

ON THE REVERSIBLE QUADRATIC CENTERS WITH MONOTONIC PERIOD FUNCTION

J. VILLADELPRAT

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ABSTRACT. This paper is devoted to studying the period function of the quadratic reversible centers. In this context the interesting stratum is the family of the so-called Loud's dehomogenized systems, namely

$$\begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + Dx^2 + Fy^2. \end{cases}$$

We determine several regions in the parameter plane for which the corresponding center has a monotonic period function. To this end we first show that any of these systems can be brought by means of a coordinate transformation to a potential system. Then we apply a monotonicity criterium of R. Schaaf.

1. INTRODUCTION AND STATEMENT OF THE RESULT

The present paper deals with the period function of the quadratic centers. The literature has used different terminology to classify these centers but essentially there are four families: Hamiltonian, reversible Q_3^R , codimension four Q_4 and generalized Lotka-Volterra systems Q_3^{LV} . According to Chicone's conjecture [1] the reversible centers have at most two critical periods, and the centers of the three other families have a monotonic period function. In fact there is much analytic evidence that the conjecture is true. Indeed, Coppel and Gavrilov [7] proved that the period function of any Hamiltonian quadratic center is monotonous and, more recently, Zhao [23] showed that the codimension four centers have the same property. Concerning the Q_3^{LV} centers there are very few results. In the middle 80s several authors [13, 16, 21] showed independently the monotonicity of the classical Lotka-Volterra centers (which constitute a hypersurface inside the Q_3^{LV} family), and more recently the same property has been proved in [20] for two other hypersurfaces.

From the point of view of the study of the period function it is clear therefore that the most interesting family of centers is the reversible one. By an affine transformation and a constant rescaling of time (see [26] for instance), any reversible quadratic center can be brought to *Loud's normal form*

$$\begin{cases} \dot{x} = -y + Bxy, \\ \dot{y} = x + Dx^2 + Fy^2. \end{cases}$$

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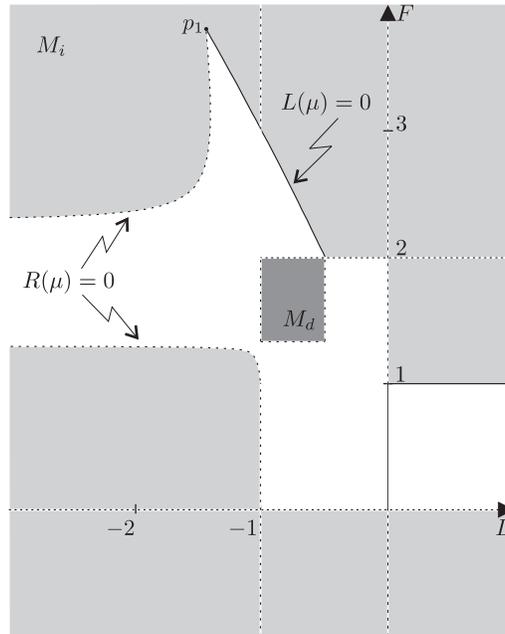


FIGURE 1. Monotonicity regions according to Theorem A.

It is proved in [9] that if $B = 0$, then the period function of the center at the origin of the above system is monotonous increasing. The remaining cases, namely $B \neq 0$, can be brought with a rescaling to $B = 1$, i.e.

$$(1) \quad \begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + Dx^2 + Fy^2. \end{cases}$$

As the excellent paper of Chicone and Jacobs [4] shows, all the interesting phenomena concerning the period function of quadratic centers occur in this family. This is precisely the family of quadratic centers that we study and, following the terminology in [4], we call them *dehomogenized Loud's systems*. The bifurcation diagram of the period function of these centers is studied extensively in [12]. Among other results, the authors determine several regions in the parameter plane for which the corresponding center has at least one critical period (and so its period function is not monotonous). In this paper we focus on the opposite situation, namely the parameters that give rise to a center with a monotonous period function. In short, we find regions of monotonicity in the parameter plane.

Theorem A is our main result and in its statement we refer to some regions in the parameter plane of the dehomogenized Loud's systems (1). To be more precise, see Figure 1, M_i stands for the union of the light grey regions and M_d is the rectangle $(-1, -\frac{1}{2}) \times (\frac{4}{3}, 2)$ in dark grey. The functions $R(\mu)$ and $L(\mu)$ used to define some boundary components of M_i are introduced in (7) and (9) respectively. Here and in what follows we use the notation $\mu := (D, F)$ for the sake of shortness. In particular the point p_1 is approximately $\mu = (-1.44, 3.81)$. We prove the following result:

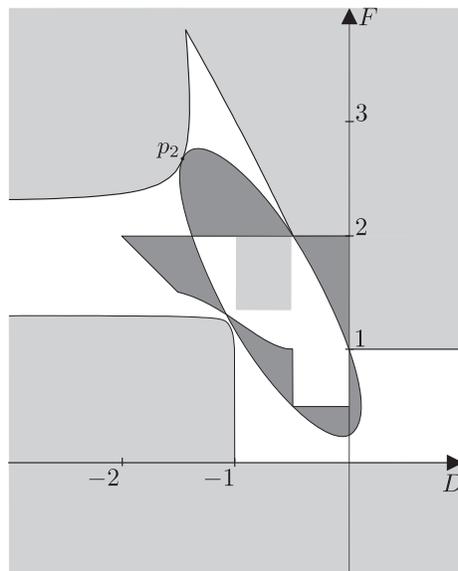


FIGURE 2. Regions of monotonicity (light grey) and regions of non-monotonicity (dark grey).

Theorem A. *The period function of the center at the origin of system (1) is monotonous increasing (respectively, decreasing) in the case that $\mu \in M_i$ (respectively, $\mu \in M_d$).*

At this point it is interesting to locate the monotonicity regions M_d and M_i in Theorem A with respect to the regions of non-monotonicity established by Theorem 5.1 in [12]. This is done in Figure 2, where the ellipse that constitutes several boundary components of the regions of non-monotonicity is given by $10D^2 + 10DF - D + 4F^2 - 5F + 1 = 0$. The intersection point p_2 is approximately $\mu = (-1.47, 2.66)$.

There are two other results on the dehomogenized Loud's systems that should be referred to. Thus, by using a Picard-Fuchs approach, Y. Zhao can completely describe the behaviour of the period function in two straight lines of the parameter plane, namely $F = 3/2$ in [24] and $F = 2$ in [25]. There is a number of different authors that have treated the general question of monotonicity of the period function (see [2, 3, 8, 14, 17, 22] and references there in). Apart from the monotonicity problem, several other questions related to the behaviour of the period function have been extensively studied. Let us quote, for instance, the isochronicity problem (see [5, 6, 10]) and the bifurcation of critical periods (see [11, 15, 19]).

Note added in proof. After this paper was accepted we knew about the publication of [R. Chouikha, *Monotonicity of the period function for some planar differential systems. Part I: Conservative and quadratic systems*, Appl. Math. (Warsaw) **32** (2005), 305–325. MR2213617]. In that paper the author shows the monotonicity for the parameters in the straight lines $F + 2D = 1$ and $F = -1$ and some segments inside $D = -1/2$, $D = 0$, $F = 1$ and $F = 2$.

2. DEFINITIONS AND PROOF OF THE RESULT

A critical point p of a planar differential system is a *center* if it has a punctured neighbourhood that consists entirely of periodic orbits surrounding p . The largest punctured neighbourhood with this property is called the *period annulus* of the center and, in what follows, it will be denoted by \mathcal{P} . The *period function* of the center assigns to each periodic orbit in \mathcal{P} its period. We say that the period function of a center is *monotonous increasing* (respectively, *decreasing*) if for any pair of periodic orbits inside \mathcal{P} , say γ_1 and γ_2 with $\gamma_1 \subset \text{Int}(\gamma_2)$, we have that the period of γ_2 is greater (respectively, smaller) than the one of γ_1 . (Here by $\text{Int}(\gamma)$ we mean the bounded connected component of $\mathbb{R}^2 \setminus \{\gamma\}$.)

If V is a smooth function with $V'(0) = 0$ and $V''(0) > 0$, then the *potential system*

$$(2) \quad \begin{cases} \dot{u} = -v, \\ \dot{v} = V'(u), \end{cases}$$

has a center at the origin. Let \mathcal{I} denote the projection of its period annulus on the u -axis. R. Schaaf proves in [17] the following monotonicity criterium:

Theorem 2.1 (Schaaf’s criterium). *The center at the origin of system (2) has a monotonous increasing period function in the case that*

$$(I_1) \quad 5V'''(u)^2 - 3V''(u)V^{(4)}(u) > 0 \text{ for any } u \in \mathcal{I} \text{ with } V''(u) > 0$$

and

$$(I_2) \quad V'(u)V'''(u) < 0 \text{ for any } u \in \mathcal{I} \text{ with } V''(u) = 0.$$

On the other hand, in the case that

$$(D) \quad 5V'''(u)^2 - 3V''(u)V^{(4)}(u) < 0 \text{ for any } u \in \mathcal{I} \text{ with } V''(u) \geq 0,$$

then the period function is monotonous decreasing.

Although (1) is not a potential system, the following result will allow us to apply Schaaf’s criterium to study the period function of its center.

Lemma 2.2. *If $F \neq 0$, then the coordinate transformation $\left\{ \begin{aligned} u &= \frac{(1-x)^{-F}-1}{F}, \\ v &= (1-x)^{-F}y \end{aligned} \right\}$ brings (1) to the potential system*

$$(3) \quad \begin{cases} \dot{u} = -v, \\ \dot{v} = D(Fu + 1) \left((Fu + 1)^{\frac{-1}{F}} - 1 \right) \left((Fu + 1)^{-\frac{1}{F}} - \frac{D+1}{D} \right). \end{cases}$$

Proof. An easy computation shows that the change of variables given in the statement brings (1) to (3). However, for the sake of completeness, let us say a few words about how we obtain it. In short, if a system with a non-degenerated center has a first integral which is quadratic in y and the corresponding integrating factor depends only on x , then Lemma 4.1 in [9] shows that it can be brought to a potential system and it also provides the coordinate transformation that achieves it. One can check that system (1) verifies these conditions. For instance if $F \notin \{0, 1/2, 1\}$, then $H(x, y) = (1-x)^{-2F}(\frac{1}{2}y^2 + q(x))$, where

$$(4) \quad q(x) := \frac{D}{2(F-1)}x^2 - \frac{D-F+1}{(F-1)(2F-1)}x + \frac{D-F+1}{2F(F-1)(2F-1)}$$

is a first integral and $f(x) = (1 - x)^{-2F-1}$ is its corresponding integrating factor (see [18]). \square

Note that system (1) has $\{x = 1\}$ as an invariant straight line and so the period annulus of the center at the origin is contained in the half-plane $\{x < 1\}$. Accordingly the coordinate transformation in Lemma 2.2 is well defined on the whole period annulus.

On the other hand if $H(u, v) = \frac{1}{2}v^2 + V(u)$ is the Hamiltonian of system (3), then one can check that

$$\begin{aligned}
 (5) \quad & V'(u) = D(Fu + 1) \left((Fu + 1)^{\frac{-1}{F}} - 1 \right) \left((Fu + 1)^{-\frac{1}{F}} - \frac{D+1}{D} \right), \\
 & V''(u) = -(2D + 1)(F - 1)(Fu + 1)^{\frac{-1}{F}} + D(F - 2)(Fu + 1)^{\frac{-2}{F}} + F(D + 1), \\
 & V'''(u) = (2D + 1)(F - 1)(Fu + 1)^{\frac{-1}{F}-1} - 2D(F - 2)(Fu + 1)^{\frac{-2}{F}-1}, \\
 & V^{(4)}(u) = -(2D + 1)(F^2 - 1)(Fu + 1)^{\frac{-1}{F}-2} + 2D(F^2 - 4)(Fu + 1)^{\frac{-2}{F}-2}.
 \end{aligned}$$

Remark 2.3. The key point to succeed in applying Schaaf’s monotonicity criterium to our problem was that, setting $z = (Fu + 1)^{\frac{-1}{F}}$, from (5) we have that

$$(6) \quad V'(u) = (Fu + 1)V_1(z), \quad V''(u) = V_2(z), \quad V'''(u) = \frac{V_3(z)}{Fu + 1}, \quad V^{(4)}(u) = \frac{V_4(z)}{(Fu + 1)^2}$$

for some *polynomial* V_i . For instance, it is clear that $V_2(z) := D(F - 2)z^2 - (2D + 1)(F - 1)z + F(D + 1)$. Taking this into account it follows that

$$5V'''(u)^2 - 3V''(u)V^{(4)}(u) = \frac{5V_3(z)^2 - 3V_2(z)V_4(z)}{(Fu + 1)^2},$$

and observe that the numerator of the last expression is polynomial in z as well. More concretely one can check that $5V_3(z)^2 - 3V_2(z)V_4(z) = zS(z)$, where

$$\begin{aligned}
 S(z) := & -2D^2(3F - 4)(F - 2)^2z^3 + D(9F - 5)(F - 1)(F - 2)(2D + 1)z^2 \\
 & + \frac{1}{2} \left((F^2(16 - 9F) - 2F + 4)(2D + 1)^2 + 3F(F^2 - 4) \right) z \\
 & + 3F(D + 1)(2D + 1)(F^2 - 1).
 \end{aligned}$$

This fact simplifies the application of Schaaf’s criterium because it enables us to verify its conditions by using polynomial functions in z . For instance, since $z = (Fu + 1)^{\frac{-1}{F}}$ is positive, condition I_1 is equivalent to showing that $S(z) > 0$ for any $z = (Fu + 1)^{\frac{-1}{F}}$ with $u \in \mathcal{I}$ and $V_2(z) > 0$.

For the sake of convenience we shall first study the parameters that fulfill condition I_2 . Henceforth we use the notation

$$\kappa(\mu) := (F - 1)^2 + (2D + 1)^2 - 1.$$

The ellipse $\kappa(\mu) = 0$ divides the rectangle $[-1, 0] \times [0, 2]$ into five connected components; see Figure 3. The sets A_1 and A_2 in the following result are two of these components.

Lemma 2.4. *If $\mu \notin A_1 \cup A_2$ and $FD(F - 2)(D + 1)\kappa(\mu) \neq 0$, then the center at the origin of system (3) verifies condition I_2 of Schaaf’s criterium.*

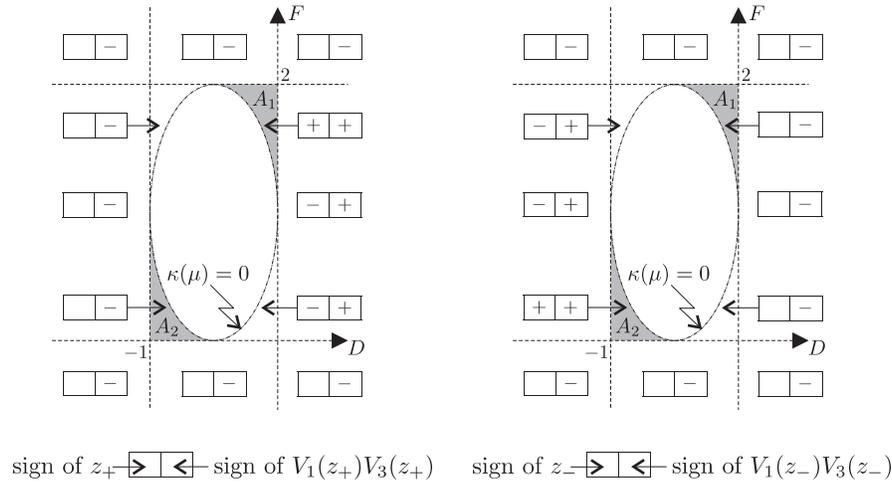


FIGURE 3. Study for the proof of Lemma 2.4.

Proof. Note that $V'(u)V'''(u)$ is polynomial in z because, from (6), we have that $V'(u)V'''(u) = V_1(z)V_3(z)$. In addition, $V''(u) = V_2(z)$. Thus, since $z = (Fu + 1)^{\frac{-1}{F}}$ is positive, it suffices to determine those parameters such that $V_1(z)V_3(z) < 0$ for any $z > 0$ with $V_2(z) = 0$. One can check that $V_2(z) = 0$ if and only if

$$z = z_{\pm} := \frac{(2D + 1)(F - 1) \pm \sqrt{\kappa(\mu)}}{2D(F - 2)}.$$

To compute the sign of $V_1(z_{\pm})V_3(z_{\pm})$ at any parameter μ we take advantage of the resultant between the polynomials $V_2(z)$ and $V_1(z)V_3(z)$, which is equal to $FD^2(F - 2)(D + 1)^2\kappa(\mu)$. It is clear then that $V_1(z_{\pm})V_3(z_{\pm})$ vanishes if and only if $FD(F - 2)(D + 1)\kappa(\mu) = 0$. This curve splits up the parameter plane into 13 connected components. By continuity, in each of these components the sign of $V_1(z_{\pm})V_3(z_{\pm})$ does not change, and so it is enough to compute it in 13 concrete parameters. We proceeded in this way to obtain Figure 3, which gathers all the information needed to show the result. For instance condition I_2 is verified in the strip $(0, +\infty) \times (0, 2)$ although $V_1(z_+)V_3(z_+) > 0$ because z_+ is negative there. Let us mention finally that the region $\kappa(\mu) < 0$ is included as well because z_{\pm} are not real for these parameters. \square

Next we shall study which parameters verify condition I_1 . This is done in Lemma 2.5, and to prove it we use the bifurcation diagram of the graphics of V_2 and S (recall Remark 2.3). To be more precise, we need the distribution of the roots and their location with respect to $z = 0$. The distribution of roots will provide us the necessary information about the sign of V_2 and S . We are also interested in their location with respect to $z = 0$ because $z = (Fu + 1)^{\frac{-1}{F}}$ is positive. The leading coefficient of V_2 and S vanishes at

$$D(F - 2) = 0 \text{ and } D(3F - 4)(F - 2) = 0$$

respectively. On the other hand, the independent term of V_2 and S becomes zero at

$$F(D + 1) = 0 \text{ and } F(F^2 - 1)(D + 1)(2D + 1) = 0$$

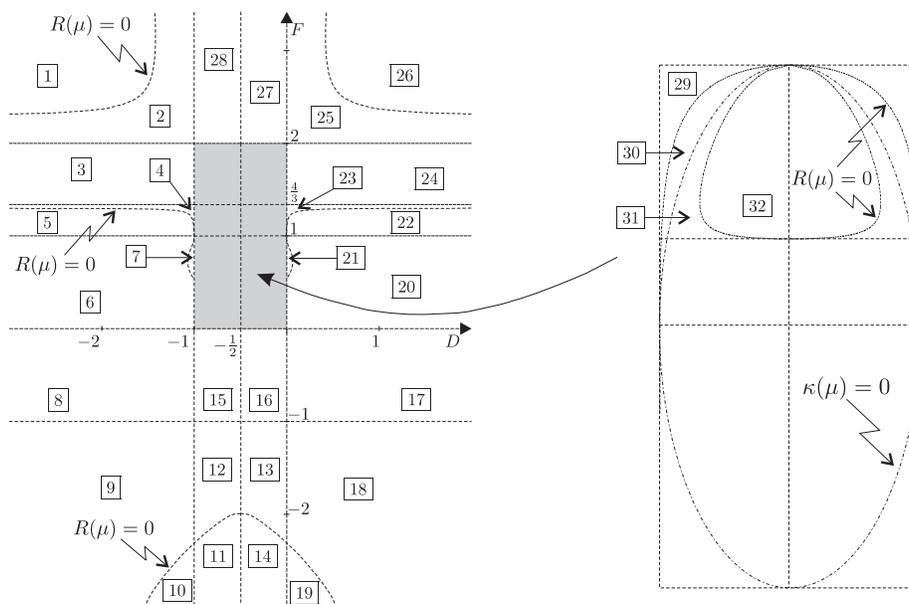


FIGURE 4. Bifurcation diagram of the graphics of V_2 and S .

respectively. The discriminant of V_2 is $\kappa(\mu)$ and the one of S is $D^2(F-2)^2\kappa(\mu)R(\mu)$, where

$$\begin{aligned}
 (7) \quad R(\mu) := & -(745F^4 - 2800F^3 + 2688F^2 - 442F + 52)(2D + 1)^4 \\
 & - 3F(F - 2)(125F^4 - 518F^3 + 174F^2 + 382F - 1)(2D + 1)^2 \\
 & + 9F^2(3F - 4)(F - 2)^2(F + 2)^3.
 \end{aligned}$$

Finally the resultant between V_2 and S is $D^2F(D + 1)(F - 2)^2\kappa(\mu)^2$. It is clear therefore that the above-mentioned diagram has

$$DF(D + 1)(2D + 1)(F^2 - 1)(F - 2)(3F - 4)\kappa(\mu)R(\mu) = 0$$

as a bifurcation curve. This curve splits up the parameter plane into several connected components and, as in the proof of Lemma 2.4, to obtain the bifurcation diagram it suffices to study one concrete parameter in each region. We compile all this in Figures 4 and 5. However, for the sake of brevity, inside the rectangle $[-1, 0] \times [0, 2]$ we only show the graphics of those parameters verifying condition I_1 . This saves us having to draw $14 + 4 = 18$ graphics because, although imperceptible in Figure 4, the curve $R(\mu) = 0$ goes into the square $[-1, 0] \times [0, 1]$ giving rise to 4 more regions. It is important to mention that this curve can be easily studied because R is a second degree polynomial in $(2D + 1)^2$. Observe in particular that it is symmetric with respect to the straight line $D = -1/2$. We can now prove the following.

Lemma 2.5. *The center at the origin of system (3) verifies condition I_1 of Schaaf's criterium if μ is inside one of the following open sets (see Figure 4):*

- (a) Regions 1, 5–8, 11–18, 22–24 and 27.
- (b) Regions 9, 10, 25 and 26.

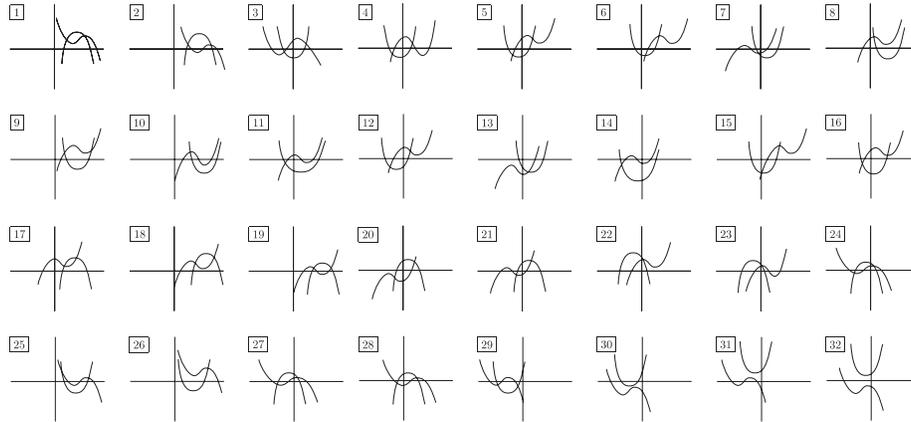


FIGURE 5. The graphics of V_2 and S corresponding to Figure 4.

On the other hand, the center verifies condition D if μ is inside the regions 29–32.

Proof. As usual let \mathcal{I} denote the projection on the u -axis of the period annulus of the center at the origin of system (3). From Remark 2.3, the condition I_1 is verified for a parameter μ if and only if

$$(8) \quad S(z) > 0 \text{ for all } z \text{ such that } V_2(z) > 0 \text{ and } z = (Fu + 1)^{\frac{-1}{F}} \text{ with } u \in \mathcal{I}.$$

In view of Figure 5, the parameters inside the regions listed in (a) verify that $S(z) > 0$ for all $z > 0$ with $V_2(z) > 0$. Thus, since $(Fu + 1)^{\frac{-1}{F}}$ is positive, it is clear that (8) holds.

Let us turn now to the parameters listed in (b). A simple computation shows that if $D \notin [-1, 0]$, then system (3) has a saddle at the point $(u^*, 0)$, where $(Fu^* - 1)^{\frac{-1}{F}} = \frac{D+1}{D}$. Accordingly $u^* \notin \mathcal{I}$. It is easy to check that $u^* < 0$ for $D > 0$ and $u^* > 0$ for $D < -1$. Therefore $\mathcal{I} \subset (u^*, +\infty)$ for $D > 0$ and $\mathcal{I} \subset (-\infty, u^*)$ for $D < -1$. Thus, since $u \mapsto (Fu + 1)^{\frac{-1}{F}}$ is decreasing, it suffices to verify (8) for $z \in (0, \frac{D+1}{D})$ in the case that $D > 0$ and for $z > \frac{D+1}{D}$ in the case that $D < -1$. Taking this into account, and using the fact that $V_2(\frac{D+1}{D}) = -\frac{D+1}{D}$ is negative for $D \notin [-1, 0]$, it is clear from the graphics in Figure 5 that the parameters inside the regions listed in (b) verify (8).

Finally, again from Remark 2.3, condition D is satisfied for a parameter μ if and only if $S(z) < 0$ for all z such that $V_2(z) \geq 0$ and $z = (Fu + 1)^{\frac{-1}{F}}$ with $u \in \mathcal{I}$. Since for the parameters inside the regions 29–32 (see Figure 5) we have that $S(z) < 0$ for all $z > 0$, the result follows. \square

The next result is also devoted to investigating which parameters verify condition I_1 . However, since the approach is different from the one in Lemma 2.5, we prefer to treat it separately. The sets \hat{U}_1 and \hat{U}_2 that appear in the statement are drawn in Figure 6.

Lemma 2.6. *The center at the origin of system (3) verifies condition I_1 of Schaaf’s criterium if $\mu \in \hat{U}_1 \cup \hat{U}_2$.*

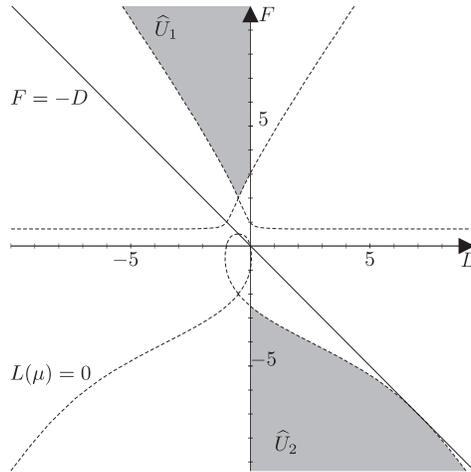


FIGURE 6. The sets \widehat{U}_1 and \widehat{U}_2 in Lemma 2.6.

Proof. In fact we will show that if $\mu \in \widehat{U}_1 \cup \widehat{U}_2$, then $V^{(4)}(u) < 0$ for all $u \in \mathcal{I}$, which obviously implies condition I_1 . From (5) we have that

$$V^{(4)}(u) = (Fu + 1)^{\frac{-1}{F}-2} \left(-(2D + 1)(F^2 - 1) + 2D(F^2 - 4)(Fu + 1)^{\frac{-1}{F}} \right).$$

Thus it is clear that the fourth derivative of V only changes its sign once, at $u = u^*$, where

$$(Fu^* + 1)^{\frac{-1}{F}} = \frac{(2D + 1)(F^2 - 1)}{2D(F^2 - 4)}.$$

Consequently to prove the result it suffices to verify that if $\mu \in \widehat{U}_1 \cup \widehat{U}_2$, then $V^{(4)}(0) < 0$ and $u^* \notin \mathcal{I}$. The phase portraits of system (1) are very well known (see [12, 18] for instance). In particular, recall (4), the conic $\frac{1}{2}y^2 + q(x) = 0$ is invariant. Moreover for the parameters inside

$$U := \{F > 1, -F < D < 0\} \cup \{-F > D > 0\},$$

one can check that the conic is a hyperbola with the focal points in the x -axis and $q(0) < 0$. Taking this into account and using the coordinate transformation in Lemma 2.2 to bring system (3) to (1), it follows that $q(1 - (Fu + 1)^{\frac{-1}{F}}) < 0$ is a necessary condition for $u \in \mathcal{I}$ in the case that $\mu \in U$. In particular, if $\mu \in U$, then $q(1 - (Fu^* + 1)^{\frac{-1}{F}}) \geq 0$ guarantees that $u^* \notin \mathcal{I}$. Some computations show that

$$q\left(1 - (Fu^* + 1)^{\frac{-1}{F}}\right) = \frac{(F^3 + 7F^2 + 17F - 16)(2D + 1)^2 - (2F - 1)(F^2 - 4)^2}{8DF(2F - 1)(F^2 - 4)^2}$$

and $V^{(4)}(0) = 1 - 6D - F^2$. In short, we can assert that if $\mu \in U$ verifies

$$6D + F^2 - 1 > 0 \text{ and } \frac{(F^3 + 7F^2 + 17F - 16)(2D + 1)^2 - (2F - 1)(F^2 - 4)^2}{DF(2F - 1)} \geq 0,$$

then $V^{(4)}(u) < 0$ for all $u \in \mathcal{I}$ as desired. Straightforward arguments show that the parameters inside U satisfying these two inequalities are precisely $\widehat{U}_1 \cup \widehat{U}_2$, see Figure 6 where

$$(9) \quad L(\mu) := (F^3 + 7F^2 + 17F - 16)(2D + 1)^2 - (2F - 1)(F^2 - 4)^2.$$

This completes the proof of the result. \square

Proof of Theorem A. Thanks to the coordinate transformation in Lemma 2.2, to show the result we can consider the potential system (3). Then, by applying Schaaf's criterium, the combination of Lemmas 2.4 and 2.5 proves the result except for some parameters in the regions 2, 19 and 28 (compare Figures 1 and 4). There are parameters inside these regions that verify condition I_1 , but to determine them we use Lemma 2.6 instead. To this end it is only necessary to study where the curves $L(\mu) = 0$ and $R(\mu) = 0$ intersect. (This is easy because L and R have degree one and two in $(2D + 1)^2$ respectively.) One can show in this way (see Figure 6) that \widehat{U}_2 contains the region 19 and that \widehat{U}_1 intersects with the region 1 as shown in Figure 1. \square

Let us conclude with some technical comments about the definition of the sets M_i and M_d in Figure 1. Following the standard convention, Theorem A does not apply at those parameters in dotted curves. For instance, it is not valid at

$$DF(D + 1)(F - 2) = 0$$

because Lemma 2.4 says nothing in these straight lines. This should also be the case for the parameters at $(2D + 1)(3F - 4)(F^2 - 1)R(\mu)\kappa(\mu) = 0$ because Lemma 2.5 does not apply there. However an accurate analysis of the graphics of V_2 and S shows that most of these parameters do verify condition I_1 . Since this is rather long to explain and it does not significantly improve the result, we prefer not to prove it. However we do include these parameters in the sets M_i and M_d because it simplifies the statement of Theorem A.

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DEPARTAMENT D'ENGINYERIA INFORMÀTICA I MATEMÀTIQUES, ETSE, UNIVERSITAT ROVIRA I VIRGILI, 43007 TARRAGONA, SPAIN