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METRIC GEODESICS OF ISOMETRIES IN A HILBERT SPACE AND THE EXTENSION PROBLEM

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Abstract. We consider the problem of finding *short* smooth curves of isometries in a Hilbert space $\mathcal{H}$. The length of a smooth curve $\gamma(t)$, $t \in [0,1]$, is measured by means of $\int_0^1 \|\dot{\gamma}(t)\|\, dt$, where $\| \ \|$ denotes the usual norm of operators. The initial value problem is solved: for any isometry V_0 and each tangent vector at V_0 (which is an operator of the form iXV_0 with $X^* = X$) with norm less than or equal to π, there exist curves of the form $e^{itZ}V_0$, with initial velocity $iZV_0 = iXV_0$, which are short along their path. These curves, which we call metric geodesics, need not be unique, and correspond to the so called extension problem considered by M.G. Krein and others: in our context, given a symmetric operator

$$X_0|_{R(V_0)} : R(V_0) \to \mathcal{H},$$

find all possible $Z^* = Z$ extending $X_0|_{R(V_0)}$ to all $\mathcal{H}$, with $\|Z\| = \|X_0\|$. We also consider the problem of finding metric geodesics joining two given isometries V_0 and V_1. It is well known that if there exists a continuous path joining V_0 and V_1, then both ranges have the same codimension. We show that if this number is finite, then there exist metric geodesics joining V_0 and V_1.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators acting on $\mathcal{H}$, and $\mathcal{U}(\mathcal{H})$ the group of unitary operators. We denote $\mathcal{I} = \{V \in \mathcal{B}(\mathcal{H}) : V^*V = I\}$ as the set of isometries of $\mathcal{H}$. The unitary group $\mathcal{U}(\mathcal{H})$ acts on $\mathcal{I}$ by means of

$$U \times V \mapsto UV.$$

This action is locally transitive. More precisely, it is well known (see [8]) that two isometries V_1, V_2 such that $\|V_1 - V_2\| < 1$ are conjugate by this action: there exists a unitary $U = U(V_1, V_2)$ such that $UV_1 = V_2$. The unitary U can be chosen as a C^∞ map in terms of V_1, V_2.

If $V_2 = UV_1$, then the final projections (onto the ranges) of V_1 and V_2 are unitarily equivalent: $V_2V_2^* = UV_1V_1^*U^*$. Since the unitary group is connected, it follows that the connected components of $\mathcal{I}$ coincide with the orbits of this action, lie at distance at least 1, and are parametrized by the codimension of the ranges of

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the isometries:

$$\mathcal{I}(\mathcal{H}) = \mathcal{I} = \bigcup_{0 \leq n \leq \infty} \mathcal{I}_n$$

where

$$\mathcal{I}_n = \{V \in \mathcal{I} : \dim R(V)^{\perp} = n\}.$$

It is also known that $\mathcal{I}_n, \mathcal{I}$ are (infinite dimensional) differentiable submanifolds of $\mathcal{B}(\mathcal{H})$, and homogeneous spaces of $\mathcal{U}(\mathcal{H})$. The tangent spaces of $\mathcal{I}$ can be regarded as complemented subspaces of $\mathcal{B}(\mathcal{H})$ [1]. Namely,

$$(T\mathcal{I})_V = \{iXV : X^* = X\}.$$

Suppose we endow $\mathcal{I}$ with the Finsler metric consisting of the usual norm of $\mathcal{B}(\mathcal{H})$ in each tangent space. That is, we measure the length of a curve γ in $\mathcal{I}$ by means of

$$length(\gamma) = \int_a^b \|\dot{\gamma}(t)\| \, dt$$

if the curve is parametrized in the interval $[a, b]$.

We study the problem, of a geometrical as well as a variational nature, of finding curves of minimal length in $\mathcal{I}$.

We obtain the following results:

1. For each $V \in \mathcal{I}$ and each tangent vector $\mathcal{V} \in (T\mathcal{I})_V$ with $\|\mathcal{V}\| \leq \pi$, there exists a curve $\nu(t) = e^{itX}V$, $t \in [0, 1]$, satisfying the initial conditions

$$\nu(0) = V, \dot{\nu}(0) = iXV = \mathcal{V},$$

which has *minimal* length along its path, among all smooth curves in $\mathcal{I}$. By this we mean: for any $t_0, t_1 \in [0, 1]$, the curve ν restricted to the interval $[t_0, t_1]$ has length which is less than or equal to the length of any other smooth curve in $\mathcal{I}$ joining $\nu(t_0)$ and $\nu(t_1)$.

2. For each pair of elements $V_0, V_1 \in \mathcal{I}_n$, with n **finite**, there exists a curve $\nu(t) = e^{itX}V_0$ with

$$\nu(0) = V_0, \nu(1) = V_1$$

with minimal length among smooth curves of $\mathcal{I}$ with the same endpoints. This curve is also minimizing along its path.

In both situations, neither of the metric geodesics need be unique. On the contrary, even for arbitrary close elements (or short velocity vectors) there may exist multiple solutions.

If $n = \infty$, we obtain metric geodesics joining two given isometries V_0 and V_1 in two special cases: if either the ranges of V_0 and V_1 are equal or orthogonal.

Note that if $n = 0$, $\mathcal{I}_0 = \mathcal{U}(\mathcal{H})$. It is a folklore fact (see for example [2]) that the curves of the form $\mu(t) = e^{itX}$, $X^* = X$, have minimal length among smooth curves of unitaries with the same endpoints, provided that $\|X\| \leq \pi$. We shall use this fact. On the other hand, since $\mathcal{I}_n$ is a homogeneous space of $\mathcal{U}(\mathcal{H})$, it follows that any curve $\nu(t)$ in $\mathcal{I}_n$ can be lifted to a curve of unitaries: $\nu(t) = \mu(t)V_0$ for $\mu(t) \in \mathcal{U}(\mathcal{H})$ and a fixed $V_0 \in \mathcal{I}_n$. Therefore the results described appear natural.

These results are related to [4] and [5], where the problem of the existence of metric geodesics is studied in the context of abstract homogeneous spaces. In those papers a quotient Finsler metric is defined. Here we consider the usual operator norm. However, some techniques used there work fine in the present context.

2. Minimality and the extension problem

Let $V \in \mathcal{I}$, $X = X^*$ and $\nu(t) = e^{itX}V$. Note that the length of ν is

$$length(\nu) = \|XV\|.$$

If one searches curves of this type, subject to given initial conditions, which should be as short as possible, it turns out to be useful to try to minimize $\|X\|$: find, if possible, the minimum of the set

$$\{\|Z\| : Z^* = Z, ZV = XV\}.$$

Denote by $P = VV^*$ the final projection of V (equal to the orthogonal projection onto the range of V). Note that $XV = ZV$ if and only if $XP = ZP$. Indeed, $XV = ZV$ implies $XVV^* = ZVV^*$; conversely, $XP = ZP$ implies $XV = XPV = ZPV = ZV$. It follows that the problem above is equivalent to finding the mimimum of

$$\{\|Z\| : Z^* = Z, ZP = XP\},$$

for a given self-adjoint X. This in turn is equivalent to solving the extension problem for self-adjoint operators, which can be stated as follows. Given a 2×2 (incomplete) block operator matrix of the form

$$\begin{pmatrix} A & B \\ B^* & ? \end{pmatrix},$$

complete the $2, 2$ entry in order to obtain a self-adjoint operator with the least possible norm. Indeed, in our case the entries are $A = PXP$ and $B = PX(I - P)$. This problem was solved in a much broader setting (for unbounded symmetric operators) by M. G. Krein in [7]. There is an elementary and beautiful construction of a solution in [11], page 336. In general, the solution is not unique; in [3] there are descriptions of all solutions. Let us state the existence result in our framework:

Lemma 2.1. *Given $X = X^* \in \mathcal{B}(\mathcal{H})$, there exists $Z = Z^* \in \mathcal{B}(\mathcal{H})$ such that*

1. *$ZV = XV$ and*
2. *$\|ZV\| = \|Z\|$.*

Such Z may not be unique.

The solutions of the extension problem give minimizing geodesics of isometries starting at V:

Theorem 2.2. *Let $V \in \mathcal{I}$ and $iXV \in (T\mathcal{I})_V$ $(X = X^*)$ with $\|XV\| \leq \pi$. Let Z be a solution of the extension problem, i.e.*

$$ZV = XV \text{ and } \|ZV\| = \|Z\|.$$

Then the curve $\nu(t) = e^{itZ}V$, $t \in [0, 1]$ which verifies $\nu(0) = V$ and $\dot{\nu}(0) = iXV$, has minimal length along its path among smooth curves in $\mathcal{I}$.

Proof. Let $P = VV^*$ be the final projection of V. Fix a solution Z of the extension problem. Note that $ZP = XP$, $\|ZP\| = \|Z\|$ and $\|Z\| \leq \pi$. Given a positive element A of a C^*-algebra, there exists a faithful representation of the algebra (for instance, the universal representation) and a unit vector ξ in the Hilbert space $\mathcal{H}$ of this representation, such that $A\xi = \|A\|\xi$ (here we identify A with its image under the representation). Let us call such a vector ξ a *norming eigenvector* for A. Let us apply this fact to the positive operator PZ^2P. Let ξ be a (unit) norming eigenvector for PZ^2P. Again we identify operators with their images under this

representation, and regard them as operators in this new, eventually bigger, Hilbert space. Clearly ξ lies in the range of P. We claim that ξ is a norming eigenvector for Z^2 as well. Indeed,

$$Z^2\xi = Z^2P\xi = PZ^2P\xi + (I-P)Z^2P\xi = \|PZ^2P\|\xi + \xi_1,$$

where $\xi_1 = (I-P)Z^2P\xi$ is orthogonal to ξ. Note that

$$\|PZ^2P\| = \|ZP\|^2 = \|Z\|^2 = \|Z^2\|.$$

Then

$$\|Z^2\|^2 \geq \|Z^2\xi\|^2 = \|PZ^2P\|^2 + \|\xi_1\|^2 = \|Z^2\|^2 + \|\xi_1\|^2.$$

It follows that $\xi_1 = 0$, and our claim is proved. Since $R(P) = R(V)$ there exists a unit vector $\eta \in \mathcal{H}$ such that $V\eta = \xi$. Consider the curve $\nu(t)\eta = e^{itZ}V\eta = e^{itZ}\xi$. Clearly $\|\nu(t)\eta\| = 1$, i.e. $\nu(t)\eta$ is a curve in the unit sphere $S_{\mathcal{H}}$ of the Hilbert space $\mathcal{H}$. We claim that it is a minimizing geodesic of this Riemann-Hilbert manifold. Indeed,

$$\ddot{\nu}(t) = -e^{itZ}Z^2\xi = -\|Z\|^2e^{itZ}\xi = -\|Z\|^2\nu(t).$$

That is, ν satisfies the differential equation of the geodesics of the sphere $S_{\mathcal{H}}$. Moreover, the length of $\nu\eta$ is

$$length(\nu\eta) = \int_0^1 \|\dot{\nu}(t)\eta\|\ dt = \|Z\xi\| \leq \pi.$$

It follows that $\nu\eta$ is a minimizing geodesic of the unit sphere. Note also that

$$\|Z\xi\|^2 = \langle Z\xi, Z\xi\rangle = \langle Z^2\xi, \xi\rangle = \|Z^2\| = \|Z\|^2.$$

Clearly, if $[t_0, t_1] \subset [0,1]$, the length of $\nu\eta$ restricted to $[t_0, t_1]$ (or shortly $\nu\eta|_{[t_0,t_1]}$) is $(t_1 - t_0)\|Z\|$. On the other hand,

$$length(\nu|_{[t_0,t_1]}) = \int_{t_0}^{t_1} \|\dot{\nu}\|\ dt = (t_1 - t_0)\|ZV\| = (t_1 - t_0)\|Z\|.$$

It follows that $length(\nu\eta) = length(\nu)$ on any subinterval of $[0,1]$.

Suppose now that $\gamma : [a,b] \to \mathcal{I}$ is a smooth curve joining $\nu(t_0)$ and $\nu(t_1)$. Clearly the curve $\gamma\eta$ is a smooth curve in $S_{\mathcal{H}}$ joining $\nu(t_0)\eta$ and $\nu(t_1)\eta$. Therefore

$$length(\gamma\eta|_{[t_0,t_1]}) \geq length(\nu\eta|_{[t_0,t_1]}) = length(\nu|_{[t_0,t_1]}).$$

On the other hand,

$$length(\gamma\eta|_{[t_0,t_1]}) = \int_{t_0}^{t_1} \|\dot{\gamma}(t)\eta\|\ dt \leq \int_{t_0}^{t_1} \|\dot{\gamma}(t)\|\ dt = length(\gamma|_{[t_0,t_1]}).$$

It follows that ν is not longer than any other smooth curve in $\mathcal{I}$ joining any two points along their path. □

The proof of this theorem is based on the fact that geodesics of length at most π of the unit sphere of a Hilbert space are minimizing curves. A similar argument was used in [10] to characterize minimizing curves of projections in a C*-algebra.

The converse of the statement in the above theorem does not hold. Consider the following example. Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0$ be of infinite dimension, let P be the orthogonal projection onto the first copy of $\mathcal{H}_0$, and let V be an isometry with final space P. Consider the following operators given as block matrices in terms of P:

$$Z = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

Note that $ZV = (Z + D)V$ and $\|Z\| = \|ZV\| = 2$. Therefore $\delta(t) = e^{itZ}V$ is minimizing along its path, for $t \in [0, 1]$. Also note that $X = Z + D$ has norm 3, and therefore it is not a solution of the corresponding extension problem. However $e^{itX}V = e^{itZ}V$ is minimizing along its path. Perhaps a proper way to state a converse of Theorem 2.2 would be the following: suppose that $\gamma(t) = e^{itX}V$, with $X^* = X$ and $\|X\| \leq \pi$, is minimizing along its path for $t \in [0, 1]$; then there exists $Z^* = Z$ with $\|ZV\| = \|Z\|$ and $e^{itZ}V = e^{itX}V$ for all t. Or equivalently, a self-adjoint operator Z with $\|ZV\| = \|Z\|$ and $Z^nV = X^nV$ for all $n \geq 1$. We will consider this question elsewhere.

3. Existence of geodesics joining given endpoints

In the previous section it was shown that for any pair of initial conditions $(V, \mathcal{V})$ in the tangent bundle of $\mathcal{I}$, there exist (eventually many) short geodesics satisfying the initial conditions. In this section we consider the problem of finding short geodesics joining a given pair of elements in $\mathcal{I}$. Again, our solution of this problem consists of minimizing the norm on a certain set of operators. Namely, let $V_0, V_1 \in \mathcal{I}_n$, $0 \leq n \leq \infty$, and consider the set

$$\mathcal{L}_{V_0,V_1} = \{Z \in \mathcal{B}(\mathcal{H}) : Z^* = Z, e^{iZ}V_0 = V_1\}.$$

Recall that by the local transitivity of the action, there exists a unitary U such that $UV_0 = V_1$, and that such U is of the form $U = e^{iZ}$ for Z as above, i.e. $\mathcal{L}_{V_0,V_1}$ is nonempty. The following result is an adaptation of Theorem 3.2 in [5], to our particular context, where the Finsler metric is given by the norm of $\mathcal{B}(\mathcal{H})$ (in [5] quotient norms are considered).

Theorem 3.1. *Let $V_0, V_1 \in \mathcal{I}_n$, and suppose that there exists $Z_0 \in \mathcal{L}_{V_0,V_1}$ such that*

$$\|Z_0\| = \inf\{\|Z\| : Z \in \mathcal{L}_{V_0,V_1}\}.$$

Then $\nu(t) = e^{itZ_0}V_0$ is shorter than any other piecewise smooth curve joining V_0 and V_1 in $\mathcal{I}$. Moreover, $\nu(t)$ is minimizing along its path.

Proof. The proof, as in 3.2 of [5], proceeds in three steps:

a) Let $Z_0 \in \mathcal{L}_{V_0,V_1}$ with $\|Z_0\| = \inf\{\|Z\| : Z \in \mathcal{L}_{V_0,V_1}\}$, fix $s \in (0, 1)$ and denote $V_s = e^{isZ_0}V_0$. Then $sZ_0 \in \mathcal{L}_{V_0,V_s}$ and $s\|Z_0\| = \inf\{\|Z\| : Z \in \mathcal{L}_{V_0,V_s}\}$.

b) Suppose that X, Y are self-adjoint operators of small norms in order that $e^{iX}e^{iY}$ lies in the domain of the power series of the logarithm log defined on a neighbourhood of I with antihermitic values (for intance, $\|e^{iX}e^{iY} - I\| < 1$). Then

$$\log(e^{iX}e^{iY}) = iX + iY + R_2(X, Y),$$

where

$$\lim_{s\to 0} \frac{R_2(sX, sY)}{s} = 0.$$

c) Let $P_0 = V_0V_0^*$. For any $Y = Y^*$ such that $Y = (I - P_0)Y(I - P_0)$, one has that

$$\|Z_0\| \leq \|Z_0 + Y\|.$$

Let us prove these steps, and show how they prove our result.

Step a):

For an element $X = X^*$, denote it by $\gamma_X(t) = e^{itX}$. Note that condition $\|Z_0\| = \inf\{\|Z\| : Z \in \mathcal{L}_{V_0,V_1}\}$ implies that the curve γ_{Z_0} is the shortest among piecewise smooth curves of unitaries joining I to the set $\{U \in \mathcal{U}(\mathcal{H}) : UV_0 = V_1\}$. Indeed, if $\mu(t)$ is another smooth curve of unitaries with $\mu(0) = I$ and $\mu(1)V_0 = V_1$, then there is a curve of the form e^{itW}, $W^* = W$ and $\|W\| \leq \pi$, with $e^{iW} = \mu(1)$, which is shorter than μ. Note that such a W lies in $\mathcal{L}_{V_0,V_1}$, and therefore $\|W\| \geq \|Z_0\|$. Then $length(\mu) \geq \|W\| \geq length(\gamma_{Z_0})$.

Let us show that $s\|Z_0\| = \inf\{\|Z\| : Z \in \mathcal{L}_{V_0,V_s}\}$. Suppose that there exists $X \in \mathcal{L}_{V_0,V_s}$ such that $\|X\| < s\|Z_0\|$. Consider the curve $\delta(t) = e^{i(1-t)sZ_0+itZ_0}$ which joins e^{isZ_0} with e^{iZ_0} in $\mathcal{U}(\mathcal{H})$, and $\sigma(t) = \delta(t)e^{-isZ_0}e^{iX}$, joining e^{iX} and $e^{i(1-s)Z_0}e^{iX}$ (in both cases $t \in [0,1]$). Note that they have the same length, for they differ on an element of $\mathcal{U}(\mathcal{H})$: $length(\delta) = length(\sigma) = (1-s)\|Z_0\|$. Note also that the endpoint of σ satisfies $\sigma(1)V_0 = V_1$. Let $\tilde{\gamma}$ be the piecewise smooth curve which consists of the curve γ_X followed by σ. Then $\tilde{\gamma}$ joins I and the fiber $\{U \in \mathcal{U}(\mathcal{H}) : UV_0 = V_1\}$ in $\mathcal{U}(\mathcal{H})$, and therefore by the fact remarked above, $length(\tilde{\gamma}) \geq \|Z_0\|$. On the other hand,

$$\begin{aligned} length(\tilde{\gamma}) &= length(\gamma_X) + length(\sigma) = \|X\| + (1-s)\|Z_0\| \\ &< s\|Z_0\| + (1-s)\|Z_0\| = \|Z_0\|. \end{aligned}$$

Step b):

The linear part of the series of $\log(e^{iX}e^{iY})$ is $iX + iY$, so that

$$\log(e^{iX}e^{iY}) = iX + iY + R_2(X,Y),$$

where the remainder term $R_2(X,Y)$ is an infinitesimal of the order $\|X\| + \|Y\|$. Therefore

$$\lim_{s\to 0} \frac{R_2(sX,sY)}{s} = 0.$$

Step c):

By step a), for any $s \in (0,1)$, $s\|Z_0\| = \inf\{\|Z\| : Z \in \mathcal{L}_{V_0,V_s}\}$. Let $Y = Y^*$ such that $Y = (I-P_0)Y(I-P_0)$. Then clearly $e^{iY}V_0 = e^{iY}P_0V_0 = V_0$. Therefore $\log(e^{iZ_0}e^{iY}) \in \mathcal{L}_{V_0,V_1}$. Analogously, $\log(e^{isZ_0}e^{isY}) \in \mathcal{L}_{V_0,V_s}$. Then

$$\begin{aligned} s\|Z_0\| &\leq \|\log(e^{isZ_0}e^{isY})\| = \|isZ_0 + isY + R_2(sZ_0,sY)\| \\ &\leq s\|Z_0 + Y\| + \|R_2(sZ_0,sY)\| \end{aligned}$$

and then

$$\|Z_0\| \leq \|Z_0 + Y\| + \frac{\|R_2(sZ_0,sY)\|}{s}.$$

Taking limits, $\|Z_0\| \leq \|Z_0 + Y\|$, which proves step c).

The theorem follows, for the set $\{Z_0 + Y : Y^* = Y, (I-P_0)Y(I-P_0) = Y\}$ parametrizes the set of all Z such that $ZP_0 = Z_0P_0$. This means that Z_0 is a solution of the extension problem described in the previous section, and therefore, by Theorem 2.2, $\nu(t) = e^{itZ_0}V_0$ is a minimizing geodesic, joining V_0 and V_1. □

As a particular case, this result solves the problem of existence of metric geodesics joining isometries of finite codimension.

Corollary 3.2. *Let $V_0, V_1 \in \mathcal{I}_n$ with $n < \infty$. Then there exists a curve of the form $\nu(t) = e^{itZ_0}V_0$, $t \in [0,1]$, joining V_0 and V_1 which is shorter than any other piecewise smooth curve in $\mathcal{I}$ with the same endpoints. Moreover, it is also minimizing along its path.*

Proof. As in the beginning of the proof of the preceding result, one can always find an element in $\mathcal{L}_{V_0,V_1}$ with norm less than or equal to π. Then $i_0 = \inf\{\|Z\| : Z \in \mathcal{L}_{V_0,V_1}\} \leq \pi$. Also, from this remark it is clear that if $i_0 = \pi$, there exists a minimum. Suppose $i_0 < \pi$. Fix $U \in \mathcal{U}(\mathcal{H})$ such that $UV_1 = V_0$. Consider the set $\exp(\mathcal{L}_{V_0,V_1}) = \{e^{iX} : X \in \mathcal{L}_{V_0,V_1}\}$. Note that $U \cdot \exp(\mathcal{L}_{V_0,V_1})$ consists of unitary operators W satisfying $WP_0 = P_0$. Indeed, it is clear that $WV_0 = V_0$ if and only if $WP_0 = P_0$. If one writes such unitaries as 2×2 matrices in terms of P_0, they are of the form

$$\begin{pmatrix} P_0 & 0 \\ 0 & (I-P_0)W(I-P_0) \end{pmatrix}.$$

It follows that $U \cdot \exp(\mathcal{L}_{V_0,V_1})$ is homeomorphic to the unitary group of $R(P_0)^\perp = R(V_0)^\perp$, which is finite dimensional. Therefore $\exp(\mathcal{L}_{V_0,V_1})$ is compact. Let Z_n be a sequence in $\mathcal{L}_{V_0,V_1}$ such that $\|Z_n\| \to i_0$. Then e^{iZ_n} has a convergent subsequence $e^{iZ_{n_k}} \to W_0$. Since $i_0 < \pi$, there exists l such that $k \geq l$ implies $\|Z_{n_k}\| < \pi$. It is well known that the exponential map,

$$\begin{aligned} \exp : \{X \in \mathcal{B}(\mathcal{H}) : X^* = X, \|X\| < \pi\} \\ \to \{W \in \mathcal{U}(\mathcal{H}) : \|W - I\| < 2\}, \quad \exp(X) = e^{iX}, \end{aligned}$$

is a homeomorphism. It follows that Z_{n_k} converges to a self-adjoint operator Z_0 in $\mathcal{L}_{V_0,V_1}$ with $\|Z_0\| = i_0$. □

There are two special situations when the problem of finding a minimizing geodesic joining two given isometries can be solved explicitly.

Proposition 3.3. *Suppose that V_0, V_1 are two isometries in the same connected component, whose ranges are either equal or orthogonal. Then there exists a minimizing geodesic of the form $\nu(t) = e^{itX}V_0$ joining them.*

Proof. Suppose first that $R(V_0) = R(V_1)$, or equivalently, $P_0 = P_1 = P$. Then $V_1V_0^*$ is a unitary operator in $R(V_0)$. Therefore $U = V_1V_0^* + I - P$ is a unitary operator in $\mathcal{H}$, satisfying $UV_0 = V_1V_0^*V_0 = V_1$. Moreover, regarded as a 2×2 matrix in terms of P, U is of the form

$$\begin{pmatrix} V_1V_0^* & 0 \\ 0 & I-P \end{pmatrix}.$$

Therefore $U = e^{iX}$ with X self-adjoint, $\|X\| \leq \pi$ and with matrix

$$\begin{pmatrix} X' & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly $\|X\| = \|XP_0\|$, and therefore $\nu(t) = e^{itX}V_0$ is a minimizing geodesic with $\nu(0) = V_0$ and $\nu(1) = V_1$.

Suppose now that $R(V_0) \perp R(V_1)$. Note that this implies that the ranges have infinite codimension. Also it is clear that $V_1^*V_0 = V_0^*V_1 = 0$. Denote by Q the orthogonal projection onto $R(V_0) \oplus R(V_1)$, i.e. $Q = P_0 + P_1$. This implies that the element $J = -V_0V_1^* + V_1V_0^*$ satisfies

$$J^* = -J, \ J^2 = -Q, \ QJ = JQ = J, \ JV_0 = V_1.$$

Indeed, these are straightforward computations. Also it is clear that

$$\|JV_0\| = 1 = \|J\|.$$

Let $X = -i\frac{\pi}{2}J$. Then X is self-adjoint with $\|X\| = \|XV_0\| = \pi/2$. Note also that

$$e^{iX} = e^{\frac{\pi}{2}J} = I + \frac{\pi}{2}J - \frac{1}{2}(\frac{\pi}{2})^2 Q - \frac{1}{6}(\frac{\pi}{2})^3 J + \frac{1}{24}(\frac{\pi}{2})^4 Q + \cdots = J + I - Q.$$

Then $e^{iX}V_0 = JV_0 + (I-Q)V_0 = V_1$. Therefore $\nu(t) = e^{itX}V_0$ is a minimizing geodesic joining V_0 and V_1. □

Note that if $R(V_0) \perp R(V_1)$, then $d(V_0, V_1) = \pi/2$ (here d denotes the rectifiable metric).

4. A linear connection in $\mathcal{I}$

It was remarked that $\mathcal{I}$ is a submanifold of $\mathcal{B}(\mathcal{H})$. In this section we try to perform a classic differential geometric approach to the problem of finding curves of isometries of minimal length. We introduce a covariant derivative in the tangent bundle, and consider the question of how the geodesics of this connection relate to the metric geodesics found in the previous sections. Since $\mathcal{I}$ is a submanifold of $\mathcal{B}(\mathcal{H})$, its tangent spaces $(T\mathcal{I})_V$ are complemented closed (real) linear subspaces of $\mathcal{B}(\mathcal{H})$. A simple way to introduce a connection is to find a smooth distribution of supplements of $(T\mathcal{I})_V$ for each $V \in \mathcal{I}$. Or equivalently, a smooth distribution $\mathcal{I} \ni V \mapsto \mathcal{P}_V \in \mathcal{B}_{\mathbb{R}}(\mathcal{B}(\mathcal{H}))$ (= the space of real linear operators acting on $\mathcal{B}(\mathcal{H})$), verifying that $\mathcal{P}_V^2 = \mathcal{P}_V$ and $R(\mathcal{P}_V) = (T\mathcal{I})_V$. Let us define

$$\mathcal{P}_V(X) = \frac{1}{2}PX - \frac{1}{2}VXV^* + (I-P)X \ , \quad X \in \mathcal{B}(\mathcal{H}), \tag{4.1}$$

where as before, $P = VV^*$. Let us first show that these maps are real linear idempotents whith ranges equal to $(T\mathcal{I})_V$, and afterwards justify this choice.

Proposition 4.1. *The map $\mathcal{P}_V : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a real linear idempotent with range equal to $(T\mathcal{I})_V$. The map $\mathcal{I} \to \mathcal{B}_{\mathbb{R}}(\mathcal{B}(\mathcal{H}))$, $V \mapsto \mathcal{P}_V$ is smooth.*

Proof. It is apparent that $\mathcal{P}_V$ is real linear and bounded. Also it is clear that the map $V \mapsto \mathcal{P}_V$ is smooth. In order to prove that $\mathcal{P}_V$ is an indempotent with range $T(\mathcal{I})_V$, it suffices to show that $R(\mathcal{P}_V) \subset (T\mathcal{I})_V$ and $\mathcal{P}_V|_{(T\mathcal{I})_V} = id_{(T\mathcal{I})_V}$. The second fact is a straightforward computation. Recall that $(T\mathcal{I})_V = \{ZV : Z^* = -Z\}$. Then

$$\mathcal{P}_V(ZV) = \frac{1}{2}PZV - \frac{1}{2}VV^*Z^*V + (I-P)ZV = PZV + (I-P)ZV = ZV.$$

The first fact is less obvious. Let $Y \in R(\mathcal{P}_V)$; then

$$Y = \frac{1}{2}PX - \frac{1}{2}VX^*V + (I-P)X = Y_1 + Y_2,$$

where $Y_1 = \frac{1}{2}PX - \frac{1}{2}VX^*V$ and $Y_2 = (I-P)X$. First note that, using $P = VV^*$ and $PV = V$

$$Y_1 = \frac{1}{2}VV^*X - \frac{1}{2}VX^*PV = \frac{1}{2}\{PXV^* - VX^*P\}V \in \{ZV : Z^* = -Z\}.$$

Accordingly, using $V^*V = 1$ and $(I-P)V = 0$,

$$Y_2 = (I-P)XV^*V = \{(I-P)XV^* - VX^*(I-P)\}V \in \{ZV : Z^* = -Z\}.$$

□

The choice of $\mathcal{P}_V$ may seem at first sight arbitrary. In fact, there exist many undistinguished supplements for $(T\mathcal{I})_V$. However, if $\mathcal{H}$ is finite dimensional, $\mathcal{B}_{\mathbb{R}}(\mathcal{H})$ becomes a real Hilbert space with the natural inner product given by the real part of the trace:

$$\langle X, Y\rangle = Re\ (Tr(Y^*X)).$$

In this case, $\mathcal{I} = \mathcal{I}_0$ is the unitary group of $\mathcal{H}$. A straightforward computation shows that $\mathcal{P}_V$ is symmetric for this inner product. It follows that the supplement chosen for $(T\mathcal{I})_V$ is its orthogonal complement, and therefore the linear connection it induces on $\mathcal{I}$ is the Levi-Civita connection of the metric induced by the (real part of the) trace on every tangent space.

Let us proceed with the computation of this connection in the general case. If $\mathcal{X}(t)$ is a smooth tangent vector field along a curve $\gamma : [a, b] \to \mathcal{I}$, which is a smooth curve of operators taking values in $(T\mathcal{I})_{\gamma(t)}$ for each t, then the covariant derivative is given by

$$\frac{D\mathcal{X}}{dt} = \mathcal{P}_{\gamma(t)}(\dot{\mathcal{X}}(t)).$$

Therefore a curve $\mu(t) \in \mathcal{I}$ is a geodesic if it satisfies the second order differential equation

$$0 = \mathcal{P}_\mu(\ddot{\mu}) = \frac{1}{2}\mu\mu^*\ddot{\mu} - \frac{1}{2}\mu\ddot{\mu}^*\mu + (I - \mu\mu^*)\ddot{\mu} = \ddot{\mu} - \frac{1}{2}\mu\mu^*\ddot{\mu} - \frac{1}{2}\mu\ddot{\mu}^*\mu.$$

This expression can be made simpler. If one differentiates twice the identity $\mu^*\mu = 1$ one obtains, first $\dot{\mu}^*\mu + \mu^*\dot{\mu} = 0$ and next $\ddot{\mu}^*\mu + 2\dot{\mu}^*\dot{\mu} + \mu^*\ddot{\mu} = 0$. Multiplying this relation by μ on the left, one obtains

$$\mu\mu^*\ddot{\mu} + \mu\ddot{\mu}^*\mu = -2\mu\dot{\mu}^*\dot{\mu}.$$

This can be replaced in the differential equation above to obtain

$$0 = \ddot{\mu} + \mu\dot{\mu}^*\dot{\mu} \tag{4.2}$$

which we shall adopt. Insofar, we have considered metric geodesics of the form $\delta(t) = e^{itX}V$. It is natural to ask whether these can be geodesics of this linear connection.

Proposition 4.2. *A curve $\delta(t) = e^{itX}V$ is a geodesic of the linear connection just defined if and only if X^2 commutes with $P = VV^*$.*

Proof. Note that $\dot{\delta}(t) = iXe^{itX}V = ie^{itX}XV$, $\dot{\delta}^* = -iV^*Xe^{itX}$ and $\ddot{\delta}(t) = -e^{itX}X^2V$. Then, replacing it in Proposition 4.2, one gets that δ is a geodesic if

$$-e^{itX}X^2V + e^{itX}VV^*Xe^{-itX}e^{itX}XV = 0,$$

i.e.

$$e^{itX}(-X^2V + PX^2V) = 0$$

which, multiplying by V^* on the right is equivalent to $X^2P = PX^2P$. Since X^2 is self-adjoint, this is clearly equivalent to X^2 commuting with P. □

Remark 4.3. The operator X^2 commutes with P in two special cases, if, regarded as a 2×2 matrix in terms of P, the element X has a diagonal or codiagonal matrix.

1. In the first case, X itself commutes with P. Note that $X = PXP + (I - P)X(I - P)$, where both summands commute. Then $e^{it(I-P)X(I-P)}V = V$. Therefore the geodesic δ is $\delta(t) = e^{itX}V = e^{itPXP}e^{it(I-P)X(I-P)}V = e^{it(I-P)X(I-P)}V$. In other words, it suffices to consider $X = PXP$. In particular, $\|X\| = \|XP\|$, so that if additionally $\|X\| \leq \pi$, the geodesic δ of the linear connection is also minimizing. Also note that for all t, $R(\delta(t)) = R(V)$, i.e. these tangent vectors iXV correspond to (all possible) directions inside $R(V)$.
2. If X has codiagonal matrix, $X = PX(I - P) + (I - P)XP$. It is apparent that also in this case one has $\|X\| = \|XP\|$, and therefore (if again $\|X\| \leq \pi$) these give minimizing geodesics. Note that in this case the range of the tangent vector iXV is orthogonal to $R(V)$. Indeed,
$$iXV = i(PX(I - P)V + (I - P)XP)V = i(I - P)XV,$$
whose range is clearly orthogonal to $R(P) = R(V)$.

Summarizing, geodesics starting at V, which either stay within the range of V, or start orthogonally to it, are of the form $\delta(t) = e^{itX}V$, with X diagonal (respectively, codiagonal) with respect to P. If $\|X\| \leq \pi$, they are minimizing.

There is another natural way to induce a linear connection in $\mathcal{I}$ [9]. Recall that $\mathcal{I}$ is a homogeneous space of the unitary group $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$. The isotropy group $\mathrm{Iso}_V = \{W \in \mathcal{U}(\mathcal{H}) : WP = P\}$ has been identified as the unitary operators with matrices of the form
$$\begin{pmatrix} P & 0 \\ 0 & W' \end{pmatrix}$$
where $W' = (I-P)W(I-P)$ is a unitary operator of $R(V)^\perp$. The isotropy Banach-Lie algebra is therefore given by $\mathcal{I}so_V = \{iZ \in \mathcal{B}(\mathcal{H}) : Z^* = Z, ZP = 0\}$, which in matrix form are
$$\begin{pmatrix} 0 & 0 \\ 0 & iZ' \end{pmatrix}$$
with Z' self-adjoint in $R(V)^\perp$. One can introduce a linear connection in the homogeneous space by means of an invariant supplement $\mathcal{K}_V$ of $\mathcal{I}so_V$ inside the Banach-Lie algebra of $\mathcal{U}(\mathcal{H})$, which is the anti-Hermitic part of $\mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{H})_{ah} = \{Y \in \mathcal{B}(\mathcal{H}) : Y^* = -Y\}$. Such $\mathcal{K}_V$ should satisfy:

1. $\mathcal{K}_V \oplus \mathcal{I}so_V = \mathcal{B}(\mathcal{H})_{ah}$.
2. Iso_V-invariance: $W \cdot \mathcal{K}_V \cdot W^* = \mathcal{K}_V$, for all $W \in \mathrm{Iso}_V$.
3. Smoothness: the map $\mathcal{I} \to \mathcal{B}_{\mathbb{R}}(\mathcal{B}(\mathcal{H})_{ah})$, $V \mapsto \Pi_V$ is smooth, where Π_V is the idempotent with range $\mathcal{K}_V$, corresponding to the decomposition above.

A possible choice would be to take $\mathcal{K}_V$ consisting of matrices (in terms of P) of the form
$$\begin{pmatrix} A & B \\ -B^* & 0 \end{pmatrix}$$
with $A^* = -A$. However this choice behaves poorly in the metric sense. The advantage of a linear connection introduced by means of a reductive structure (see [6]) is that the geodesics can be explicitly computed, they are of the form $\delta(t) = e^{itK}V$, with $K \in \mathcal{K}_V$. If these curves were of minimal length, then the extension problem would admit the trivial answer, namely putting 0 in the $2, 2$ entry. This is not necessarily the case [3].

The question remains open as to whether there exists a reductive supplement consisting of solutions to the extension problem. That is, if there exists a supplement $\mathcal{K}_V$ which additionally satisfies

4. $\|\Pi_V\| \leq 1$.

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