ARTINIANNESS OF GRADED LOCAL COHOMOLOGY MODULES

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Abstract. Let \( R = \bigoplus_{n \in \mathbb{N}} R_n \) be a Noetherian homogeneous ring with local base ring \( (R_0, \mathfrak{m}_0) \) and let \( M \) be a finitely generated graded \( R \)-module. Let \( a \) be the largest integer such that \( H^n_{R_+}(M) \) is not Artinian. We will prove that \( H^n_{R_+}(M)/\mathfrak{m}_0 H^n_{R_+}(M) \) are Artinian for all \( i \geq a \) and there exists a polynomial \( P \in \mathbb{Q}[x] \) of degree less than \( a \) such that length\( _{R_0}(H^n_{R_+}(M)/\mathfrak{m}_0 H^n_{R_+}(M)a) = P(n) \) for all \( n < 0 \). Let \( s \) be the first integer such that the local cohomology module \( H^n_{R_+}(M) \) is not \( R_+ \)-cofinite. We will show that for all \( i \leq s \) the graded module \( \Gamma_{\mathfrak{m}_0}(H^n_{R_+}(M)) \) is Artinian.

1. Introduction

Let \( R = \bigoplus_{n \in \mathbb{N}_0} R_n \) be a Noetherian homogeneous ring with local base ring \((R_0, \mathfrak{m}_0)\). So \( R_0 \) is a Noetherian ring and there are finitely many elements \( l_1, \ldots, l_r \in R_1 \) such that \( R = R_0[l_1, \ldots, l_r] \). Let \( R_+ := \bigoplus_{n \in \mathbb{N}} R_n \) denote the irrelevant ideal of \( R \) and let \( \mathfrak{m} := \mathfrak{m}_0 \oplus \hat{R}_+ \) denote the graded maximal ideal of \( R \). Moreover let \( q_0 \subseteq \mathfrak{m}_0 \) be an \( \mathfrak{m}_0 \)-primary ideal. Finally let \( M = \bigoplus_{n \in \mathbb{Z}} M_n \) be a finitely generated graded \( R \)-module.

Let \( *E := *E_R(R/m) \) be the \(*\)-injective envelope of the graded \( R \)-module \( R/m \), and let \( E_0 := E_{R_0}(R_0/m_0) \) be the injective envelope of the \( R_0 \)-module \( R_0/m_0 \). Moreover, for a graded \( R \)-module \( T = \bigoplus_{n \in \mathbb{Z}} T_n \) and an \( R_0 \)-module \( U \), let \( *D(T) := *\text{Hom}_R(T, *E) \) and \( D_0(U) := \text{Hom}_{R_0}(U, E_0) \) denote the \(*\)-Matlis dual of \( T \) and the Matlis dual of \( U \) respectively (cf. [BS Exercise 13.4.5], [BH Theorem 3.6.17]). Let \( A \) be a graded Artinian \( R \)-module and let \( \hat{R}_0 \) be the completion of \( R_0 \) with respect to \( \mathfrak{m}_0 \)-adic topology. Then \( A \) carries a natural structure as a graded \( \hat{R} \)-module where \( \hat{R} = \hat{R}_0 \otimes_{R_0} R \). Clearly, as an \( \hat{R} \)-module \( A \) is again Artinian with \( \text{length}_{\hat{R}_0}(A_n) = \text{length}_{\hat{R}_0}(A_n) \) for all \( n \in \mathbb{Z} \). In particular by [K] the length of graded components of an \( \hat{R} \)-module \( A \) has polynomial growth and as an \( \hat{R} \)-module \( A \) has the same polynomial as it has over \( R \). We now take the \(*\)-Matlis dual of the \( \hat{R} \)-module \( A \) and denote it by \( *\hat{D}(A) \).

The modules \( H^n_{R_+}(M) \) and their graded components are closely related to sheaf cohomology over projective schemes (cf. [BS Chap. 20]). Then it is very important...
to study the Artinianess of these graded modules which has a very close relation with the minimal generators of their components.

Brodmann, Fumasoli and Tajarod in [BFT] showed that if the local base ring $R_0$ is of dimension one, then for all $i$ and for all $m_0$-primary ideals $q_0$ the graded $R$-modules $H^i_{R_+}(M)/q_0 H^i_{R_+}(M), (0: H^i_{R_+}(M)) q_0$ are Artinian and hence the length of the components of these graded modules have polynomial growth. Next, the authors in [BRS] showed that the degrees of these polynomials are independent of the choice of $q_0$. In the case $\dim(R_0) = 2$, the situation changes drastically. Here, the graded $R$-modules $(0: H^i_{R_+}(M)) m_0$ and $H^i_{R_+}(M)/m_0 H^i_{R_+}(M)$ need not be Artinian in general (cf. [BFT, Examples 4.1, 4.2]). Moreover the above numerical functions need not be polynomial in this case, as shown by examples of Katzman and Sharp.

Let $g = g(M)$ (referred to as the cohomological finite length dimension) be the least integer $i$ such that the $R_0$-module $H^i_{R_+}(M)_n$ is of infinite length for infinitely many integers $n$. Authors in [BRS] showed that if $i \leq g$, then $\Gamma_{m_0}(H^i_{R_+}(M))$ is Artinian. In this paper we will obtain a parallel conclusion to this result. Let $c = c(M)$ (referred to as the cohomological dimension of $M$ with respect to $R_+$) be the largest integer $i$ such that $H^i_{R_+}(M) \neq 0$. Rotthaus and Şega in [RS] proved that $H^i_{R_+}(M)/m_0 H^i_{R_+}(M)$ is Artinian. We will also extend this result for the largest $a$ such that $H^a_{R_+}(M)$ is not Artinian.

Let $a = a_{R_+}(M)$ be the largest integer such that $H^a_{R_+}(M)$ is not Artinian. We will prove that $H^i_{R_+}(M)/m_0 H^i_{R_+}(M)$ is Artinian for all $i \geq a$. We will also show that there exists a polynomial $\tilde{P} \in \mathbb{Q}[x]$ of degree less than $a$ such that $\text{length}_{R_0}(H^a_{R_+}(M)_n/m_0 H^a_{R_+}(M)_n) = \tilde{P}(n)$ for all $n \ll 0$. Next we deduce that $H^i_{R_+}(M)/q_0 H^i_{R_+}(M)$ is Artinian for all $m_0$-primary ideals $q_0$, the length of the components of this graded module has polynomial growth and the degrees of these polynomials are independent of $q_0$.

For any graded ideal $a$ of $R$ and any graded $R$-module $N$ we say that $N$ is $a$-cofinite if $\text{Supp}(N) \subset V(a)$ and $\text{Ext}^1_R(R/a, N)$ is finitely generated graded for all $i \geq 0$. Also, we define $s = c_a(N)$ as the first integer such that the local cohomology module $H^i_a(N)$ is not $a$–cofinite. We will show that for all $i \leq s = c_{R_+}(M)$ the graded module $\Gamma_{m_0}(H^i_{R_+}(M))$ is Artinian and there exists a polynomial $\tilde{P} \in \mathbb{Q}[x]$ of degree $\dim_R(\tilde{D}(0 : \Gamma_{m_0}(H^a_{R_+}(M)_0)) = \tilde{P}(n)$ for all $n \ll 0$.

2. The results

2.1. Definition. For any graded ideal $a$ and any finitely generated graded $R$-module $M$ we define

$$a_a(M) = \sup \{i : H^i_a(M) \text{ is not Artinian} \}.$$

In view of this definition we have the following lemma.

2.2. Lemma. Let $x \in m$ be a homogeneous non-zero divisor of $M$. Then we have

$$a_{R_+}(M/xM) \leq a_{R_+}(M).$$

Proof. Let $\deg x = d$. As $x$ is a non-zero divisor of $M$, there is an exact sequence

$$0 \to M(-d) \overset{x}{\to} M \to M/xM \to 0$$
Proof. We proceed by induction on $\dim R = d$. If $A$ is Artinian, and then in view of [BFT, Lemma 2.2], flat base change property of local cohomology $H^{i}_{R_{x}}$ above remark, it suffices show that we may replace $\mathfrak{m}$ by $\mathfrak{m}_0$. So we assume that $H^{i}_{R_{x}}(M) = 0$. Then by the flat base change property of local cohomology $R_{0} \otimes_{R_{x}} H^{i}_{R_{x}}(M) = 0$. Then by the flat base change property of local cohomology $R_{0} \otimes_{R_{x}} H^{i}_{R_{x}}(M) = 0$. Therefore, the result follows.

2.3. Remark. a) Any local flat morphism of local Noetherian rings is faithfully flat. So, if $R_{0}$ is flat over $R$ and $\mathfrak{m}_0 = \mathfrak{m}_0 R_{0}$, then $R_{0}$ is faithfully flat over $R_{0}$. Moreover, it follows from [K, Theorem 1] that if $(R_{0}, \mathfrak{m}_0)$ be a faithfully flat local $R_0$-algebra, then $A$ is a graded Artinian $R$-module if and only if $A := R_{0} \otimes_{R_{0}} A$ is a graded Artinian module over $R' := R_{0} \otimes_{R_{0}} R$.

b) Let $(R_{0}', \mathfrak{m}_0')$ be a faithfully flat local $R_0$-algebra. One can easily show that $a_{R_{x}}(M) = a_{R_{0}'} R_{x} (\mathfrak{m}_0 R_{x})$. Application of the functor $H^{i}_{R_{x}}(-)$ to it induces the following exact sequence $H^{i}_{R_{x}}(M) \rightarrow H^{i}_{R_{x}}(M/x_{M}) \xrightarrow{\delta} H^{i+1}_{R_{x}}(M)$. If $a_{R_{x}}(M) = t$, then for all $i \geq t$ the $R$-modules $H^{i}_{R_{x}}(M)$ are Artinian and so $H^{i}_{R_{x}}(M/x_{M})$ are Artinian. Therefore the result follows.

2.4. Theorem. Let $a = a_{R_{x}}(M)$. Then $H^{i}_{R_{x}}(M)/\mathfrak{m}_0 H^{i}_{R_{x}}(M)$ is Artinian for all $i \geq a$.

Proof. We proceed by induction on $d = \dim R_{x}$. At first if $i > a$, then $H^{i}_{R_{x}}(M)$ is Artinian, and so it follows from [BFT, Lemma 2.2] that $H^{i}_{R_{x}}(M)/\mathfrak{m}_0 H^{i}_{R_{x}}(M)$ is Artinian. So we assume that $i = a$. If $d = 1$, then the result follows by [BFT, Theorem 2.5]. Suppose inductively that the result has been proved for all values smaller than $d$ and so we prove it for $d$. Let $x$ be an indeterminate and let $R_{0} := R_{0} [x]_{\mathfrak{m}_0 R_{0}[x]}$, $\mathfrak{m}_0 := \mathfrak{m}_0 R_{0}$, $R' = R_{0} \otimes_{R} R$ and $M' := R_{0} \otimes_{R} M$. Then by the flat base change property of local cohomology $R_{0} \otimes_{R_{x}} H^{i}_{R_{x}}(M)/\mathfrak{m}_0 H^{i}_{R_{x}}(M) \cong H^{i}_{R_{x}}(M)/\mathfrak{m}_0 H^{i}_{R_{x}}(M')$. As $R_{0}'$ is a faithfully flat local $R_0$-algebra, in view of the above remark, it suffices show that $H^{i}_{R_{x}}(M)/\mathfrak{m}_0 H^{i}_{R_{x}}(M')$ is Artinian. Therefore, we may replace $R$ and $M$ by $R'$ and $M'$, respectively and hence we assume that $R_{0}/\mathfrak{m}_0$ is infinite residue field. Consider the exact sequence $0 \rightarrow \Gamma_{\mathfrak{m}_0}(M) \rightarrow M \rightarrow M/\Gamma_{\mathfrak{m}_0}(M) \rightarrow 0$. Application of the functor $H^{i}_{R_{x}}(-)$ induces the following exact sequence:

$$H^{i}_{R_{x}}(\Gamma_{\mathfrak{m}_0}(M)) \xrightarrow{\alpha} H^{i}_{R_{x}}(M) \xrightarrow{\beta} H^{i}_{R_{x}}(M/\mathfrak{m}_0(M)) \xrightarrow{\gamma} H^{i+1}_{R_{x}}(\Gamma_{\mathfrak{m}_0}(M)).$$

It should be noted that $H^{i}_{R_{x}}(\Gamma_{\mathfrak{m}_0}(M))$ is Artinian for each $i$ by [BFT, Lemma 2.3]. Now consider $U = \text{Im} \alpha$, $V = \text{Im} \beta$, and $W = \text{Im} \gamma$. It follows from [BFT, Lemma 2.2] that both $\text{Tor}^{R_{x}}_{0}(R_{0}/\mathfrak{m}_0, U)$ and $\text{Tor}^{R_{x}}_{0}(R_{0}/\mathfrak{m}_0, W)$ are Artinian for all $i$. Now, let $H^{i}_{R_{x}}(M/\Gamma_{\mathfrak{m}_0}(M))/\mathfrak{m}_0 H^{i}_{R_{x}}(M/\Gamma_{\mathfrak{m}_0}(M))$ be Artinian. It implies that $V/\mathfrak{m}_0 V$ is Artinian and then we can conclude that $H^{i}_{R_{x}}(M)/\mathfrak{m}_0 H^{i}_{R_{x}}(M)$ is Artinian. One can also easily show that $a_{R_{x}}(M) = a_{R_{x}}(M/\Gamma_{\mathfrak{m}_0}(M))$. So we may assume that $\Gamma_{\mathfrak{m}_0}(M) = \Gamma_{\mathfrak{m}_0}(M') = 0$. Now, let $x_0$ be a non-zero divisor of $M$ and a part of a system of parameters of $\mathfrak{m}_0$, and consider the exact sequence $0 \rightarrow M/x_0 M \rightarrow M/x_0 M \rightarrow 0$ of $R$-modules. Application of the functor $H^{i}_{R_{x}}(-)$ induces the exact sequence:

$$H^{i}_{R_{x}}(M) \xrightarrow{\delta} H^{i}_{R_{x}}(M/x_0 M) \xrightarrow{\delta} H^{i+1}_{R_{x}}(M).$$

Consider $\ker \delta = X$ and $\text{Im} \delta = Y$. As $H^{i+1}_{R_{x}}(M)$ is Artinian, the graded $R$-module $Y$ is Artinian, and then in view of [BFT, Lemma 2.2], $H^{i}_{R_{x}}(M/x_0 M)/\mathfrak{m}_0 H^{i}_{R_{x}}(M/x_0 M)$ is Artinian if and only if $X/\mathfrak{m}_0 X$ is Artinian. On the other hand, application of
the functor $R_0/m_0 \otimes R_0$ induces the exact sequence
\[ R_0/m_0 \otimes_R H^a_{R_+}(M) \xrightarrow{id_{R_0/m_0} \otimes m_0 \cdot x_0} R_0/m_0 \otimes_R H^a_{R_+}(M) \rightarrow X/m_0X \rightarrow 0. \]
As $x_0 \in m_0$, the map $id_{R_0/m_0} \otimes m_0 \cdot x_0$ is zero and then $H^a_{R_+}(M)/m_0H^a_{R_+}(M) \cong X/m_0X$. Now, in view of the above arguments one can conclude that $H^a_{R_+}(M)/m_0H^a_{R_+}(M)$ is Artinian if and only if $H^a_{R_+}(M)/m_0H^a_{R_+}(M)$ is Artinian. We note that the local base ring of the graded ring $R_0 \otimes_R R$ is $(R_0, m_0)$ and dim$R_0 = d - 1$. It is easy to see that
\[ H^a_{R_+}(M/x_0M)/m_0H^a_{R_+}(M/x_0M) \cong H^a_{R_0}(M/x_0M)/m_0H^a_{(R_0 \otimes_R R)_+}(M/x_0M). \]
Now, in view of Lemma 2.2 and using induction hypotheses this module is Artinian.

\[ \square \]

2.5. Proposition. Let $a_{R_+}(M) = a$ and $m \notin \text{Att}_R(H^a_{R_+}(M)/m_0H^a_{R_+}(M))$; then there exists an element $x \in R_1$ such that $a_{R_+}(M/xM) = a - 1$.

Proof. As for any finitely generated graded $R$-module $M$ and any non-negative integer $i$ there is an isomorphism $H^i_{R_+}(M) \cong H^i_{R_+}(M/\Gamma_{R_+}(M))$; we may assume that $\Gamma_{R_+}(M) = 0$. Since, by Theorem 2.4, $H^a_{R_+}(M)/m_0H^a_{R_+}(M)$ is Artinian, the set of its attached prime ideal is finite, and so we set
\[ \mathcal{P} = \text{Att}_R(H^a_{R_+}(M)/m_0H^a_{R_+}(M)) \bigcup \text{Ass}_R(M) \setminus \text{Var}(R_+). \]
We note that $R_1 \not\subseteq \bigcup_{p \in \mathcal{P}} p$, otherwise for some $p \in \text{Att}_R(H^a_{R_+}(M)/m_0H^a_{R_+}(M))$ we should have $R_1 \subseteq p$ and hence $R_+ \subseteq p$. On the other hand, since
\[ \text{Att}_R(H^a_{R_+}(M)/m_0H^a_{R_+}(M)) \subseteq \text{Var}(m_0R), \]
one can conclude that $p = m$ and this is a contradiction. Now, consider $x \in R_1 \setminus \bigcup_{p \in \mathcal{P}} p$ and the exact sequence $0 \rightarrow M(-1) \rightarrow M \rightarrow M/xM \rightarrow 0$. Application of the functor $H^i_{R_+}(-)$ induces the following exact sequence:
\[ H^i_{R_+}(M)(-1) \xrightarrow{x} H^i_{R_+}(M) \rightarrow H^i_{R_+}(M/xM) \rightarrow H^{i+1}_{R_+}(M)(-1). \]
Set $i = a$. This fact that $m \notin \text{Att}_R(H^a_{R_+}(M)/m_0H^a_{R_+}(M))$ implies that any prime ideal in $\text{Att}_R(H^a_{R_+}(M)/m_0H^a_{R_+}(M))$ belongs to $\mathcal{P}$. So using the same proof mentioned in [BF1, Lemma 3.2], we deduce that $	ext{Coker}x = 0$, and hence $H^a_{R_+}(M/xM)$ is embedded in the Artinian module $H^{a+1}_{R_+}(M)$. Thus $a_{R_+}(M/xM) \leq a_{R_+}(M) - 1$. Conversely, for $i > a_{R_+}(M/xM)$ the graded module $H^i_{R_+}(M/xM)$ is Artinian and so $(0 : H^i_{R_+}(M) x)$ is Artinian. Now, since $H^{a+1}_{R_+}(M)$ is $x$-torsion, using the Melkersson Lemma the result follows.

\[ \square \]

2.6. Theorem. Let $a = a_{R_+}(M)$. Then there exists a polynomial $\tilde{P} \in \mathbb{Q}[x]$ of degree less than $a$ such that $\text{length}_{R_0}(H^a_{R_+}(M)/m_0H^a_{R_+}(M)) = \tilde{P}(n)$ for all $n \ll 0$. 

Proof. By the fact that for each $R_0$-module $T$ and any Noetherian local flat $R_0$-algebra $(R_0', m_0')$ we have length$_{R_0}(M) = length_{R_0}(R_0' \otimes_{R_0} M)$, we may assume that $R_0/m_0$ is infinite and $\Gamma_{R_+}(M) = 0$. As $H^a_{R_+}(M)/m_0H^a_{R_+}(M)$ is Artinian, by using [K], there exists a polynomial $\bar{P} \in \mathbb{Q}[x]$ such that length$_{R_0}(H^a_{R_+}(M)_n/m_0H^a_{R_+}(M)_n) = \bar{P}(n)$ for all $n \ll 0$. Now, we prove $\deg \bar{P} < a$. Since $H^a_{R_+}(M)/m_0H^a_{R_+}(M)$ is Artinian, it has a graded secondary representation. Let $H^a_{R_+}(M)/m_0H^a_{R_+}(M) = S^1 + \cdots + S^l$ be a minimal graded secondary representation with $p_j = \sqrt{(0 : R S^j)}$ for all $1 \leq j \leq l$. Let $p_1 = m$. So in this case $S^l$ is a graded $R$-module of finite length and hence it is concentrated in finitely many degrees. If we take $\delta = \begin{cases} \deg S^j - 1 \end{cases} - 1$ where $\deg S^j$ is the beginning degree of $S^j$, then $H^a_{R_+}(M)_n/m_0H^a_{R_+}(M)_n = (S^1(n) + \cdots + (S^{l-1})(n))$ for all sufficiently small $n$ such that $n < d$. So

$$length_{R_0}(H^a_{R_+}(M)_n/m_0H^a_{R_+}(M)_n) = \bar{P}(n) = length_{R_0}((S^1 + \cdots + S^{l-1})_n)$$

for all $n \ll 0$. Therefore we may assume that $m \notin \text{Att}(H^a_{R_+}(M)/m_0H^a_{R_+}(M))$.

Now, we prove the assertion by induction on $a$. Using Proposition 2.6 there exists an element $x \in R_1$ such that $a_{R_+}(M/xM) = a - 1$ and $x$ is a non-zero divisor of $M$. By the same proof mentioned in [BFT] Lemma 3.2, for all $n \ll 0$ there exists an exact sequence of $R_0$-modules

$$H^{a-1}_{R_+}(M/xM)_{n+1} \longrightarrow H^a_{R_+}(M)_n \xrightarrow{x} H^a_{R_+}(M)_{n+1} \longrightarrow 0.$$

Application of the functor $R_0/m_0 \otimes_{R_0}$ induces the following exact sequence:

$$H^{a-1}_{R_+}(M/xM)_{n+1}/m_0H^{a-1}_{R_+}(M/xM)_{n+1} \longrightarrow H^a_{R_+}(M)_n/m_0H^a_{R_+}(M)_n \xrightarrow{x} H^a_{R_+}(M)_{n+1}/m_0H^a_{R_+}(M)_{n+1} \longrightarrow 0.$$

If $a = 1$, then we have the following exact sequence:

$$\Gamma_{R_+}(M/xM)_{n+1}/m_0\Gamma_{R_+}(M/xM)_{n+1} \longrightarrow H^1_{R_+}(M)_n/m_0H^1_{R_+}(M)_n \longrightarrow 0.$$

We note that $\Gamma_{R_+}(M/xM)/m_0\Gamma_{R_+}(M/xM)$ is Artinian and finitely generated, and hence length$_{R_0}(\Gamma_{R_+}(M/xM)_{n+1}/m_0\Gamma_{R_+}(M/xM)_{n+1}) = 0$ for all $n \ll 0$. Therefore $P(n + 1) = P(n) \leq P(n + 1) + length_{R_0}(\Gamma_{R_+}(M/xM)_{n+1}/m_0\Gamma_{R_+}(M/xM)_{n+1}) = \bar{P}(n + 1) \leq \bar{P}(n + 1)$ for all $n \ll 0$. It implies that $\deg \bar{P} < 1$. Let $a > 1$ and the result be true for all values smaller than $a$. In view of the exact sequence ($\dagger$), since $H^{a-1}_{R_+}(M/xM)/m_0H^{a-1}_{R_+}(M/xM)$ is Artinian, using induction there is a polynomial $Q \in \mathbb{Q}[x]$ of degree less than $a - 1$ such that

$$length_{R_0}(H^{a-1}_{R_+}(M/xM)_n/m_0H^{a-1}_{R_+}(M/xM)_n) = Q(n)$$

for all $n \ll 0$. So we have $\bar{P}(n) \leq \bar{P}(n + 1) + Q(n + 1)$ for all $n \ll 0$, and this implies that $\deg \bar{P} < a$. □

2.7. Corollary. Let $a = a_{R_+}(M)$ and let $q_0$ be a an $m_0$-primary ideal of $R_0$. Then $H^a_{R_+}(M)/q_0H^a_{R_+}(M)$ is Artinian and there is a polynomial $\bar{T} \in \mathbb{Q}[x]$ such that $deg \bar{T} = \deg \bar{P}$ and length$_{R_0}(H^a_{R_+}(M)_n/q_0H^a_{R_+}(M)_n) = \bar{T}(n)$ for all $n \ll 0$.

Proof. The proof of Artinianess of $H^a_{R_+}(M)/q_0H^a_{R_+}(M)$ is similar to [BFT] Corollary 2.6 and $\deg \bar{T} = \deg \bar{P}$ by [BRS] Proposition 2.6.] □
2.8. Definition. Let $M$ be a finitely generated graded $R$-module and let $a$ be a graded ideal of $R$. We say that $M$ is $a$-cofinite if $\text{Supp}(M) \subset V(a)$ and $\text{Ext}_R^i(R/a, M)$ is finitely generated graded for all $i \geq 0$. We also introduce the non-negative integer $s$ as follows:

$$s = c_a(M) = \inf \{i | H^i_a(M) \text{ is not } a\text{-cofinite}\}.$$ 

Moreover, if there in no such integer, then we define $c_a(M) = -\infty$.

2.9. Theorem. Let $s = c_{R_+}(M)$ and $i \in \mathbb{N}_0$ with $i \leq s$. Then $\Gamma_{m_0}(H^{i}_{R_+}(M))$ is Artinian.

Proof. The strategy of the proof for all $i \leq s$ is the same, and hence we consider the case $i = s$. As $M$ is a finitely generated graded $R$-module, using [DY, Theorem 2.1], $\text{Hom}_R(R/R_+, H^s_{R_+}(M))$ is a finitely generated graded $R$-module. One can easily show that $\Gamma_{m_0}(\text{Hom}_R(R/R_+, H^s_{R_+}(M))) \cong (0 : H^s_{R_+}(M)) R_+ = (0 : \Gamma_{m_0}^*(H^s_{R_+}(M))) R_+ \subset \text{Ann}_{R_m} R_+$. As $\text{Hom}_R(R/R_+, H^s_{R_+}(M))$ is $(0 : \Gamma_{m_0}(H^s_{R_+}(M))) R_+$ it is Artinian, and now since $\Gamma_{m_0}(H^s_{R_+}(M))$ is $R_+$-torsion, the result follows by the Melkersson Lemma. 

2.10. Corollary. Let $s < \infty$. Then we have the following conditions:

(a) There exists a polynomial $P \in \mathbb{Q}[x]$ such that

$$\text{length}_{R_0}(\Gamma_{m_0}(H^s_{R_+}(M))) = \tilde{P}(n) \quad \text{for all } n < 0.$$

(b) If $q_0$ is an $m_0$-primary ideal of $R_0$, then there is a polynomial $\tilde{P} \in \mathbb{Q}[x]$ such that $\text{deg}(\tilde{P}) = \text{deg}(P) = \dim_{R_0}(\text{Ann}_{R_0} (H^s_{R_+}(M)))$ and

$$\text{length}_{R_0}(0 : \Gamma_{m_0}^*(H^s_{R_+}(M)), q_0)) = \tilde{P}(n) \quad \text{for all } n < 0.$$

Proof. (a) As $\Gamma_{m_0}(H^s_{R_+}(M))$ is Artinian, the result is obtained by [K].

(b) Apply [BRS, Corollary 2.5] with $A = \Gamma_{m_0}(H^s_{R_+}(M))$. 

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