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LINEAR BIJECTIONS PRESERVING THE HÖLDER SEMINORM

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ABSTRACT. Let (X,d) be a compact metric space and let α be a real number with $0<\alpha<1$. The aim of this paper is to solve a linear preserver problem on the Banach algebra $C^\alpha(X)$ of Hölder functions of order α from X into $\mathbb{K}$. We show that each linear bijection $T: C^\alpha(X)\to C^\alpha(X)$ having the property that $\alpha(T(f))=\alpha(f)$ for every $f\in C^\alpha(X)$, where

$$\alpha(f)=\sup\left\{\frac{|f(x)-f(y)|}{d^\alpha(x,y)}: x,y\in X,\ x\neq y\right\},$$

is of the form $T(f)=\tau f\circ\varphi+\mu(f)1_X$ for every $f\in C^\alpha(X)$, where $\tau\in\mathbb{K}$ with $|\tau|=1$, $\varphi: X\to X$ is a surjective isometry and $\mu: C^\alpha(X)\to\mathbb{K}$ is a linear functional.

1. Introduction

Let (X,d) be a compact metric space and let α be a real number with $0<\alpha<1$. A function $f: X\to\mathbb{K}$ is said to be a Hölder function of order α if there is a constant k such that

$$|f(x)-f(y)|\leq kd^\alpha(x,y),\quad \forall x,y\in X.$$

The smallest constant k for which the above inequality holds is called the Hölder seminorm of order α of f and it is denoted by $\alpha(f)$, that is,

$$\alpha(f)=\sup\left\{\frac{|f(x)-f(y)|}{d^\alpha(x,y)}: x,y\in X,\ x\neq y\right\}.$$

Let $C^\alpha(X)$ be the vector space of all the functions f from X into $\mathbb{K}$ such that $\alpha(f)$ is finite. This supremum does not define a norm on $C^\alpha(X)$, since $\alpha(f)=0$ if and only if f is constant on X.

It is said that a linear bijection $T: C^\alpha(X)\to C^\alpha(X)$ preserves the Hölder seminorm if

$$\alpha(T(f))=\alpha(f),\quad \forall f\in C^\alpha(X).$$

By brevity, these maps will be called Hölder seminorm preserving.

The most common way to actually obtain a norm on $C^\alpha(X)$ is to define

$$\|f\|=\max\{\alpha(f),\|f\|_\infty\},$$

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where

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

An advantage of this approach is that the Banach space of Hölder functions obtained is actually a Banach algebra. This space of functions and other closely related spaces have been the subject of considerable study (see, for example, [4, 5, 6, 9, 10, 11]).

Linear preserver problems concern the question of determining all linear maps on algebras of matrices or operators which leave invariant a given set, function or relation (see, for example, the survey papers [7, 8]), but similar questions can be raised on arbitrary algebras.

Let Y be a compact Hausdorff space and let $C(Y)$ be the space of real or complex continuous functions on Y equipped with the supremum norm. With regard to $C(Y)$, the main linear preserver problems concern the characterization of linear bijections preserving some given seminorm or norm. The classical Banach-Stone theorem determines the linear bijections preserving the sup-norm on $C(Y)$. Recently, when Y is a first countable space, Györy and Molnar [3] have given a complete description of linear bijections of $C(Y)$ which preserve the seminorm

$$f \mapsto \operatorname{diam}(f(Y)) = \sup\{|f(x) - f(y)| : x, y \in Y\}.$$

Their result has been extended to the case of a general compact Hausdorff space by González and Uspenskij [2] and independently by Cabello Sánchez [1].

Motivated by this, the purpose of this paper is to determine all Hölder seminorm preserving linear bijections of $C^\alpha(X)$.

2. The linear bijections which preserve the Hölder seminorm

Our characterization of these maps is the following.

Theorem 2.1. *A linear bijection $T : C^\alpha(X) \to C^\alpha(X)$ is Hölder seminorm preserving if and only if there is a surjective isometry $\varphi : X \to X$, a linear functional $\mu : C^\alpha(X) \to \mathbb{K}$ and a number τ with $|\tau| = 1$ and $\mu(1_X) + \tau \neq 0$ such that $T(f) = \tau f \circ \varphi + \mu(f)1_X$ for every $f \in C^\alpha(X)$.*

To prove Theorem 2.1, we follow a similar process to the one of Félix Cabello Sánchez in [1]. Our approach depends on the analysis of the isometry group of certain Banach spaces of Hölder functions in which the Hölder seminorm becomes a norm.

There is another way to obtain a norm on the vector space of Hölder functions, and that is to identify a distinguished point of the metric space and consider only the functions which are zero at this point.

Let x_0 be an arbitrarily chosen point of X. The set $C^\alpha(X, x_0)$ of all the functions $f \in C^\alpha(X)$ such that $f(x_0) = 0$ becomes a Banach space endowed with the norm $f \mapsto \alpha(f)$. The map $\rho : C^\alpha(X) \to C^\alpha(X, x_0)$ defined by

$$\rho(f) = f - f(x_0)1_X, \quad \forall f \in C^\alpha(X),$$

is surjective linear with $\ker\rho$ equal to the space of the constant functions on X and

$$\alpha(\rho(f)) = \alpha(f), \quad \forall f \in C^\alpha(X).$$

Suppose that T is a Hölder seminorm preserving linear bijection of $C^\alpha(X)$. Then there is a (unique) surjective linear isometry T_α of $C^\alpha(X, x_0)$ such that the diagram

$$\begin{array}{ccc} C^\alpha(X) & \xrightarrow{T} & C^\alpha(X) \\ \rho\downarrow & & \downarrow\rho \\ C^\alpha(X,x_0) & \xrightarrow{T_\alpha} & C^\alpha(X,x_0) \end{array}$$

commutes.

We derive Theorem 2.1 from the following characterization of the isometries of $C^\alpha(X, x_0)$. We shall denote by $S_{\mathbb{K}}$ the set of all the unimodular scalars of $\mathbb{K}$.

Theorem 2.2. *A linear map $T_\alpha : C^\alpha(X, x_0) \to C^\alpha(X, x_0)$ is a surjective isometry if and only if there is a surjective isometry φ of X and a number τ in $S_{\mathbb{K}}$ such that $T_\alpha(\rho(f)) = \rho(\tau f \circ \varphi)$ for all $f \in C^\alpha(X)$.*

Proof of Theorem 2.1. It is straightforward to check that every linear map T of the form $T(f) = \tau f \circ \varphi + \mu(f)1_X$ for every $f \in C^\alpha(X)$ with τ, φ, μ being as in Theorem 2.1, is a Hölder seminorm preserving linear bijection of $C^\alpha(X)$.

Now, suppose that $T : C^\alpha(X) \to C^\alpha(X)$ is a linear bijection which preserves the Hölder seminorm and let T_α be the corresponding isometry of $C^\alpha(X, x_0)$ such that $T_\alpha \circ \rho = \rho \circ T$. By Theorem 2.2 there is a surjective isometry φ of X and a number $\tau \in S_{\mathbb{K}}$ such that $T_\alpha(\rho(f)) = \rho(\tau f \circ \varphi)$ for all $f \in C^\alpha(X)$. Then $T(f) - \tau f \circ \varphi \in \ker \rho$ for all $f \in C^\alpha(X)$, and therefore there exists a linear functional $\mu : C^\alpha(X) \to \mathbb{K}$ such that $T(f) = \tau f \circ \varphi + \mu(f)1_X$ for every $f \in C^\alpha(X)$. Since the relation $\mu(1_X) \neq -\tau$ is obvious, the proof of Theorem 2.1 is complete. □

For the proof of Theorem 2.2 we shall need a description of the following points.

3. The extreme points of the unit ball of $C^\alpha(X, x_0)^*$

These extreme points play a key role in our characterization of the isometries of $C^\alpha(X, x_0)$. To identify them, we construct a linear isometric imbedding of $C^\alpha(X, x_0)$ into a suitable space of continuous functions supplied with the supremum norm.

Let W be the complement of the diagonal in the cartesian product space $X \times X$, that is, $W = \{(x, y) \in X \times X : x \neq y\}$ and let βW be the Stone-Čech compactification of W.

If $C(\beta W)$ denotes the Banach space of continuous functions from βW into $\mathbb{K}$ endowed with the supremum norm, we define the map $r : C^\alpha(X, x_0) \to C(\beta W)$ by

$$r(f)(w) = \beta f^*(w), \quad \forall f \in C^\alpha(X, x_0),\ \forall w \in \beta W,$$

where

$$f^*(x, y) = \frac{f(x) - f(y)}{d^\alpha(x, y)}, \quad \forall (x, y) \in W,$$

and βf^* is its norm-preserving extension to βW. Clearly r is linear and

$$\|r(f)\|_\infty = \|\beta f^*\|_\infty = \|f^*\|_\infty = \alpha(f)$$

for every $f \in C^\alpha(X, x_0)$.

For every $w \in \beta W$, we shall denote by δ_w the evaluation functional on $C(\beta W)$ given by $\delta_w(f) = f(w)$. Moreover, $\widetilde{\delta}_w$ will stand for the functional $\delta_w \circ r$, that is,

$$\widetilde{\delta}_w(f) = \delta_w(r(f)), \quad \forall f \in C^\alpha(X, x_0).$$

Clearly $\widetilde{\delta}_w$ is linear and

$$\left|\widetilde{\delta}_w(f)\right| = |\beta f^*(w)| \leq \|\beta f^*\|_\infty = \alpha(f), \quad \forall f \in C^\alpha(X, x_0).$$

Hence $\widetilde{\delta}_w \in C^\alpha(X, x_0)^*$ and $\left\|\widetilde{\delta}_w\right\| \leq 1$.

Lemma 3.1. *Each extreme point of the unit ball of $C^\alpha(X, x_0)^*$ is of the form $\tau\widetilde{\delta}_w$ for some $\tau \in S_{\mathbb{K}}$ and some $w \in \beta W$.*

Proof. The linear map r imbeds $C^\alpha(X, x_0)$ isometrically as a subspace of $C(\beta W)$. As a consequence, the adjoint map $r^* : C(\beta W)^* \to C^\alpha(X, x_0)^*$ sends the unit ball of $C(\beta W)^*$ onto the unit ball of $C^\alpha(X, x_0)^*$. Thus the Krein-Milman Theorem implies that each extreme point of the unit ball of $C^\alpha(X, x_0)^*$ is the image under r^* of some extreme point of the unit ball of $C(\beta W)^*$. By the Arens-Kelley Theorem, the extreme points of the unit ball of $C(\beta W)^*$ are of the form $\tau\delta_w$ where $\tau \in S_{\mathbb{K}}$ and $w \in \beta W$.

Therefore, if Φ is an extreme point of the unit ball of $C^\alpha(X, x_0)^*$, we have

$$\Phi = r^*(\tau\delta_w) = \tau r^*(\delta_w) = \tau(\delta_w \circ r) = \tau\widetilde{\delta}_w$$

for some $\tau \in S_{\mathbb{K}}$ and some $w \in \beta W$. □

Observe that for each (x, y) in W, the functional $\widetilde{\delta}_{(x,y)}$ comes given by

$$\widetilde{\delta}_{(x,y)}(f) = \delta_{(x,y)}(r(f)) = \beta f^*(x, y) = f^*(x, y) = \frac{f(x) - f(y)}{d^\alpha(x, y)}$$

for every $f \in C^\alpha(X, x_0)$.

Next we shall study the relations between the extreme points of the unit ball of $C^\alpha(X, x_0)^*$ and the functionals of the form $\widetilde{\delta}_{(x,y)}$ with $(x, y) \in W$. Mayer-Wolf states these relations in [9] without proof (see remark after Theorem 2.3). We include the proofs for the sake of completeness, but we shall detail essentially those steps which do not appear in [9].

We first prove that every functional $\widetilde{\delta}_{(x,y)}$ is an extreme point of the unit ball of $C^\alpha(X, x_0)^*$. We shall need the following technical lemma ([12], Lemma 2.4.4).

Lemma 3.2. *Let $0 < \alpha < \beta < 1$ and let $A, \varepsilon > 0$. Then there exists $\delta > 0$ such that for any $a, b, c, d \geq 0$ and $B > 0$ which satisfy the three conditions:*

$$|a - b|, |c - d| \leq A \leq a + b, c + d,$$
$$|b - c|, |a - d| \leq B \leq b + c, a + d,$$
$$\varepsilon \leq a + c, b + d,$$

we have

$$\frac{\left|a^\beta - b^\beta + c^\beta - d^\beta\right|}{2A^{\beta-\alpha}B^\alpha} \leq 1 - \delta.$$

Lemma 3.3. *Let τ be in $S_{\mathbb{K}}$ and (x, y) in W. Then $\tau\widetilde{\delta}_{(x,y)}$ is an extreme point of the unit ball of $C^\alpha(X, x_0)^*$.*

Proof. The proof that $\tau\widetilde{\delta}_{(x,y)}$ (or $\widetilde{\delta}_{(x,y)}$ equivalently) is an extreme point of the unit ball of $C^\alpha(X, x_0)^*$ will be based on the following fact which is a special case of a more general result proved by de Leeuw ([6], see Lemma 3.2):

Let X be a compact Hausdorff space and A a closed subspace of $C(X)$. If $x \in X$ and δ_x is the evaluation functional $f \in C(X) \to f(x) \in \mathbb{K}$, a sufficient condition for $\delta_x|_A$ to be an extreme point of the unit ball of A^* is that there exists a function f in the unit ball of A such that $f(x) = 1$, and $|f(y)| = 1$ iff there exists a $\theta = \pm 1$ such that $g(y) = \theta g(x)$ for all $g \in A$. In this case we say that f peaks at x relative to A.

Taking into account this result, we only have to find a function $f_{(x,y)} \in C^\alpha(X, x_0)$ such that $r(f_{(x,y)})$ peaks at (x, y) relative to $r(C^\alpha(X, x_0))$. To see this we follow ([12], Proposition 2.4.5). Choose a real number β with $\alpha < \beta < 1$ and consider the function $f_{(x,y)} = \rho(g_{(x,y)})$, where

$$g_{(x,y)}(z) = \frac{d^\beta(z, y) - d^\beta(z, x)}{2d^{\beta-\alpha}(x, y)}, \quad \forall z \in X.$$

Clearly

$$r(f_{(x,y)})(x, y) = \frac{f_{(x,y)}(x) - f_{(x,y)}(y)}{d^\alpha(x, y)} = 1.$$

Let U be any open subset of W (in the topology induced by the sup metric) containing (x, y) and (y, x). Then there exists $\varepsilon > 0$ such that

$$(z, w) \notin U \quad \Rightarrow \quad d(x, z) + d(y, w) \geq \varepsilon,\ d(x, w) + d(y, z) \geq \varepsilon.$$

For this ε and $A = d(x, y)$, let δ be given by the above lemma. For any $(z, w) \notin U$, let $a = d(y, z)$, $b = d(x, z)$, $c = d(x, w)$, $d = d(y, w)$ and $B = d(z, w)$. A simple calculation shows that the hypotheses of the lemma are satisfied and thus we have

$$\left|r(f_{(x,y)})(z, w)\right| = \frac{\left|f_{(x,y)}(z) - f_{(x,y)}(w)\right|}{d^\alpha(z, w)} = \frac{\left|a^\beta - b^\beta + c^\beta - d^\beta\right|}{2A^{\beta-\alpha}B^\alpha} \leq 1 - \delta.$$

This shows that $f_{(x,y)} \in C^\alpha(X, x_0)$ with

$$\left\|r(f_{(x,y)})\right\|_\infty = \alpha(f_{(x,y)}) = 1,$$

and $\left|r(f_{(x,y)})(w)\right| < 1$ for all $w \in \beta W$ except at $w = (x, y)$ and $w = (y, x)$. According to the definition, $r(f_{(x,y)})$ peaks at (x, y) relative to $r(C^\alpha(X, x_0))$. $\square$

Let $\Delta : W \to C^\alpha(X, x_0)^*$ be the map defined by

$$\Delta(x, y) = \widetilde{\delta}_{(x,y)}.$$

Lemma 3.4. *The map $\Delta : W \to (\Delta(W), w^*)$ is a local homeomorphism.*

Proof. The proof will be carried out through a series of steps.

Step 1. Let $(x_1, y_1), (x_2, y_2)$ be in W and suppose that $\widetilde{\delta}_{(x_1,y_1)} = \widetilde{\delta}_{(x_2,y_2)}$. Then $\{x_1, y_1\} = \{x_2, y_2\}$.

Proof. Let us assume, to obtain a contradiction, that $\{x_1, y_1\} \neq \{x_2, y_2\}$. Then there exists at least a point in $\{x_2, y_2\}$ which is not in $\{x_1, y_1\}$. We can suppose, without loss of generality, that such a point is x_2. Let $g : X \to \mathbb{R}$ be the function defined by

$$g(z) = \frac{d^\alpha(z, \{x_1, y_1, y_2\})}{d^\alpha(z, x_2) + d^\alpha(z, \{x_1, y_1, y_2\})}, \quad \forall z \in X.$$

It is easy to check that $g \in C^\alpha(X)$. Therefore the function $f = \rho(g) \in C^\alpha(X, x_0)$ and for every $(x, y) \in W$, it is clear that

$$\widetilde{\delta}_{(x,y)}(f) = \frac{g(x) - g(y)}{d^\alpha(x, y)}.$$

In particular we obtain that $\widetilde{\delta}_{(x_1,y_1)}(f) = 0$ and $\widetilde{\delta}_{(x_2,y_2)}(f) = \frac{1}{d^\alpha(x_2,y_2)}$. Since $\widetilde{\delta}_{(x_1,y_1)} = \widetilde{\delta}_{(x_2,y_2)}$, a contradiction follows. So $\{x_1, y_1\} = \{x_2, y_2\}$.

Step 2. Given (x, y) in W, there exists a neighbourhood $B_\varepsilon(x, y)$ of (x, y) in W for a suitable $\varepsilon > 0$ such that $(w, z) \notin B_\varepsilon(x, y)$ if $(z, w) \in B_\varepsilon(x, y)$.

Proof. For each $\varepsilon > 0$ the set

$$B_\varepsilon(x, y) = \{(z, w) \in W : d^\alpha(z, x) < \varepsilon,\ d^\alpha(w, y) < \varepsilon\}$$

is a neighbourhood of (x, y) in W with the topology induced by the sup metric. Let ε be a real number such that $0 < \varepsilon < d^\alpha(x, y)/2$. Then the neighbourhoods $B_\varepsilon(x, y)$ and $B_\varepsilon(y, x)$ are disjoint. If $(z, w) \in B_\varepsilon(x, y)$, then clearly $(w, z) \in B_\varepsilon(y, x)$ and therefore $(w, z) \notin B_\varepsilon(x, y)$.

Step 3. Given (x, y) in W, the map Δ is injective on $B_\varepsilon(x, y)$.

Proof. This follows immediately from Steps 1 and 2.

Step 4. The map $\Delta : W \to (\Delta(W), w^*)$ is continuous.

Proof. Let $(x, y) \in W$. If $\{(x_i, y_i)\}$ is a net in W converging to (x, y), then $\{f(x_i, y_i)\}$ converges to $f(x, y)$ for every f in $C(\beta W)$. This says that $\{\delta_{(x_i,y_i)}\}$ converges to $\delta_{(x,y)}$ in $(C(\beta W)^*, w^*)$. Since r^* (as an adjoint mapping) is continuous from $(C(\beta W)^*, w^*)$ into $(C^\alpha(X, x_0)^*, w^*)$, then $\{r^*(\delta_{(x_i,y_i)})\} = \{\widetilde{\delta}_{(x_i,y_i)}\}$ converges to $r^*(\delta_{(x,y)}) = \widetilde{\delta}_{(x,y)}$ in $(C^\alpha(X, x_0)^*, w^*)$.

Step 5. The map $\Delta : W \to (\Delta(W), w^*)$ is open.

Proof. Let U be an open subset of W and let $(x, y) \in U$. We take a suitable neighbourhood $B_\varepsilon(x, y) \subset U$ and we can find a function $f \in C^\alpha(X, x_0)$ such that $r(f)(x, y) = 1$ and $r(f)(z, w) = 0$ for all (z, w) in $W - B_\varepsilon(x, y)$. Let $V_1 = \{\Phi \in C^\alpha(X, x_0)^* : \Phi(f) \neq 0\}$. Then V_1 is w^*-open in $C^\alpha(X, x_0)^*$. Hence $V_1 \cap \Delta(W)$ is w^*-open in $\Delta(W)$ and we have that $\widetilde{\delta}_{(x,y)} \in V_1 \cap \Delta(W) \subset \Delta(B_\varepsilon(x, y)) \subset \Delta(U)$.

Step 6. Given (x, y) in W, the map Δ is a homeomorphism from $B_\varepsilon(x, y)$ into $(\Delta(B_\varepsilon(x, y)), w^*)$.

Proof. The proof is deduced from the above steps. □

From now on $F(W)$ stands for the set

$$\{\tau\widetilde{\delta}_{(x,y)} : \tau \in S_{\mathbb{K}},\ (x, y) \in W\}.$$

Every element of $F(W)$ is an extreme point of the unit ball of $C^\alpha(X, x_0)^*$, but the reciprocal is not true. However we have the following fact:

Lemma 3.5. *Every extreme point of the unit ball of $C^\alpha(X, x_0)^*$ belongs to the w^*-closure of $F(W)$.*

Proof. Let Φ be an extreme point of the unit ball of $C^\alpha(X, x_0)^*$. Then $\Phi = \tau\widetilde{\delta}_w$ for some $\tau \in S_{\mathbb{K}}$ and some $w \in \beta W$ by Lemma 3.1. If $w \in W$, there is nothing to prove. If $w \in \beta W \backslash W$, since W is dense in βW, we can select a net $\{w_i\}$ from W converging to w in βW. Reasoning as in the proof of Step 4 in Lemma 3.4, we show that $\{\widetilde{\delta}_{w_i}\}$ converges to $\widetilde{\delta}_w$ in $(C(X, x_0)^*, w^*)$ and so Φ is in the w^*-closure of $F(W)$. □

A complete description of the extreme points of the unit ball of $C^\alpha(X,x_0)^*$ of the form $\widetilde{\delta}_w$ with $w \in \beta W\backslash W$ appears difficult ([5], see page 185). For example, Johnson ([4], Theorem 2.8) states that if the compact metric space X is an infinite set, then the unit ball of the space $C^\alpha(X)^*$ has extreme points of this type, and Sherbert ([10], Chapter III) shows that these functionals must be point derivations.

However the w^*-topology supplies a property which permits us to distinguish the extreme points of the form $\widetilde{\delta}_{(x,y)}$ with $(x,y) \in W$ from those which are of the form $\widetilde{\delta}_w$ with $w \in \beta W\backslash W$.

Lemma 3.6. *Let Φ be an extreme point of the unit ball of $C^\alpha(X,x_0)^*$. Then Φ belongs to $F(W)$ if and only if Φ has a w^*-metrizable neighbourhood in the w^*-closure of $F(W)$.*

Proof. The proof of the necessity is immediate since $(\Delta(W), w^*)$ and W are locally homeomorphic by Lemma 3.4.

To prove the sufficiency suppose that $\Phi \notin F(W)$ and that Φ has a w^*-metrizable neighbourhood in the w^*-closure of $F(W)$. The existence of this neighbourhood implies that there is a sequence $\{\Phi_n\}$ in $F(W)$ such that $\{\Phi_n\}$ converges to Φ in the w^*-topology.

Since $\Phi \notin F(W)$, then $\Phi = \tau\widetilde{\delta}_w$ for some $\tau \in S_{\mathbb{K}}$ and $w \in \beta W\backslash W$ by Lemma 3.1. Since W is dense in βW, there is a net $\{w_i\} = \{(x_i, y_i)\}$ in W converging to w in βW. By ([10], Lemma 9.6) the nets $\{x_i\}$ and $\{y_i\}$ converge in X to the same point $x_w \in X$ and this point x_w is independent of the choice of net $\{w_i\}$ from W.

In view of these comments we can suppose without loss of generality that $\Phi_n = \tau\widetilde{\delta}_{(x_n,y_n)}$ with $\{x_n\}$ and $\{y_n\}$ converging to x_w. Following the proof of ([9], Theorem 2.3), we construct a function $f \in C^\alpha(X,x_0)$ such that $\lim_{n\to\infty} \widetilde{\delta}_{(x_n,y_n)}(f)$ does not exist and so we obtain a contradiction. $\square$

4. The isometry group of $C^\alpha(X,x_0)$

Now we are ready to characterize the isometries of $C^\alpha(X,x_0)$. The isometries between pairs of spaces of this type have been studied in [9].

Proof of Theorem 2.2. It is easy to verify that the map T of the form $T(\rho(f)) = \rho(\tau f \circ \varphi)$ for all $f \in C^\alpha(X)$ with τ, φ under the assumptions of Theorem 2.2, is a surjective linear isometry of $C^\alpha(X,x_0)$.

Now suppose that T is a surjective isometry of $C^\alpha(X,x_0)$. Then the adjoint map T^* is a surjective isometry of $C^\alpha(X,x_0)^*$ as well, and therefore T^* is a bijection of the set of the extreme points of the unit ball of $C^\alpha(X,x_0)^*$. Moreover, as T^* is a homeomorphism of $(C^\alpha(X,x_0)^*, w^*)$, T^* carries points which possess a w^*-metrizable neighbourhood in the w^*-closure of $F(W)$ into points with the same property.

Let (x,y) be in W. By Lemmas 3.3 and 3.6, $\widetilde{\delta}_{(x,y)}$ is an extreme point of the unit ball of $C^\alpha(X,x_0)^*$ which has a w^*-metrizable neighbourhood in the w^*-closure of $F(W)$. In view of the above comments, $T^*(\widetilde{\delta}_{(x,y)})$ satisfies the same properties. Applying again Lemma 3.6, $T^*(\widetilde{\delta}_{(x,y)})$ belongs to $F(W)$. Therefore there are $\sigma \in S_{\mathbb{K}}$ and $(u,v) \in W$ such that

$$T^*(\widetilde{\delta}_{(x,y)}) = \sigma\widetilde{\delta}_{(u,v)}.$$

Let X_2 be the collection of all subsets of X having exactly two elements. Clearly T^* defines a bijection $\Phi : X_2 \to X_2$ by

$$\Phi(\{x,y\}) = \operatorname{supp}\left(T^*(\widetilde{\delta}_{(x,y)})\right).$$

This definition of Φ makes sense because if $\tau, \sigma \in S_{\mathbb{K}}$, $(x,y), (u,v) \in W$ and $\tau\widetilde{\delta}_{(x,y)} = \sigma\widetilde{\delta}_{(u,v)}$, then $\{x,y\} = \{u,v\}$. This follows as in Step 1 of Lemma 3.4.

There is a bijection $\varphi : X \to X$ such that $\Phi(\{x,y\}) = \{\varphi(x), \varphi(y)\}$ for every $x, y \in X$ (see [1], Lemma 3). Plainly

$$T^*(\widetilde{\delta}_{(x,y)}) = \sigma(x,y)\widetilde{\delta}_{(\varphi(x),\varphi(y))},$$

where $\sigma(x,y) \in S_{\mathbb{K}}$. If for each x in X, δ_x denotes the evaluation functional

$$\delta_x(f) = f(x), \quad \forall f \in C^\alpha(X, x_0),$$

we can write the preceding equality as

$$T^*\left(\frac{\delta_x - \delta_y}{d^\alpha(x,y)}\right) = \sigma(x,y)\frac{\delta_{\varphi(x)} - \delta_{\varphi(y)}}{d^\alpha(\varphi(x),\varphi(y))}.$$

Let us see that $\sigma(x,y)\frac{d^\alpha(x,y)}{d^\alpha(\varphi(x),\varphi(y))}$ does not depend on x, y. Let $z \notin \{x,y\}$. Then

$$\begin{aligned}
\sigma(x,y)&\frac{d^\alpha(x,y)}{d^\alpha(\varphi(x),\varphi(y))}(\delta_{\varphi(x)} - \delta_{\varphi(y)}) = T^*(\delta_x - \delta_y) \\
&= T^*(\delta_x - \delta_z + \delta_z - \delta_y) = T^*(\delta_x - \delta_z) + T^*(\delta_z - \delta_y) \\
&= \sigma(x,z)\frac{d^\alpha(x,z)}{d^\alpha(\varphi(x),\varphi(z))}(\delta_{\varphi(x)} - \delta_{\varphi(z)}) + \sigma(z,y)\frac{d^\alpha(z,y)}{d^\alpha(\varphi(z),\varphi(y))}(\delta_{\varphi(z)} - \delta_{\varphi(y)}),
\end{aligned}$$

so that

$$\sigma(x,y)\frac{d^\alpha(x,y)}{d^\alpha(\varphi(x),\varphi(y))} = \sigma(x,z)\frac{d^\alpha(x,z)}{d^\alpha(\varphi(x),\varphi(z))} = \sigma(z,y)\frac{d^\alpha(z,y)}{d^\alpha(\varphi(z),\varphi(y))}.$$

Hence there exists a constant $k > 0$ such that

$$d(\varphi(x),\varphi(y)) = kd(x,y)$$

and clearly $\operatorname{diam}(\varphi(X)) = k \operatorname{diam}(X)$. Since φ is surjective, then $k = 1$ and so

$$d(\varphi(x),\varphi(y)) = d(x,y).$$

Hence φ is an isometry of X. It follows that

$$\sigma(x,y) = \sigma(x,z) = \sigma(z,y)$$

and therefore there exists a $\tau \in S_{\mathbb{K}}$ such that $\sigma(x,y) = \tau$.

Finally, given $f \in C^\alpha(X)$ we have

$$\begin{aligned}
\widetilde{\delta}_{(x,y)}(T(\rho(f)) &= T^*(\widetilde{\delta}_{(x,y)})(\rho(f)) = \tau\widetilde{\delta}_{(\varphi(x),\varphi(y))}(\rho(f)) = \tau\frac{f(\varphi(x)) - f(\varphi(y))}{d^\alpha(\varphi(x),\varphi(y))} \\
&= \frac{(\tau f \circ \varphi)(x) - (\tau f \circ \varphi)(y)}{d^\alpha(x,y)} = \widetilde{\delta}_{(x,y)}(\rho(\tau f \circ \varphi))
\end{aligned}$$

for every $(x,y) \in W$. By Lemma 3.5 it follows that $\Phi(T(\rho(f))) = \Phi(\rho(\tau f \circ \varphi))$ for every extreme point Φ of the unit ball of $C^\alpha(X, x_0)^*$. The Krein-Milman Theorem implies that $\Phi(T(\rho(f))) = \Phi(\rho(\tau f \circ \varphi))$ for every $\Phi \in C^\alpha(X, x_0)^*$ and so

$$T(\rho(f)) = \rho(\tau f \circ \varphi).$$

□

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