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ACTIONS OF POINTED HOPF ALGEBRAS WITH REDUCED PI INVARIANTS

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ABSTRACT. Let R be an H-module algebra, where H is a pointed Hopf algebra acting on R finitely of dimension N. Suppose that $L^H \neq 0$ for every nonzero H-stable left ideal of R. It is proved that if R^H satisfies a polynomial identity of degree d, then R satisfies a polynomial identity of degree dN provided at least one of the following additional conditions is fulfilled:

(1) R is semiprime and R^H is almost central in R,
(2) R is reduced.

If we also assume that R^H is central, then R satisfies the standard polynomial identity of degree $2[\sqrt{N}]$, where $[\sqrt{N}]$ is the greatest integer in $\sqrt{N}$.

1. INTRODUCTION

This paper is motivated by the following general question: if H is a finite-dimensional Hopf algebra over the field K, and R is a left H-module algebra such that the algebra of invariants R^H satisfies a polynomial identity, must R also satisfy a polynomial identity? The answer to this question is positive in many concrete situations, e.g.,

(1) when $H = K[G]$, where G is a finite group, and either $|G|^{-1} \in K$ or R is reduced (see [K1] and [K2]);
(2) when $H = K[G]^*$ (see [BC] and [BaZ]);
(3) when $H = u(L)$, where L is a finite-dimensional restricted Lie algebra of derivations of a prime ring R with $\operatorname{char} R = p > 0$ such that R^H is semiprime and the elements inducing the X-inner part of L generate a quasi-Frobenius algebra (see [K3]);
(4) when H is such that for every H-module algebra R such that R^H is nilpotent, also R is nilpotent (see [BaL]);
(5) when H is pointed and R contains an element γ such that $t \cdot \gamma = 1$, for some $0 \neq t \in \int_H^l$, the space of left integrals of H (see [BeT]).

However for actions of finite groups, where $|G|R = 0$, it is known that the answer can be negative. In an example of Bergman, there is an action of a group G of order p^2 (where p is the characteristic of K) on the algebra $R = M_2(K[x, y])$ of 2×2 matrices over a free algebra $K[x, y]$ such that R^G is commutative. Recall that in

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this example, G is generated by the inner automorphisms induced by

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.$$

Then R is a prime ring where every nonzero G-stable left (right) ideal of R contains nontrivial invariants. This shows that the assumption that every nonzero H-stable left ideal of R contains nontrivial invariants is not sufficient to obtain a positive answer to the above question. Notice that in the above example R^H contains nilpotent elements. The main goal of this paper is to present a condition, which guarantees, for a semiprime algebra R, that if R^H satisfies a PI, then R also satisfies a PI. We will show that if H is pointed and every nonzero H-stable left ideal contains a nontrivial central invariant, then R^H satisfying a PI implies that R satisfies a PI. This extends a situation considered in [C] and [CW], where actions of Hopf algebras with central invariants are studied. The PI condition for prime rings with central rings of invariants under the action of p-nilpotent groups, nilpotent Lie algebras and Lie superalgebras was also considered in [BCF] and [BG]. In the second main result, we show that if R has no nonzero nilpotent elements, then the assumption that nonzero H-stable left ideals contain nontrivial invariants is sufficient for lifting the PI property from R^H to R. Note that the most typical nontrivial examples of pointed Hopf algebras, which are neither group algebras nor universal enveloping algebras, are given by Lusztig's finite-dimensional Hopf algebras $\mathfrak{u}_q(\mathfrak{g})$ arising from quantized enveloping algebras at roots of unity for semisimple Lie algebras $\mathfrak{g}$ (see [AS]).

Throughout the paper K will be a field, H a pointed Hopf algebra over K, and R an algebra over K. We let $\Delta : H \to H \otimes H$ be the comultiplication of H, $\epsilon : H \to K$ is the counit of H, and $S : H \to H$ the antipode of H. We say that R is a left H-module algebra if R is a left H-module such that $h \cdot ab = \sum (h_1 \cdot a)(h_2 \cdot b)$ and $h \cdot 1_R = \epsilon(h) 1_R$, where $h \in H$, $\Delta(h) = \sum h_1 \otimes h_2$, $a, b \in R$. If A is a subset of R such that $h \cdot A \subseteq A$, for all $h \in H$, then we say that A is H-stable. When R is a left H-module algebra one can consider the smash product $R\#H$. As a vector space $R\#H$ is $R \otimes H$. The elements of $R\#H$ can be written as finite sums $\sum a_h h$, where $h \in H$ and $a_h \in R$. Then the multiplication in $R\#H$ is determined by the formula $(ah)(bl) = \sum a(h_1 \cdot b)h_2 l$, for all $a, b \in R$ and $h, l \in H$. The ring of invariants R^H is defined as $\{r \in R \mid h \cdot r = \epsilon(h) r, \text{ for all } h \in H\}$.

If R is a left H-module algebra, then R becomes a left $R\#H$-module using the left action $(ah).r = a(h \cdot r)$, where $a, r \in R$ and $h \in H$. Then the commuting ring $\mathrm{End}_{R\#H}(R)$ is isomorphic to R^H and the submodules of R over $R\#H$ are precisely left H-stable ideals of R.

If M is a left H-module, then there is a homomorphism $\pi : H \to \mathrm{End}_K(M)$ defined by $\pi(h)(m) = hm$, for all $h \in H$ and $m \in M$. If $\dim_K \pi(H) = N < \infty$, then we say that H acts **finitely** of dimension N. Clearly $\dim_K \pi(H) \leq \dim_K H$, so if H is finite dimensional, then H acts finitely on each H-module.

If R is semiprime, we let $Q = Q(R)$ denote the symmetric Martindale quotient ring. Its center, known as the extended centroid of R, we denote by C. The following properties of Q, when R is acted on by a Hopf algebra, are proved in Propositions 1, 2 and 5 of [GH].

Lemma 1. *Let R be a semiprime H-module algebra such that the H-action on R extends to an H-action on Q and any nonzero H-stable ideal of R contains nontrivial invariants. Then*

1. *the ring $C^H = C \cap Q^H$ is von Neumann regular and selfinjective.*
2. *For any nonempty subset X of Q there exists an idempotent $\widehat{e}_X \in C^H$ such that $\mathrm{ann}_{C^H}(X) = (1 - \widehat{e}_X)C^H$. If X is an injective C^H-submodule of Q, then there exists $x \in X$ such that $\mathrm{ann}_{C^H}(X) = \mathrm{ann}_{C^H}(x) = (1 - \widehat{e}_x)C^H$.*
3. *If $L \subseteq Q$ is an H-stable subalgebra of Q which is injective as a C^H-module, then L^H is also injective as a C^H-module.*
4. *If a nonempty subset $S \subseteq C^H \setminus \{0\}$ is closed under a multiplication, then the localization Q_S of Q at S is semiprime and $Z(Q_S) = C_S$.*
5. *If H acts finitely on Q and $S = C^H \setminus M$, where M is a maximal ideal of C^H, then the H-action on Q extends to an H-action on Q_S and $(Q^H)_S = (Q_S)^H$, where $(C^H)_S = (C_S)^H = C_S \cap (Q_S)^H$ is a field contained in the center of Q_S.*

It is also known ([GH, Proposition 2]) that under the assumptions of Lemma 1, the ring Q is nonsingular and injective as a C^H-module. This immediately implies that if $\varphi : M \to N$ is an onto C^H-module map, where $0 \neq N \subseteq Q$ and M is injective, then N is also an injective C^H-module. In particular, any principal left ideal Qq of Q is nonsingular and injective as a C^H-module. Hence each finitely generated left ideal of Q (finitely generated as a left Q-module) is also injective over C^H.

An important role will be played by the following result of Bergen, Cohen and Fischman on irreducible actions of Hopf algebras (see [BCF], Theorem 2.2).

Theorem 2. *Let A be a left H-module algebra such that $A\#H$ acts irreducibly on A, A has a finite left Goldie rank, and H acts finitely of dimension N on A. Then $[A : A^H]_r \leq N$, where $[A : A^H]_r$ is the dimension of A as a right vector space over the division ring A^H.*

2. Main results

Throughout this section H will be a pointed Hopf algebra over a field K. Recall that a ring R is said to be **reduced** if it does not contain nonzero nilpotent elements. It is well known that if $r_1, r_2, \ldots, r_n$ are elements of a reduced ring R such that $r_1 r_2 \ldots r_n = 0$, then $r_{f(1)} r_{f(2)} \ldots r_{f(n)} = 0$ for any bijection $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

If R is a left H-module algebra with center $Z(R)$, we say that the ring of invariants is **almost central** in R if $L^H \cap Z(R) \neq 0$ for every nonzero H-stable left ideal L of R. Notice that if R is semiprime and R^H is almost central in R, then R^H is reduced. Indeed, suppose there exists $0 \neq a \in R^H$ such that $a^2 = 0$. The left ideal Ra is H-stable, so one can find a nonzero element $ra \in (Ra)^H \cap Z(R)$. Then $ara = a(ra) = (ra)a = 0$ and thus $(ra)^2 = 0$, which is impossible since $Z(R)$ is reduced.

Our first main goal is to prove the following.

Theorem 3. *Let R be a semiprime K-algebra with center Z and suppose R is a left H-module algebra, where H is a pointed Hopf algebra acting on R finitely of dimension N. If the subalgebra of invariants R^H is almost central in R, and R^H*

satisfies a polynomial identity of degree d, then R satisfies the standard polynomial identity of degree dN. If in addition $R^H \subseteq Z$*, then R satisfies the standard polynomial identity of degree* $2[\sqrt{N}]$*, where* $[\sqrt{N}]$ *is the greatest integer in* $\sqrt{N}$*.*

Our next result concerns the situation when the algebra R is reduced.

Theorem 4. *Let R be a reduced H-module K-algebra, where H is a pointed Hopf algebra acting on R finitely of dimension N. Suppose that $L^H \neq 0$ for every nonzero H-stable left ideal L of R. If R^H satisfies a polynomial identity of degree d, then R satisfies the standard polynomial identity of degree dN.*

The proofs require some preparation. Recall that a module M is called uniform if the intersection of any two nonzero submodules is nonzero. We start with the following general observation.

Lemma 5. *Let M be an irreducible (uniform) left $R\#H$-module and suppose that H acts finitely on M. Then M has finite length (finite Goldie rank) as a left R-module.*

Proof. Let M be an arbitrary (not necessarily irreducible) left $R\#H$-module. Let $\pi : H \to \mathrm{End}_K(M)$ be a homomorphism of algebras induced by the action of H on M. By using the Taft-Wilson Theorem (see [M1, Theorem 5.4.1]) we can decompose H as a finite union $\bigcup_{i=1}^{N} H_i$ of an increasing chain of subspaces $\{H_i\}$ such that

(i) $\pi(H_i) = \pi(H_{i-1}) + K \cdot \pi(h_i)$, where $h_1 = 1_H$ and $h_i \in H$ for $2 \leq i \leq N$,

(ii) $\Delta(h_i) \in \sigma \otimes h_i + h_i \otimes \tau + H_{i-1} \otimes H_{i-1}$, where $\sigma, \tau \in G$ and $2 \leq i \leq N$.

Moreover, we can assume in (ii) that if $h_i \neq \tau$ (that is, if h_i is not a group-like element), then $\tau \in H_{i-1}$.

If A is an R-submodule of M and $j \geq 1$, let

$$A_{(j)} = \{m \in M \mid h_1 m, \ldots, h_j m \in A\}.$$

If $h_i \in H$ satisfies (ii), then

$$h_i(rm) = \sigma(r) h_i m + (h_i \cdot r)\tau m + \sum (h_{i1} \cdot r) h_{i2} m,$$

where $r \in R$, $m \in M$ and $h_{i1}, h_{i2} \in H_{i-1}$. Thus an easy induction argument shows that $A_{(j)}$ is also an R-submodule of M. Since $\{\pi(h_1), \ldots, \pi(h_N)\}$ is a K-basis of $\pi(H)$, we obtain immediately that $hA_{(N)} \subseteq A_{(N)}$, for all $h \in H$; thus $A_{(N)}$ is an $R\#H$-submodule. In fact $A_{(N)}$ is the largest $R\#H$-submodule contained in A. Now if $\{A_\alpha\}$ is a chain of R-submodules of M, each of which contains no nonzero $R\#H$-submodule, then $\bigcup A_\alpha$ also contains no nonzero $R\#H$-submodule. Indeed, if $B \subseteq \bigcup A_\alpha$ is a nonzero $R\#H$-submodule and $0 \neq b \in B$, then $\{h_1 b, \ldots, h_N b\} \subseteq A_{\alpha_0}$ for some α_0. Therefore $(R\#H)b \subseteq A_{\alpha_0}$, and so A_{α_0} contains a nonzero $R\#H$-submodule, a contradiction. Consequently, by Zorn's Lemma, there exists an R-submodule $\widehat{A}$ of M which is maximal with respect to containing no nonzero $R\#H$-submodule. We can now consider the chain of R-submodules

$$M \supseteq \widehat{A} = \widehat{A}_{(1)} \supseteq \widehat{A}_{(2)} \supseteq \cdots \supseteq \widehat{A}_{(N-1)} \supseteq \widehat{A}_{(N)} = 0.$$

Now suppose that M is irreducible (resp. uniform) as a left $R\#H$-module. Since $B_{(N)} \neq 0$, for any R-submodule B properly containing $\widehat{A}$, we see that the factor R-module $M/\widehat{A}$ is irreducible (resp. uniform). If $1 \leq i \leq N-1$, then we can consider the maps

$$\varphi_i : \widehat{A}_{(i)} \to M/\widehat{A}$$

defined as $\varphi_i(a) = h_{i+1}a + \widehat{A}$, for all $a \in \widehat{A}_{(i)}$. By (ii) there exist $\sigma, \tau \in G$ such that $\Delta(h_{i+1}) - \sigma \otimes h_{i+1} - h_{i+1} \otimes \tau \in H_i \otimes H_i$. Hence if $r \in R$ and $a \in \widehat{A}_{(i)}$, then since $H_i a \subseteq \widehat{A}$, we have

$$\begin{aligned}\varphi_i(ra) &= h_{i+1}(ra) + \widehat{A} = (h_{i+1}r)a + \widehat{A} = \sigma(r)h_{i+1}a + (h_{i+1} \cdot r)\tau a + \widehat{A} \\ &= \sigma(r)h_{i+1}a + \widehat{A} = \sigma(r)\varphi_i(a).\end{aligned}$$

It is easy to see that $\ker \varphi_i = \widehat{A}_{(i+1)}$. Therefore each φ_i induces an embedding of the lattice of R-submodules of $\widehat{A}_{(i)}/\widehat{A}_{(i+1)}$ into the lattice of R-submodules of $M/\widehat{A}$. In our situation the R-module $M/\widehat{A}$ is irreducible, so each $\widehat{A}_{(i)}/\widehat{A}_{(i+1)}$ is either the zero module or irreducible (resp. uniform) as an R-module. Therefore M has a finite length (resp. finite Goldie rank), not exceeding N, as an R-module. □

Let $Q = Q(R)$ be the symmetric Martindale quotient ring of R. From the result of Montgomery (see [M2, Corollary 3.5]) it follows that when H is pointed, the H-action on R can be extended to an H-action on Q. Moreover, it is easy to see that if H acts finitely on R, then every essential ideal of R contains an H-stable ideal which is also essential in R (see [GH, Lemma 9]). As a consequence of some basic properties of Q, we obtain the following.

Lemma 6. *Let H be a pointed Hopf algebra and let R be a semiprime left H-module algebra such that R^H is reduced and $L^H \neq 0$ for every nonzero H-stable left ideal L of R. Suppose R^H satisfies a multilinear identity of degree d and H acts on R finitely of dimension N. Then*

(1) *$L^H \neq 0$, for every nonzero H-stable left ideal L of Q.*
(2) *$Z(R^H) \subseteq Z(Q^H)$.*
(3) *Q^H is reduced and satisfies the same multilinear identity as R^H.*
(4) *H acts finitely of dimension N on Q.*
(5) *If R^H is almost central in R, then Q^H is almost cental in Q.*
(6) *If in addition $R^H \subseteq Z$, then $Q^H \subseteq C$.*

Proof. For (1), if L is a nonzero H-stable left ideal of Q, then $\widehat{L} = L \cap R$ is a nonzero H-stable left ideal of R. By assumption $\widehat{L}^H \neq 0$, so $L^H \neq 0$.

Before proving (2), notice that if I is an H-stable essential ideal of R, then $\mathrm{l.ann}_R(I^H \cap Z(R^H)) = 0$. Indeed, it is clear that $L = \mathrm{l.ann}_R(I^H \cap Z(R^H))$ is an H-stable left ideal of R. If $L \neq 0$, then $0 \neq I \cdot L \subseteq I \cap L$ and since R^H is reduced, we obtain that $(I \cap L)^H$ is a two-sided ideal of R^H. By assumption R^H satisfies a PI, so $0 \neq Z((I \cap L)^H) \subseteq Z(R^H)$. Thus one can choose a nonzero element $c \in (I \cap L)^H \cap Z(R^H)$. But then $c^2 \in L \cdot (I^H \cap Z(R)) = 0$, which is impossible, since R^H is reduced. This also implies that $\mathrm{l.ann}_Q(I^H \cap Z(R^H)) = 0$. By using an easy induction argument, we obtain that for any $d \geq 1$,

$$\mathrm{l.ann}_Q((I^H \cap Z(R^H))^d) = 0. \tag{2.1}$$

This immediately implies that $Z(R^H) \subseteq Z(Q^H)$. To see this, take a nonzero $q \in Q^H$ and an essential H-stable ideal J of R satisfying $Jq \subseteq R$ and $qJ \subseteq R$. Then, for any $x \in J^H$ and $c \in Z(R^H)$, we have $qx \in R^H$ and

$$0 = [qx, c] = [q, c]x + q[x, c] = [q, c]x.$$

Hence $[q, c]J^H = 0$ and by (2.1), $[q, c] = 0$. Consequently, $Z(R^H) \subseteq Z(Q^H)$. This ends the proof of (2).

For the first part of (3), take $q \in Q^H$ such that $q^2 = 0$, and I an essential H-stable ideal of R satisfying $qI \cup Iq \subseteq R$. Then $(I^H \cap Z(R^H))q \subseteq R$ and using (2) we obtain $((I^H \cap Z(R^H))q)^2 = q^2(I^H \cap Z(R^H))^2 = 0$. Since R^H is reduced, $(I^H \cap Z(R^H))q = 0$ and (2.1) forces that $q = 0$. Therefore Q^H is reduced.

Now let $f(x_1, x_2, \dots, x_d) = \sum_{\sigma \in S_d} a_\sigma x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(d)}$ be a multilinear polynomial such that the identity $f(x_1, x_2, \dots, x_d) = 0$ is satisfied by R^H. Take $q_1, q_2, \dots, q_d \in Q^H$ and an H-stable essential ideal I of R such that $q_j I \subseteq R$ for $j = 1, 2, \dots, d$. Then for all $c_1, c_2, \dots, c_d \in I^H \cap Z(R^H)$ we have $c_i q_i \in R^H$, so by using (2),

$$0 = f(c_1 q_1, c_2 q_2, \dots, c_d q_d) = f(q_1, q_2, \dots, q_d) c_1 c_2 \dots c_d.$$

This means that $f(q_1, q_2, \dots, q_d) \in \operatorname{l.ann}_Q((I^H \cap Z)^d) = 0$. Thus the identity

$$f(x_1, x_2, \dots, x_d) = 0$$

is satisfied also by Q^H. This proves (3).

For (4), let $\widehat{\pi} : H \to \operatorname{End}_K(Q)$ be the natural K-algebra homomorphism, corresponding to the action of H on Q. We need to show that $\ker \pi = \ker \widehat{\pi}$. The inclusion $\ker \pi \supseteq \ker \widehat{\pi}$ is clear. Suppose $h \in \ker \pi$. Take $q \in Q$ and I an essential H-stable ideal of R such that $qI \subseteq R$. Since $\widehat{\pi}(h)$ is an R^H-bimodule map, we obtain that

$$\widehat{\pi}(h)(q)a = \widehat{\pi}(h)(qa) = \pi(h)(qa) = 0$$

for any $a \in I^H$. Hence $\widehat{\pi}(h)(q) \in \operatorname{l.ann}_Q(I^H) \subseteq \operatorname{l.ann}_Q(I^H \cap Z(R^H)) = 0$. Thus $h \in \ker \widehat{\pi}$ and consequently $\ker \widehat{\pi} = \ker \pi$. Thus $\dim_K \widehat{\pi}(H) = \dim_K \pi(H)$.

For (5), if L is a nonzero H-stable left ideal of Q, then $\widehat{L} = L \cap R$ is a nonzero H-stable left ideal of R. Since $Z(R) \subseteq C$ and $\widehat{L}^H \cap Z(R) \neq 0$, we obtain that $L^H \cap C \neq 0$. Thus Q^H is almost central in Q.

For (6), take $q \in Q^H$ and an H-stable essential ideal I of R such that $qI \subseteq R$. If $c \in I^H$, then $qc \in R^H \subseteq Z$ and hence

$$(qr - rq)c = (qr)c - r(qc) = (qc)r - (qc)r = 0,$$

for any $r \in R$. Thus $rq - qr \in \operatorname{l.ann}_Q(I^H \cap Z(R^H)) = 0$. Therefore q centralizes R, so $q \in C$. □

We are now ready to prove the first main result of the paper.

Proof of Theorem 3. By Lemma 6, all assumptions on R can be lifted to Q. Let $h_1, h_2, \dots, h_N \in H$ be such that $\{\pi(h_1), \pi(h_2), \dots, \pi(h_N)\}$ is a basis for $\pi(H) \subseteq \operatorname{End}_K(Q)$. Notice that for any $q \in Q$ the left ideal $L = \sum_{i=1}^N Q(h_i \cdot q)$ is H-stable. By applying the remarks after Lemma 1, we see that any finitely generated (as a left Q-module) left ideal of Q is contained in an H-stable finitely generated left ideal which is also injective as a C^H-module.

Let M be a maximal ideal of C^H and $\eta_M : Q \to Q_M$ be a natural ring homomorphism, where Q_M is the localization of Q at $S = C^H \setminus M$. By Lemma 1 it follows that Q_M is semiprime and Lemma 6 shows that $(Q_M)^H = (Q^H)_M$ satisfies a multilinear identity of degree d. We claim that $(Q^H)_M$ is almost central in Q_M. Take a nonzero H-stable left ideal T of Q_M and choose a finitely generated H-stable left ideal L of Q such that $0 \neq \eta_M(L) \subseteq T$. Then L is injective as a left C^H-module and by Lemma 1(3), L^H is also injective as a left C^H-module. Since C is injective over C^H, the intersection $L^H \cap C$ is injective as a C^H-module. By Lemma 1(2), there exist $x \in L^H \cap C$ and an idempotent $\widehat{e}_x \in C^H$ such that

$\mathrm{ann}_{C^H}(L^H \cap C) = \mathrm{ann}_{C^H}(x) = (1-\widehat{e}_x)C^H$. We claim that $(1-\widehat{e}_x)L = 0$. If not, then $(1-\widehat{e}_x)L$ is a nonzero H-stable left ideal of Q. Since Q^H is almost central in Q, we can choose a nonzero $c \in ((1-\widehat{e}_x)L)^H \cap C$. Then $c \in L^H \cap C$ and $c = (1-\widehat{e}_x)c \in (1-\widehat{e}_x)C^H = \mathrm{ann}_{C^H}(L^H \cap C)$. Therefore $c^2 = 0$, which is impossible because C is reduced. This proves the claim. Since $\eta_M(L) \neq 0$, $1-\widehat{e}_x \in M$. Hence $\mathrm{ann}_{C^H}(x) \subseteq M$ and thus $0 \neq \eta_M(x) \in T^H \cap C_M$. Therefore $(Q_M)^H$ is almost central in Q_M.

On the other hand by Lemma 1(5), $(Q_M)^H \cap C_M$ is a field, so Q_M does not contain proper H-stable left ideals. Thus Q_M is an irreducible left $Q_M\#H$-module. By Lemma 5, Q_M has finite length as a left Q_M-module, so in particular Q_M has finite left Goldie rank. We are now in a position to apply Theorem 2. It asserts that $(Q_M)^H$ is a division ring and $[Q_M : (Q_M)^H]_r = n \leq N$. If we let A_M denote the annihilator ideal $\{w \in Q_M\#H \mid wQ_M = 0\}$, then $Q_M\#H/A_M \simeq M_n((Q_M)^H)$. The division algebra $(Q_M)^H$ satisfies a polynomial identity of degree d, so $M_n((Q_M)^H)$ satisfies the standard polynomial identity s_{dn} of degree $dn \leq dN$. Since Q_M is semiprime we have an embedding $Q_M \hookrightarrow Q_M\#H/A_M$. Thus for any maximal ideal M of C^H, the localization Q_M satisfies the standard polynomial identity s_{dN}. The fact that C^H is von Neumann regular implies immediately the existence of an embedding $Q \hookrightarrow \prod_M Q_M$, where the product is taken over all maximal ideals of C^H. Therefore Q satisfies s_{dN}.

If we additionally assume that $R^H \subseteq Z$, then by Lemma 6(6), $Q^H \subseteq C$. Thus for a given maximal ideal M of C^H, Q_M is a semisimple finite-dimensional algebra containing a central subfield $(C_M)^H$ such that $\dim_{(C_M)^H} Q_M \leq N$. Therefore, the Amitsur-Levitzki Theorem asserts that Q_M satisfies the standard polynomial identity of degree $2[\sqrt{N}]$. As a result, if the invariants R^H are central in R, then R satisfies $s_{2[\sqrt{N}]}$, thereby concluding the proof. □

Proof of Theorem 4. Let us first consider the special case where R^H is a domain. Then, by Posner's Theorem, R^H is a Goldie ring. Furthermore, if we put $T = Z(R^H) \setminus \{0\}$, then the localization $T^{-1}R^H$ is a division algebra with center $\mathcal{Z} = T^{-1}Z(R^H)$ and $\dim_{\mathcal{Z}} T^{-1}R^H \leq \left(\frac{d}{2}\right)^2$. It is easy to see that every nonzero element $z \in Z(R^H)$ is regular in R. In fact, since R is reduced, $J = \mathrm{l.ann}_R(z) = \mathrm{r.ann}_R(z)$ is a two-sided H-stable ideal of R. If J is nonzero, then $Z(J^H) \neq 0$ (because R^H satisfies a PI), and clearly $Z(J^H) \subseteq Z(R^H)$. But $Z(J^H)z = 0$, and this contradicts our assumption that R^H is a domain. We claim that the subset T satisfies the left Ore condition in R. To see this, note that by Lemma 5, R has a finite left Goldie rank. Furthermore R, as a reduced ring, certainly has zero singular ideal. Thus R is left Goldie. Now it is enough to show that any essential left ideal of R intersects T nontrivially. Since the group $G = G(H)$ of group-like elements acts finitely, we need only consider essential left ideals which are G-stable. Let L be a G-stable essential left ideal of R and, using the notation in Lemma 5, let $L_{(j)} = \{x \in L \mid h_1 \cdot x, \ldots, h_j \cdot x \in L\}$. We will show by induction that $L_{(j)}$ is essential for all $j \geq 1$. To see this, let I be any nonzero left ideal of R. Given $0 \neq a \in L_{(j-1)} \cap I$, the left ideal $E = \bigcap_{i \leq j} \{r \in R \mid r(h_i \cdot a) \in L\}$ is essential. Since R is nonsingular, we can choose $r \in \bigcap_{\sigma \in G} E^\sigma$ with $ra \neq 0$. Then

$$h_j \cdot (ra) = \sigma(r)(h_j \cdot a) + (h_j \cdot r)\tau(a) + \sum (h_{j1} \cdot r)(h_{j2} \cdot a) \in L.$$

Thus $0 \neq ra \in L_{(j)}$, so $L_{(j)}$ is essential. In particular $\widehat{L} = L_{(N)}$ is nonzero and H-stable. By assumption $\widehat{L}^H$ is a nonzero left ideal of R^H. But R^H is a PI domain, so any nonzero left ideal intersects T nontrivially. This proves the claim.

Notice that $T^{-1}R$ is in a natural way a left H-module algebra and $(T^{-1}R)^H = T^{-1}R^H$ is a division algebra satisfying a polynomial identity of degree d. It is also clear that $T^{-1}R$ has no proper left H-stable ideals, so $T^{-1}R$ becomes an irreducible left $(T^{-1}R)\#H$-module. Applying the same argument as in the proof of our previous theorem, we obtain that $T^{-1}R$ satisfies s_{dN}. Therefore R also satisfies the standard identity s_{dN}.

For the general case, since R is reduced, the symmetric Martindale quotient ring Q is also reduced. Similarly, as in Theorem 3, let us consider a maximal ideal M of C^H and the canonical map $\eta_M : Q \to Q_M$. We claim that $(Q^H)_M$ is a domain. To this end, let $a, b \in Q^H$ be such that $ab = 0$, and let $e_a, e_b \in C^H$ be idempotents such that $\mathrm{ann}_Q(QaQ) = (1-e_a)Q$ and $\mathrm{ann}_Q(QbQ) = (1-e_b)Q$. Since Q is reduced, $\mathrm{l.ann}_Q(a) = \mathrm{r.ann}_Q(a) = \mathrm{ann}_Q(QaQ)$. Thus $\mathrm{r.ann}_Q(a) = (1-e_a)Q$. Hence $e_a b = 0$. On the other hand $\mathrm{l.ann}_Q(b) = (1-e_b)Q$, so there exists $x \in Q$ satisfying $e_a = (1-e_b)x$. Now it is clear that $e_a e_b = 0$, and thus either $e_a \in M$ or $e_b \in M$. This immediately implies that either $\eta_M(a) = 0$ or $\eta_M(b) = 0$. Therefore $(Q^H)_M$ is a domain, as claimed. Notice that the ring Q_M is reduced. Indeed, if $q \in Q$ and $c \in C^H \setminus M$ are such that $cq^2 = 0$, then $(cq)^2 = 0$ and $cq = 0$, since Q is reduced. Consequently $\eta_M(q) = 0$. As a result Q_M is a reduced left H-module algebra and its subalgebra of invariants $(Q_M)^H = (Q^H)_M$ is a domain satisfying a polynomial identity of degree d. By the previous paragraph Q_M satisfies s_{dN}. Since this holds for any maximal ideal M of C^H, the ring Q satisfies s_{dN}. □

We close the paper with a remark concerning actions on reduced algebras. We see that the result of Kharchenko for group actions (mentioned in the introduction) is a direct consequence of his fundamental result on the existence of fixed elements and Theorem 4. Moreover, Beidar and Grzeszczuk proved in [BeG] an analogous result on the existence of nontrivial constants for actions of Lie algebras. Finally, Theorem 4 now provides us with a common proof of the following.

Corollary 7. *Let R be a reduced algebra. Then*

(1) (*cf.* [K2]) *if R is acted on by a finite group G and R^G satisfies a PI of degree d, then R satisfies a PI of degree $d|G|$.*
(2) *If R is acted finitely on by a finite-dimensional Lie algebra L and R^L satisfies a PI of degree d, then R satisfies a PI of degree dN, where N is the dimension of the action.*

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