A QUASICONFORMAL MAPPING CLASS GROUP ACTING TRIVIALLY ON THE ASYMPTOTIC TEICHMÜLLER SPACE

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ABSTRACT. For an analytically infinite Riemann surface \( R \), the quasiconformal mapping class group \( \text{MCG}(R) \) always acts faithfully on the ordinary Teichmüller space \( T(R) \). However in this paper, an example of \( R \) is constructed for which \( \text{MCG}(R) \) acts trivially on its asymptotic Teichmüller space \( AT(R) \).

§1. INTRODUCTION AND STATEMENT OF RESULTS

The asymptotic Teichmüller space \( AT(R) \) of a Riemann surface \( R \) is a certain quotient space of the Teichmüller space \( T(R) \). It was first introduced in [7] when \( R \) was the hyperbolic plane and in [3], [4] when \( R \) was an arbitrary hyperbolic Riemann surface.

A quasiconformal homeomorphism \( f \) of \( R \) is called asymptotically conformal if, for every \( \varepsilon > 0 \), there exists a compact subset \( V \) of \( R \) such that the maximal dilatation \( K(f) \) of \( f \) is less than \( 1 + \varepsilon \) on \( R - V \). Two quasiconformal homeomorphisms \( f_1 \) and \( f_2 \) of \( R \) are said to be asymptotically equivalent if there exists an asymptotically conformal homeomorphism \( h : f_1(R) \to f_2(R) \) such that \( f_2^{-1} \circ h \circ f_1 : R \to R \) is homotopic to the identity relative to the ideal boundary at infinity of \( R \). The asymptotic Teichmüller space \( AT(R) \) is the set of all asymptotic equivalence classes \( [f]_A \) of quasiconformal homeomorphisms \( f \) of \( R \). Since a conformal homeomorphism is asymptotically conformal, the ordinary Teichmüller equivalence is stronger than the asymptotic equivalence. Hence there exists a natural projection \( \pi : T(R) \to AT(R) \) that maps each Teichmüller equivalence class \( [f] \in T(R) \) to the asymptotic equivalence class \( [f]_A \in AT(R) \).

For the complex conjugate \( R^* \) of \( R \), let \( B(R^*) \) be the complex Banach space of all bounded holomorphic quadratic differentials on \( R^* \) with hyperbolic supremum norm and \( B_0(R^*) \) the closed subspace of \( B(R^*) \) consisting of all bounded holomorphic quadratic differentials \( \varphi \) that vanish at infinity. Here we say that \( \varphi \) vanishes at infinity if for every \( \varepsilon > 0 \) there exists a compact subset \( V^* \subset R^* \) such that the norm of \( \varphi \) on \( V^* \) is less than \( \varepsilon \). It is proved in [3] that the asymptotic Teichmüller space \( AT(R) \) is a complex Banach manifold modeled on the quotient Banach space \( B(R^*)/B_0(R^*) \). In fact, \( AT(R) \) is embedded in \( B(R^*)/B_0(R^*) \) as a bounded contractible domain, as is proved in [5]. This is the unique complex
structure on $AT(R)$ such that the projection $\pi : T(R) \to AT(R)$ is holomorphic with respect to the complex structure on $T(R)$.

A pseudo-distance $d_A$ on $AT(R)$ is induced from the Teichmüller distance $d$ on $T(R)$ as the quotient: $d_A([f]_A, [g]_A) = \inf d([f'], [g'])$ where the infimum is taken over all $[f'] \in \pi^{-1}([f]_A)$ and all $[g'] \in \pi^{-1}([g]_A)$. It is proved in [3] that the fiber $\pi^{-1}([f]_A)$ for each $[f]_A \in AT(R)$ is a closed submanifold of $T(R)$. Also, the above infimum is coincident with the infimum taken over all $[g'] \in \pi^{-1}([g]_A)$ with $[f'] \in \pi^{-1}([f]_A)$ fixed arbitrarily. Hence $d_A$ turns out to be a complete distance which is called the asymptotic Teichmüller distance.

The asymptotic Teichmüller space $AT(R)$ is of interest only when $R$ is analytically infinite. Otherwise $AT(R)$ is trivial, that is, it consists of just one point. Conversely, if $R$ is analytically infinite, then $AT(R)$ is not trivial. This is because the Banach space $B_0(R^*)$ is a proper subspace of $B(R^*)$ whenever $R$ is analytically infinite. See the Corollary in Section 6 of [1]. In fact, $B_0(R^*)$ is a separable Banach space whereas $B(R^*)$ is not separable. Hence $B(R^*)/B_0(R^*)$ is not separable and in particular infinite dimensional.

The quasiconformal mapping class is a homotopy equivalence class $[g]$ of quasiconformal automorphisms $g$ of $R$, and the quasiconformal mapping class group $\mathrm{MCG}(R)$ is the group of all these mapping classes. Here the homotopy is also relative to the ideal boundary at infinity. Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is also isometric with respect to the Teichmüller distance $d$. In this way, $\mathrm{MCG}(R)$ acts on $T(R)$, and a representation $\iota : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ is given. Similarly, $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic isometric automorphism of $AT(R)$ by $[f]_A \mapsto [f \circ g^{-1}]_A$. Thus we have a representation $\iota_A : \mathrm{MCG}(R) \to \mathrm{Aut}(AT(R))$.

It is proved in [2] that the representation $\iota : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ is injective (faithful) if $R$ is analytically infinite, or more precisely, unless $R$ is analytically finite of either $(0, 3)$, $(0, 4)$, $(1, 1)$, $(1, 2)$ or $(2, 0)$ type. In fact, $\iota$ is bijective (see [3]). On the other hand, it is also proved in [2] that the representation $\iota_A : \mathrm{MCG}(R) \to \mathrm{Aut}(AT(R))$ is injective precisely when $R$ is either the unit disk or a once-punctured disk. This fact is natural in some sense, for different actions of mapping classes of $R$ on $T(R)$ can shrink to the same action on $AT(R)$ under the projection $\pi : T(R) \to AT(R)$. Concerning this matter, we will see in this paper that even the following extreme case can occur.

**Theorem 1.** There exists an analytically infinite Riemann surface $R^o$ such that the quasiconformal mapping class group $\mathrm{MCG}(R^o)$ acts trivially on the asymptotic Teichmüller space $AT(R^o)$, namely, the kernel of the homomorphism $\iota_A$ is entirely $\mathrm{MCG}(R^o)$.

**§2. Construction of the Riemann surface $R^o$**

In [9], we have obtained an analytically infinite Riemann surface $R$ such that the quasiconformal mapping class group $\mathrm{MCG}(R)$ consists only of a countable number of elements. This mapping class group also holds a property that every mapping class has an asymptotically conformal representative as an automorphism of the base surface $R$. However, this is weaker than the statement of Theorem 1 which is equivalent to saying that every mapping class has an asymptotically conformal representative as an automorphism of every surface $R'$ quasiconformally equivalent.
to $R$. For this property, it suffices to prove that every mapping class has an eventually trivial representative as an automorphism of the base surface $R$. Here we say that a quasiconformal automorphism $f$ of $R$ is eventually trivial if there exists a topologically finite subsurface $V$ of $R$ such that $f$ is the identity on $R - V$. The Riemann surface $R^e$ as in Theorem 1 is constructed from $R$ by removing a countable number of points appropriately so that every quasiconformal mapping class should be eventually trivial.

A pair of pants $P$ is a hyperbolic surface with geodesic boundaries $c$ whose interior is homeomorphic to a three-punctured sphere. Every pair of pants admits a canonical orientation reversing isometric involution. The fixed point loci of this involution consist of three geodesic segments $\sigma$, which we call the symmetry axes. Cutting along the symmetry axes, we have two congruent right-angled hexagons $D$. Let $P_0$ be a pair of pants the lengths of whose geodesic boundary components are $0!$ and $1!$ and $1!$. Let $P_1$ be a pair of pants with the lengths $1!$ and $2!$ and $2!$. In the same way, for every non-negative integer $n$, let $P_n$ be a pair of pants with the lengths $n!$ and $(n + 1)!$ and $(n + 1)!$. The three symmetry axes divide $P_n$ into two congruent right-angled hexagons $D_n$. The geodesic boundary components of length $n!$ and $(n + 1)!$ in $P_n$ are denoted by $c_n$ and $c_{n+1}$ (right and left) respectively.

The Riemann surface $R$ is made of $2^{n+1}$ copies of $P_n$ for all $n \geq 0$ as follows. We take 2 copies of $P_0$ and glue the geodesic boundary component $c_0$ of each $P_0$ together. The resulting hyperbolic surface with 4 geodesic boundary components $c_1$ is denoted by $R_1$. Next take 4 copies of $P_1$ and glue the geodesic boundary component $c_1$ of each $P_1$ with the 4 boundary components of $R_1$. The resulting hyperbolic surface with 8 geodesic boundary components $c_2$ is denoted by $R_2$. Continuing this process, we obtain, for every $n \geq 1$, a hyperbolic surface $R_n$ with $2^{n+1}$ geodesic boundary components $c_n$ made of $R_{n-1}$ and $2^n$ copies of $P_{n-1}$. Then we take the exhaustion of these surfaces $R_n$, which is $R = \bigcup_{n=0}^{\infty} R_n$. Each connected component of $R - R_n$ is a neighborhood of an end, which is denoted by $E_n$. At each step of gluing, we give an appropriate amount of twist along the geodesic boundaries so that $R$ is a complete hyperbolic surface without ideal boundary at infinity.

To make $R^e$ from $R$, we puncture infinitely many pairs of pants $P_n$ in $R$. The removed point $q_n$ in $P_n$ is chosen at the middle of the symmetric axis connecting the right and left $c_{n+1}$ in $P_n$. In order to choose the punctured pairs of pants, we use a family of streams as is defined in the next two paragraphs. A pair of pants $P_n$ in $R$ is punctured if no stream in the family intersects it. Otherwise, it is not punctured. In this manner, we have a punctured Riemann surface $R^e$.

An n-stream ($n \geq 0$) is a path starting from a pair of pants $P_n$ and going through $P_{n+1}, P_{n+2}, \ldots$ in order towards infinity. Each n-stream takes a right as its initial direction at $P_n$ and then continues to keep right until it reaches the turning point $P_i$ ($i > n$). The stream switches its direction from right to left once it is here at $P_i$. We call this integer $i$ the turning number for the stream. After $P_i$, it takes a left throughout.

We make a family of the streams satisfying the following conditions:

1. Two 0-streams start from both the pairs of pants $P_0$.
2. If a stream passes or starts from a pair of pants $P_n$ ($n \geq 0$), then, among the two adjacent pairs of pants $P_{n+1}$ to the $P_n$, the one where the stream
§3. Geometry of the Riemann surface $R^0$

The punctured Riemann surface $R^0$ has no ideal boundary at infinity. This can be seen from a general fact that if $R$ has no ideal boundary at infinity and $R^0$ is obtained by removing a totally disconnected closed set from $R$, then $R^0$ has no ideal boundary at infinity. Indeed, a Riemann surface has ideal boundary at infinity if and only if it has a simply connected subdomain $W$ such that the relative boundary $\partial W$ is connected and that the harmonic measure of $\partial W$ in $W$ is not full; if $R^0$ has this property then so does $R$. Remark that, since $R^0$ has no ideal boundary at infinity, homotopy in the definition of the Teichmüller equivalence for $T(R^0)$ can be regarded as free homotopy.

A boundary component of $P_n \cap R^0$ is not geodesic with respect to the intrinsic hyperbolic metric on $R^0$. For each of the three boundary components of $P_n \cap R^0$, there exists a unique simple closed geodesic in $R^0$ in its free homotopy class, which is denoted by $c_n^0$ or $c_{n+1}^0$. A subsurface of $R^0$ bounded by these three simple closed geodesics, $c_n^0$, $c_{n+1}^0$ right and $c_{n+1}^0$ left, is denoted by $P_{n}^0$, which is either a punctured or a non-punctured pair of pants. Similarly, $R_n^0$ is defined to be the subsurface of $R^0$ bounded by all $c_n^0$, and $E_n^0$ is defined to be a connected component of $R^0 - R_n^0$.

The complete hyperbolic metric on $R^0$ is not less than the hyperbolic metric on $R$. This is because of a monotone property of the hyperbolic densities with respect to the inclusion relation between domains. This in particular implies that the length $\ell(c_n^0)$ of the geodesic $c_n^0$ in $R^0$ is not less than the length $\ell(c_n)$ of the geodesic $c_n$ freely homotopic to $c_n^0$ in $R$. On the other hand, for our particular surface $R^0$, $\ell(c_n^0)$ has the following upper bound in terms of $\ell(c_n)$.

**Lemma 2.** The length $\ell(c_n^0)$ is not greater than $4\ell(c_n) = 4n!$.

**Proof.** In the original surface $R$, consider an embedded annulus

$$H_\omega(c_n) = \{ p \in E_n \mid 0 < d_R(c_n, p) < \omega \}$$

called a half-collar for the simple closed geodesic $c_n$, where $d_R$ denotes the hyperbolic distance on $R$. By Proposition 5 and Lemma 3 in [9], we see that the length of the shortest geodesic arc $\delta_n(\not\subset c_n)$ connecting $c_n$ to itself in $E_n$ is

$$\ell(\delta_n) = 2 \text{arcsinh} \left\{ \frac{\cosh((n + 1)!/2)}{\sinh(n!/4)} \right\} > n! \times n.$$  

Hence uniform width $\omega$ of the half-collar $H_\omega(c_n)$ can be taken as

$$\text{arcsinh} \left\{ \frac{\cosh((n + 1)!/2)}{\sinh(n!/4)} \right\} > \text{arcsinh} 1 = : \omega$$

for every $n \geq 1$.

Let $A_n$ be the annular cover of $R$ with respect to the $c_n$. The half-collar $H_\omega(c_n)$ can be regarded as embedded in $A_n$. The value of the width $\omega = \text{arcsinh} 1$ implies that their moduli are related as $m(H_\omega(c_n)) = m(A_n)/4$. Since $\omega$ is less than
Let \( c_n \) be the boundary component of \( \eta_n \) be one of the pairs of pants in \( R \) such that \( P_n \cap R^o \) is punctured and let \( c_n \) be the boundary component of \( P_n \). Let \( \eta_n \) be a closed or non-closed curve in \( P_n \cap R^o \). When \( \eta_n \) is closed, assume that it is freely homotopic to \( c_n \) in \( R \) but not in \( R^o \). When \( \eta_n \) is not closed, assume that both end points of \( \eta_n \) are on \( c_n \), and that \( \eta_n \) is not homotopic to a subarc of \( c_n \) in \( R^o \) fixing the end points. Then the hyperbolic length of \( \eta_n \) in \( R \) is greater than \( n! \times n \).

Proof. Consider the right-angled hexagon \( D_n \), which is the symmetric half of \( P_n \). Then the length of \( \eta_n \) can be estimated by double the distance between the side of \( D_n \) where \( \eta_n \) lies and other sides not adjacent to it, which is either \( \ell(c_{n+1}) \) or \( \ell(c_n) \). Both are greater than \( n! \times n \).

This proposition gives the following estimate on the hyperbolic length of a closed geodesic on \( R^o \), which corresponds to Lemma 3 in [9].

Lemma 4. The length of a non-trivial closed geodesic \( c^o \ (\neq c_n^o) \) in \( R^o \) that intersects \( E_n^o \) is greater than \( n! \times n \).

Proof. First assume that \( c^o \) does not cross any \( c_i \) trivially, which means that no subarcs of \( c^o \) and \( c_i \) together bound a simply connected domain. Let \( m \) be the largest integer among \( i \) for which \( c^o \) has intersection with some \( P_i \). Then the subarc \( \eta_m = c^o \cap P_m \) satisfies the conditions as in Proposition 3. Since the length of \( \eta_m \) measured by the hyperbolic metric on \( R^o \) is not less than that on \( R \), it is greater than \( n! \times m \geq n! \times n \). If \( c^o \) crosses some \( c_i \) trivially, we replace the subarc of \( c^o \) with the subarc of \( c_i \), which together bounds a simply connected domain. By this replacement, the length measured in \( R \) does not increase. Hence we may reduce our argument to the case under the first assumption.

\[\S4. Quasiconformal Automorphisms of R^o\]

In this section, we complete the proof of Theorem 1. First, we notice that the distribution of the punctures makes \( R^o \) to satisfy the following property.

Lemma 5. Let \( E_n \) and \( E_n' \) be any connected components of \( R - R_n \) possibly coincident and let \( h : E_n \to E_n' \) be a conformal homeomorphism that is not the identity. Then the image of \( E_n \cap R^o \) under \( h \) is not coincident with \( E_n' \cap R^o \).

Proof. If the \( P_n \subset E_n \) and the \( P_n' \subset E_n' \) are of different type regarding puncture, there is nothing to prove. If \( P_n \) and \( P_n' \) are both punctured, we consider one of pairs of \( (n+1) \)-streams starting from the adjacent \( P_{n+1} \) and \( P_{n+1}' \). Since their turning numbers are distinct by construction, we have an index \( i > n + 1 \) such that the corresponding \( P_i \) and \( P_i' \) under \( h \) are of different type. If neither \( P_n \) nor \( P_n' \) are punctured, we consider the streams intersecting \( P_n \) and \( P_n' \) respectively. Since
their turning numbers are distinct, we again have an index $i > n$ such that $P_i$ and $P_i'$ are of different type. Note that, in the above arguments, we mainly suppose the case where $h$ maps the right-hand side of $E_n$ to the right-hand side of $E'_n$. Indeed the claim is rather easier to prove in the other case.

By this property of $R^o$ and a property of $R$ proved by Theorem 3 in [9], we have the following theorem concerning quasiconformal automorphisms of $R^o$, which is an essential ingredient for proving Theorem 1.

**Theorem 6.** Let $g^o : R^o \to R^o$ be an $n/4$-quasiconformal automorphism of the Riemann surface $R^o$ for any integer $n \geq 5$. Then, for each connected component $E^o_n$ of $R^o - R_n^o$, the restriction $g^o|_{E^o_n} : E^o_n \to R^o$ is homotopic to the inclusion map $id|_{E^o_n} : E^o_n \to R^o$.

**Proof.** Since a puncture is a removable singularity for a quasiconformal map, $g^o : R^o \to R^o$ extends to a quasiconformal automorphism $g : R \to R$ of the original surface $R$. The maximal dilatation of $g$ is bounded by $n/4 \leq n$. Then, by Theorem 3 and its proof in [9], the restriction $g|_{E_n} : E_n \to R$ is homotopic to a conformal homeomorphism $h : E_n \to E'_n$ onto a connected component $E'_n$ of $R - R_n$.

Suppose to the contrary that $g^o|_{E^o_n}$ is not homotopic to $id|_{E^o_n}$. There are two cases to be considered: $h \neq id_{E_n}$ and $h = id_{E_n}$. In the former case, the image of $E_n \cap R^o$ under $h$ is not coincident with $E'_n \cap R^o$ by Lemma 5. Hence in both cases, there exists some $c_m = \partial E_m \subset E_n$ ($m \geq n$) such that $g(c_m)$ and $c'_m := h(c_m)$ are freely homotopic in $R$ but not in $R^o$. Then we see that either the geodesic $g(c_m)^o$ freely homotopic to $g(c_m)$ in $R^o$ has intersection with $h(E_m)^o$ or the geodesic $g^{-1}(c'_m)^o$ has intersection with $E'_n$. Without loss of generality, we may assume the former case. By Lemma 2, the geodesic length $\ell(c^o_m)$ in $R^o$ is not greater than $4m!$. By Lemma 4, the geodesic length $\ell(g(c_m)^o)$ is greater than $m! \times m$. Thus the maximal dilatation of $g$ is greater than $m/4 \geq n/4$. However, this is a contradiction. □

**Corollary 7.** Every quasiconformal mapping class $[g] \in \text{MCG}(R^o)$ has an eventually trivial representative $g_0 : R^o \to R^o$.

**Proof.** For a $K$-quasiconformal automorphism $g$ of $R^o$, take an integer $n$ satisfying $n/4 > K$. Then, applying Theorem 6 to $g$, we see that $g$ is homotopic to the inclusion map on $R^o - R_n^o$. We modify $g$ to a quasiconformal automorphism $g_0$ of $R^o$ homotopic to $g$ such that $g_0$ is precisely the identity on $R^o - R_n^o$. Since $R_n^o$ is topologically finite, such a modification is always possible. □

**Proof of Theorem 1.** Let $f : R^o \to S^o$ be any quasiconformal homeomorphism of the Riemann surface $R^o$ onto another Riemann surface $S^o$ and let $[g] \in \text{MCG}(R^o)$ be any quasiconformal mapping class. By Corollary 7, there is an eventually trivial representative $g_0$ in $[g]$. Then $f \circ g \circ f^{-1}$ is homotopic to $f \circ g_0 \circ f^{-1}$. Here $f \circ g_0 \circ f^{-1}$ is the identity on $S^o - f(R_n^o)$ for some $n$, and in particular it is an asymptotically conformal automorphism of $S^o$. This implies that the action of every $[g] \in \text{MCG}(R^o)$ fixes each asymptotic equivalence class $[f]_A \in \text{AT}(R^o)$, namely, $\text{MCG}(R^o)$ acts trivially on $\text{AT}(R^o)$.

**Remark.** Note that $R^o$ as well as $R$ satisfies the property that, for every $L > 0$, the number of simple closed geodesics whose lengths are less than $L$ is finite. Actually, it is proved in [8] that if a Riemann surface does not satisfy this property, then the quasiconformal mapping class group has no common fixed point in the asymptotic
Teichmüller space and in particular it acts non-trivially. Also Theorem 6 implies that MCG$(R^2)$ consists of a countable number of elements, as does MCG$(R)$. In general, we will prove in the forthcoming paper [10] that, if MCG$(R)$ has a common fixed point in $AT(R)$, then MCG$(R)$ must be countable.

References