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MATRIX COEFFICIENTS AND COADJOINT ORBITS OF COMPACT LIE GROUPS

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ABSTRACT. Let G be a compact Lie group. We use Weyl functional calculus (Anderson, 1969) and symplectic convexity theorems to determine the support and singular support of the operator-valued Fourier transform of the product of the j-function and the pull-back of an arbitrary unitary irreducible representation of G to the Lie algebra, strengthening and generalizing the results of Cazzaniga, 1992. We obtain as a consequence a new demonstration of the Kirillov correspondence for compact Lie groups.

1. Introduction

Let G be a tame unimodular Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the vector space dual of $\mathfrak{g}$. The group G operates on $\mathfrak{g}^*$ by the coadjoint action yielding, according to the Kirillov-Kostant philosophy, a consequent parametrization of the unitary irreducible representations of G by orbits satisfying a certain integrality condition, equivalent to the following character formula: let π be a finite-dimensional unitary irreducible representation of G related to the coadjoint orbit $\mathcal{O}$ – for almost all π in the reduced dual of G, as an equality of distributions,

$$j(X)\operatorname{Tr}\pi(\exp(X)) = \int_{\mathcal{O}} e^{i\eta(X)}\, d\mu_{\mathcal{O}}(\eta) \tag{1.1}$$

for all $X \in \mathfrak{g}$ in a sufficiently small neighbourhood of 0, where $\mu_{\mathcal{O}}$ is a Liouville measure on $\mathcal{O}$ and the G-invariant function j is the analytic square root of the jacobian of the exponential map, $j(0) = 1$.

When G is compact, (1.1) follows from the Weyl character formula and a well-known result [7, Theorem 2] of Harish-Chandra (see [15] or [12] for an exposition). It holds as an identity over all of $\mathfrak{g}$, and a unitary irreducible representation of highest weight λ corresponds to the orbit though $\lambda + \delta$, where δ is half the sum of the positive roots.

This was originally proved by Kirillov in [10], where the validity of formula (1.1) for nilpotent groups was also demonstrated and its universality conjectured. It has since been verified for many other classes of Lie groups, notable contributions being [16] and [9].

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It is remarkable in particular that the support of the Fourier tranform of the product $j(X)\mathrm{Tr}\,\pi(\exp(X))$ is a single integral coadjoint orbit. When $G = SU(2)$, on extending the Kirillov formula to the matrix coefficients, Cazzaniga [4] makes the interesting observation that the support and singular support of the Fourier transform of an arbitrary trigonometric polynomial multiplied by j also have descriptions in terms of the coadjoint orbit picture. We extend his results to the case of an arbitrary semisimple compact Lie group.

Fixing a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$, let $\mathcal{O}_\lambda$ be the coadjoint orbit through $\lambda \in \mathfrak{t}^*$ and let μ_λ be the Liouville measure on $\mathcal{O}_\lambda$; choose an ordered basis $x = (x_1, \dots, x_n)$ of $\mathfrak{g}$ which is orthonormal with respect to the Killing form and let W be the Weyl group. We have the following

Theorem 1.1. *Let G be a compact, semisimple, connected Lie group and let π be a unitary irreducible representation of G with highest weight λ. The operator-valued Fourier transform of $j(X)\pi(\exp(X))$ $(X \in \mathfrak{g})$ has support contained in the convex hull of $\mathcal{O}_{\lambda+\delta}$ and singular support equal to $\bigcup_{w\in W} \mathcal{O}_{\lambda+w\cdot\delta}$; furthermore, its entries are, in the sense of distributions, polynomial in the differential operators $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, x\cdot\frac{\partial}{\partial x}$ applied to the measure $\mu_\delta * I_\lambda$, where I_λ is the canonical measure on the moment set of π.*

Here the "moment set" of a Lie group representation is the image of the symplectic geometry moment map of the Hamiltonian action of G ([18, Proposition 1.1]) on the projectivisation of its representation space, which is afforded a symplectic structure [3, p. 345].

We remark that in [5] the first author and N.J. Wildberger produce character formulae generalizing (1.1) by wrapping central distributions supported on $\bigcup_{w\in W} \mathcal{O}_{\lambda+w\cdot\delta}$.

It follows from the arguments of [2, Lemme 15] (and in more generality in [13, Proposition VIII.1.30(i)]) that the set of extremal points of the convex hull of $\mathcal{O}_{\lambda+\delta}$ is the orbit $\mathcal{O}_{\lambda+\delta}$. In view of this, the second statement in the above theorem can be regarded as a new demonstration of the Kirillov correspondence for compact Lie groups.

By means of a new estimate for the singular support of the j-modified Fourier transform of the pull-back character, our results also yield a short route to the Kirillov formula (1.1) from the work of Harish-Chandra, independent of the Weyl character formula. Details will appear in the second author's Ph.D. thesis.

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2. Proof of the theorem

With $x = (x_1, \dots, x_n)$ a basis of the Lie algebra $\mathfrak{g}$ as above, write $X = \xi \cdot x$ $(\xi \in \mathbb{R}^n)$ for arbitrary $X \in \mathfrak{g}$ and note that

$$\pi(\exp(X)) = e^{\xi\cdot\, d\pi(x)}$$

where $d\pi(x) := (d\pi(x_j))_j$.

Since the n-tuple $d\pi(x)$ consists entirely of skew-hermitian matrices, the Fourier transform $(e^{\xi\cdot\, d\pi(x)})^\wedge$ is given by Nelson's formula [14] which in the notation of [8] is

$$(e^{\xi\cdot d\pi(x)})\widehat{}= \sum_{k=0}^{d_\lambda} \sum_{j=0}^{d_\lambda-k-1} \sum_{m=0}^{j} (-1)^k A^{(n)}_{j,m} (d\pi(x)\cdot \frac{\partial}{\partial x})^k$$
$$\cdot\, \phi_{n-j-k-1}(d\pi(x)\cdot \frac{\partial}{\partial x})(\frac{\partial}{\partial x}\cdot x)^m \mu_{d\pi(x)}$$

where

$$A^{(n)}_{j,m} := (-1)^m \binom{j}{m} \frac{i^{-n+j+1}}{(n-1-j+m)!},$$

$d_\lambda := \deg \pi$, ϕ_j is the sum of the principal minors of order j of the matrix $d\pi(x)\cdot \frac{\partial}{\partial x} := \sum_j d\pi(x_j)\frac{\partial}{\partial x_j}$ and $\mu_{d\pi(x)} := \nu\circ\Psi_\pi^{-1}$, ν being a unitarily invariant probability measure on $\Omega := \{z \in \mathbb{C}^n : |z| = 1\}$ and

$$\Psi_\pi(u) := \frac{1}{i}(\langle d\pi(x_1)u,u\rangle,\ldots,\langle d\pi(x_n)u,u\rangle)$$

for $u \in \Omega$, where $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{C}^n$, is the moment map of π [18].

By commutation relations, $\frac{\partial}{\partial x}\cdot x = x\cdot\frac{\partial}{\partial x} + nI$, where I is the identity operator, and since the Fourier transform of the j-function is μ_δ (from (1.1) with $\lambda = 0$) the second statement follows.

From the results [2] and [18], when the set $\{\lambda - w\cdot\lambda : w \in W\}$ contains no roots, the moment set I_λ is the convex hull of $\mathcal{O}_\lambda$, and furthermore the projection of I_λ onto $\mathfrak{t}^*$ is the convex hull of the Weyl orbit of λ. Similarly, by Kostant's convexity theorem [11], the projection of $\mathcal{O}_\delta$ onto $\mathfrak{t}^*$ is the convex hull of the Weyl orbit of δ. Hence,

$$\operatorname{supp} \mu_\delta * I_\lambda \subseteq \operatorname{conv}\, \mathcal{O}_{\lambda+\delta}$$

and

$$\operatorname{singsupp} \mu_\delta * I_\lambda = \bigcup_{w\in W} \mathcal{O}_{\lambda + w\cdot\delta}.$$

On the other hand, when the set $\{\lambda - w\cdot\lambda : w \in W\}$ contains a root, or equivalently, a simple root, I_λ is properly contained in the convex hull of $\mathcal{O}_\lambda$, but the extremal set of I_λ includes $W\cdot\lambda$, and the line joining $w'\cdot\lambda$ and $w''\cdot\lambda$ $(w', w'' \in W)$ is contained in I_λ whenever the difference $w'\cdot\lambda - w''\cdot\lambda$ is not a root; in fact the whole convex hull of the set of weights for which the pairwise difference is not a root is contained in I_λ (see [17, Lemma 7.1]). Hence the theorem follows as before, making use of the fact that the intersection of the image of the moment map with the positive Weyl chamber is a convex polytope [6].

3. The example of $SU(2)$

We identify the Lie algebra $\mathfrak{su}(2)$ with $\mathfrak{su}(2)^*$ by the trace form $X \longleftrightarrow -\frac{1}{2}\mathrm{Tr}(X\cdot)$. Noting that $-\frac{1}{2}\mathrm{Tr}(X^2) = \det X$, we have by invariance that the coadjoint orbits of $SU(2)$ are spheres. We write B_n for the the solid ball of radius n in $\mathbb{R}^3 \approx \mathfrak{su}(2)$.

The irreducible representations of $SU(2)$ are parameterized by the set $\mathbb{N}$. The moment set I_n is the single point $\{0\}$ when $n = 0$, the sphere ∂B_1 when $n = 1$ and the ball B_n when $n > 1$ [18, Proposition 4.1]. For π a unitary irreducible representation of highest weight n, let T_n denote the the Fourier transform of $j(X)\pi(\exp(X)$. Let δ_{B_n} be the Dirac measure of B_n and let χ_{B_n} be the characteristic function of

B_n. We note that $\frac{\partial}{\partial\nu}\chi_{B_n} = \delta_{B_n}$, where $\frac{\partial}{\partial\nu}$ is the derivative in the radial direction. In this context, Theorem 1.1 states that

$$\operatorname{supp} T_n \subseteq B_{n+1}$$

and

$$\operatorname{singsupp} T_n = \partial B_{n-1} \cup \partial B_{n+1},$$

and that for the right choice of basis for $\mathfrak{su}(2)$, the entries of T_n are polynomials in the derivatives in the root, toral and radial directions applied to the measure $\chi_{B_1} * \chi_{B_n}$.

This is slightly more precise than [4], where it is shown that $\operatorname{supp} T_n \subseteq B_{n+1}$ and $\operatorname{singsupp} T_n \subseteq \bigcup_0^{[\frac{n+1}{2}]} \partial B_{(n+1-2k)}$, and that the entries are polynomial in the derivatives in the root, toral and radial directions applied to the measure $\chi_{B_1}^{*(n+1)}$.

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