ g\rangle & -\|g\|^2 \end{pmatrix}.$$

If we denote the eigenvalues of A by λ_1 and λ_2, then

$$\lambda_1 + \lambda_2 = \|f\|^2 - \|g\|^2, \quad \lambda_1\lambda_2 = -\|f\|^2\|g\|^2 + |\langle f,\ g\rangle|^2.$$

So

$$trA^2 = \lambda_1^2 + \lambda_2^2 = \|f\|^4 + \|g\|^4 - 2|\langle f,\ g\rangle|^2.$$

□

The following fact follows directly from Corollary 4.1 and the above lemma.

Corollary 4.4. $trC^2 = 1 + 2\sum_{j=1}^{\infty} 1 - |q_j'(0)|^2$.

Recall that for an invariant subspace M, its fringe operator F is defined on $M \ominus zM$ by

$$Ff = P_{M\ominus zM} wf, \quad f \in M \ominus zM.$$

It is indicated in [Ya1] that the fringe operator is also a very useful tool for the study in this area. It is interesting to see how the fringe operator acts on $M = \bigoplus_{j=0}^{\infty} q_j H^2(z) w^j$. It is not difficult to check that in this case

$$M \ominus zM = \bigoplus_{j=0}^{\infty} \mathbb{C} q_j(z) w^j,$$

and clearly $\{q_j(z)w^j :\ j \geq 0\}$ is an orthonormal basis for $M \ominus zM$. For every j,

$$\begin{aligned} F(q_j w^j) &= P_{M\ominus zM} q_j w^{j+1} \\ &= \langle q_j w^{j+1},\ q_{j+1} w^{j+1}\rangle q_{j+1}(z) w^{j+1} \\ &= \langle q_{j+1} q_{j+1}',\ q_{j+1}\rangle q_{j+1}(z) w^{j+1} \\ &= q_{j+1}'(0) q_{j+1}(z) w^{j+1}. \end{aligned}$$

This shows that the fringe operator in this case is a weighted shift with weighs

$$q_1'(0),\ q_2'(0),\ q_3'(0),\ \cdots.$$

5. An inequality about Blaschke products

For any bounded linear operator A on a Hilbert space H, the so-called minimum modulus

$$\gamma(A) := \inf\{\|Ax\| : \ x \in (\ker A)^{\perp}, \|x\| = 1\}$$

measures the norm of A's "partial inverse". Clearly, A has closed range if and only if $\gamma(A) > 0$. When A is invertible, $\gamma^{-1}(A) = \|A^{-1}\|$.

It is shown in [Ya2] that for every invariant subspace M

$$trC^2 \leq 2\gamma^{-2}(S_z) + 2\dim\ker(S_z) - 1. \tag{5-1}$$

It is interesting to see what this inequality means for an inner-sequence-based invariant subspace. Without loss of generality we assume $q_0(0) \neq 0$. Then by (2-1), $\ker(S_z)$ is trivial. To calculate $\gamma(S_z)$, we consider a general Jordan operator $S(\theta)$ on the space $N_\theta := H^2(D) \ominus \theta H^2(D)$. One verifies that

$$I - S^*(\theta)S(\theta) = P_{N_\theta}(\bar{z}\theta) \otimes P_{N_\theta}(\bar{z}\theta).$$

Therefore, for $g \in N_\theta$,

$$\begin{aligned}\|S(\theta)g\|^2 &= \|g\|^2 - |\langle g, \ P_{N_\theta}(\bar{z}\theta)\rangle|^2 \\ &\geq (1 - \|P_{N_\theta}(\bar{z}\theta)\|^2)\|g\|^2 \\ &= |\theta(0)|^2\|g\|^2,\end{aligned}$$

and the equality is obtained when $g = P_{N_\theta}(\bar{z}\theta)$. This shows that $\gamma(S(\theta)) = |\theta(0)|$. (2-1) then implies that

$$\gamma(S_z) = \inf\{|q_j(0)| : \ j \geq 0\}.$$

But since every q_j is a factor of q_0,

$$\gamma(S_z) = |q_0(0)|. \tag{5-2}$$

Combining Corollary 4.4, (5-1) and (5-2), we have

$$\sum_{j=1}^{\infty} 1 - |q_j'(0)|^2 \leq |q_0(0)|^{-2} - 1. \tag{5-3}$$

Or equivalently, we have

Corollary 5.1. *For* $M = \bigoplus_{j=0}^{\infty} q_j H^2(z) w^j, \quad trC^2 \leq 2|q_0(0)|^{-2} - 1.$

If we let $B(z)$ be a Blaschke product with zeros $\{\lambda_j : \ 1 \leq j \leq m\}$ counting multiplicity, where m can be infinity, and set

$$q_j = \prod_{i=j+1}^{m} \frac{\lambda_i - z}{1 - \overline{\lambda_i}z}, \quad j \geq 0,$$

then (5-3) leads to an interesting inequality for Blaschke products, namely

$$\sum_{j=1}^{\infty} 1 - |\lambda_j|^2 \leq |B(0)|^{-2} - 1. \tag{5-4}$$

Let us pursue this point a little further. Clearly, (5-4) is equivalent to the general inequality

$$\sum_{j=1}^{\infty} 1 - |\lambda_j|^2 \leq (\prod_{j=1}^{\infty} |\lambda_j|)^{-2} - 1$$

for any sequence $\{\lambda_j : j \geq 1\} \subset \bar{D}$ which can be proved easily by induction, and the equality holds only in the case when both sides are zero. So substituting λ_j by its square root, (5-4) leads to the following inequality.

$$\sum_j 1 - |\lambda_j| \leq |B(0)|^{-1} - 1, \tag{5-5}$$

where $B(z)$ is a Blaschke product with zeros $\{\lambda_j\}$ counting multiplicity. (5-5) is a simple inequality, but surprisingly it appears to have been unknown before. If $f \in H^p(D)$, where $1 \leq p \leq \infty$, and $\|f\|_p \leq 1$, and $\{\lambda_j : j \geq 1\}$ are its zeros counting multiplicity, then B is a factor of f, and $|B(0)| \geq |f(0)|$. So (5-5) can be generalized to the following corollary.

Corollary 5.2. *If $f \in H^p(D)$ with $\|f\|_p \leq 1$, where $1 \leq p \leq \infty$, and $\{\lambda_j : j \geq 1\}$ are its zeros counting multiplicity, then*

$$\sum_{j=1}^{\infty} 1 - |\lambda_j| \leq |f(0)|^{-1} - 1.$$

If $\|f\|_\infty \leq 1$, and $\eta \in D$, then $\frac{f(z)-\eta}{1-\bar{\eta}f(z)}$ is also a analytic self map of D, and hence Corollary 5.2 implies

$$\sum_{f(z)=\eta} 1 - |\eta| \leq \left| \frac{1-\bar{\eta}f(0)}{f(0)-\eta} \right| -1. \tag{5-6}$$

Acknowledgments

The authors wish to express their gratitude to K. Izuchi whose discussions with the authors have nurtured an important part of Section 3 and Section 5. Thanks go to R. O'Neil for his valuable input. The authors also thank H. Bercovici and D. Sarason for references on the inequality (5-5).

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