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BLO SPACES ASSOCIATED WITH THE SECTIONS

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ABSTRACT. *BLO* spaces associated with the sections are introduced. It is shown that some properties which hold for the classical space *BLO* related to the balls (or cubes) remain valid for the space *BLO* related to the sections.

0. Introduction

In 1980, R. R. Coifman and R. Rochberg [5] introduced the space BLO of functions of bounded lower oscillation. More precisely, we say that a locally integrable function f on $\mathbb{R}^n$ is in BLO if

$$\|f\|_{BLO} = \sup_Q (m_Q(f) - f_Q) < \infty,$$

where $m_Q(f) = \frac{1}{|Q|}\int_Q f(y)dy,\ f_Q = \operatorname*{ess\,inf}_{x\in Q} f(x)$, and Q is a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

R. R. Coifman and R. Rochberg [5] discussed the relationship between BMO functions and BLO functions. Later, C. Bennett [2] obtained a criterion for the classical spaces BLO.

In this note, we will introduce the spaces $BMO_{\mathcal{F}}$ based on sections as the substitutes for the classical spaces BLO defined by R. R. Coifman and R. Rochberg [5]. We will give out a criterion for the new spaces $BLO_{\mathcal{F}}$. Meanwhile, we will also show that the singular integral operator H^* (see Section 2) is bounded from L^∞ into $BLO_{\mathcal{F}}$.

1. Preliminary results

In this section, we first introduce some notation and then state our main results.

The analysis of the Monge-Ampère equation suggests the axiomatic definition of what are called ‘sections’. These are defined as follows.

For $x \in \mathbb{R}^n$ and $t > 0$, let $S(x,t)$ denote an open and bounded convex set containing x. We call $S(x,t)$ a section if the family $\{S(x,t)\ :\ x \in \mathbb{R}^n,\ t > 0\}$ is monotone increasing in t, i.e., $S(x,t) \subset S(x,t')$ for $t \le t'$, and satisfies the following conditions:

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(A) There exist positive constants K_1, K_2, K_3 and ϵ_1, ϵ_2 such that given two sections $S(x_0, t_0)$, $S(x,t)$ with $t \le t_0$ satisfying

$$S(x_0, t_0) \cap S(x,t) \neq \emptyset,$$

and an affine transformation T that "normalizes" $S(x_0, t_0)$, that is,

$$B(0, 1/n) \subset T(S(x_0, t_0)) \subset B(0,1),$$

there exists $z \in B(0, K_3)$ depending on $S(x_0, t_0)$ and $S(x,t)$, which satisfies

$$B(z, K_2(t/t_0)^{\epsilon_2}) \subset T(S(x,t)) \subset B(z, K_1(t/t_0)^{\epsilon_1})$$

and

$$T(z) \in B(z, (1/2)K_2(t/t_0)^{\epsilon_2}).$$

Here and below $B(x,t)$ denotes the Euclidean ball centered at x with radius t.

(B) There exists a constant $\delta > 0$ such that given a section $S(x,t)$ and $y \notin S(x,t)$, if T is an affine transformation that "normalizes" $S(x,t)$, then for any $0 < \epsilon < 1$,

$$B(T(y), \epsilon^\delta) \cap T(S(x, (1-\epsilon)t)) = \emptyset.$$

(C) $\bigcap_{t>0} S(x,t) = \{x\}$ and $\bigcup_{t>0} S(x,t) = \mathbb{R}^n$.

An important example coming from the Monge-Ampère equation is where we let ϕ be a smooth solution whose graph contains no lines. Then we let $\rho(x,y) = \phi(y) - \phi(x) - \nabla\phi(x)(y-x)$ and define the sections by $S(x,t) = \{y : \rho(x,y) < t\}$. These sections satisfy the properties (A), (B) and (C); see [3].

In addition, we also assume that a Borel measure μ is finite on compact sets, $\mu(\mathbb{R}^n) = \infty$, and satisfies the following doubling property with respect to $\mathcal{F}$, where $\mathcal{F} = \{S(x,t) : x \in \mathbb{R}^n, t > 0\}$; that is, there exists a constant A such that

$$\mu(S(x,2t)) \le A\mu(S(x,t)) \quad \text{for any section } S(x,t) \in \mathcal{F}. \tag{1.1}$$

The properties (A) and (B) of the sections imply the following engulfing property proved in [1]: there exists a constant $\theta > 1$, depending only on K_1 and ϵ_1, such that for $y \in S(x,r)$ we have

(D) $S(y,r) \subset S(x, \theta r)$ and $S(x,r) \subset S(y, \theta r)$.

We use the sections to define a pseudo-distance function: $\rho(x,y) = \inf\{t > 0 : y \in S(x,t)\}$. The engulfing property of the sections implies the following two properties of ρ:

$$\rho(x,y) \le \theta\rho(y,x), \tag{1.2}$$

and

$$\rho(x,y) \le \theta^2(\rho(x,z) + \rho(z,y)); \tag{1.3}$$

see [8].

In [1], the authors proved that if a family $\mathcal{F}$ of sections satisfies the properties (A), (B) and (C), then there exists a quasi-metric $d(x,y)$ on $\mathbb{R}^n$ with respect to $\mathcal{F}$ defined by

$$d(x,y) = \inf\{r : x \in S(y,r) \text{ and } y \in S(x,r)\}.$$

It is easy to see that

(E) $\rho(x,y) \le d(x,y) \le \theta\rho(x,y)$.

The triangular constant of the quasi-metric d is just the θ appearing in the property (D), that is,

$$d(x,y) \le \theta(d(x,z) + d(z,y)) \quad \text{for any } x,y,z \in \mathbb{R}^n.$$

Moreover, we denote by $B_d(x,r) = \{y \in \mathbb{R}^n \ : \ d(x,y) < r\}$ the d-ball centered at x with radius r.

The following relationship between a section and a d-ball can be found in [1].

(F) For any $x \in \mathbb{R}^n$ and any $r > 0$, $S(x, \frac{r}{2\theta}) \subset B_d(x,r) \subset S(x,r)$.

It should be pointed out that the quasi-metric d and the Borel measure μ satisfying the doubling condition (1.1) create a space of homogeneous type; see also [1].

In the rest of this paper, we assume that the Borel measure μ satisfies the doubling condition (1.1). The maximal function Mf of f is defined by

$$Mf(x) = \sup_{x\in S\in\mathcal{F}} \frac{1}{\mu(S)} \int_S f(x)d\mu(x).$$

Let f be a real-valued function defined on $\mathbb{R}^n$. We say that $f \in BMO_{\mathcal{F}}$ if

$$\|f\|_{BMO_{\mathcal{F}}} := \sup_{S\in\mathcal{F}} \frac{1}{\mu(S)} \int_S |f(x) - m_S(f)|d\mu(x) < \infty,$$

where and in what follows, $m_S(f) = \frac{1}{\mu(S)} \int_S f(x)d\mu(x)$.

We are ready to define the $BLO_{\mathcal{F}}$ space if there exists some constant C such that for any section $S \in \mathcal{F}$,

$$m_S(f) - \inf_{x\in S} f(x) \le C.$$

The smallest constant C will be denoted by $\|f\|_{BLO_{\mathcal{F}}}$.

Obviously, from properties (D), (E) and (F) of the sections, we know that the $BMO_{\mathcal{F}}$ space coincides with the BMO space related to the d-balls. But, a BLO space defined by sections need not coincide with a BLO space defined by a metric. This will be seen from Lemma 2.1 below.

We shall consider kernels $k(x,y)$ that can be represented in the form

$$k(x,y) = \sum_i k_i(x,y), \tag{1.4}$$

where the k_i's satisfy the following properties:

$$\text{supp } k_i(\cdot,y) \subset S_i(y), \ \forall y; \tag{1.5}$$

$$\text{supp } k_i(x,\cdot) \subset S_i(x), \ \forall y; \tag{1.6}$$

$$\int_{\mathbb{R}^n} k_i(x,y)d\mu(y) = \int_{\mathbb{R}^n} k_i(x,y)d\mu(x) = 0, \ \forall x,y; \tag{1.7}$$

$$\sup_i \int_{\mathbb{R}^n} |k_i(x,y)|d\mu(y) \le C_1, \ \forall x; \tag{1.8}$$

$$\sup_i \int_{\mathbb{R}^n} |k_i(x,y)|d\mu(x) \le C_1, \ \forall y, \tag{1.9}$$

where $S_i(x) = S(x, 2^i)$ for any $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$. If T is an affine transformation that normalizes the section $S_i(y)$, then

$$|k_i(u,y) - k_i(v,y)| \le C_2 \frac{1}{\mu(S_i(y))} |Tu - Tv|, \tag{1.10}$$

and finally, if T is an affine transformation that normalizes the section $S_i(x)$, then

$$|k_i(x,u) - k_i(x,v)| \le C_2 \frac{1}{\mu(S_i(x))} |Tu - Tv|. \tag{1.11}$$

The operator associated with the kernel k is defined by

$$Hf(x) = \int_{\mathbb{R}^n} k(x,y) f(y) d\mu(y).$$

The maximal operator H^* associated with the operator H is defined by

$$H^* f(x) = \sup_{\epsilon > 0} \left| \int_{d(x,y) > \epsilon} k(x,y) f(y) d\mu(y) \right|,$$

for $f \in L^p, 1 \le p < \infty$.

It was proved in [3] that the operator H is the strong type (2,2). Subsequently, the weak type (1,1) of H was proved in [8].

The main results of this paper can be stated as follows.

Theorem 1.1. *If $f \in L^\infty \cap L^p$ for some $1 \le p < \infty$, then $H^* f \in BLO_{\mathcal{F}}$.*

Theorem 1.2. *If $f \in BMO_{\mathcal{F}}$ and if S is any section in $\mathcal{F}$, then*

$$m_S(Mf) \le C \|f\|_{BMO_{\mathcal{F}}} + \inf_S Mf,$$

where C is a constant. In particular, if Mf is finite for μ a.e., then $Mf \in BLO_{\mathcal{F}}$ and

$$\|Mf\|_{BLO_{\mathcal{F}}} \le C \|f\|_{BMO_{\mathcal{F}}}.$$

In the sequel, C is a positive constant which is independent of the main parameters and not necessarily the same at each occurrence. For a measurable set E, we denote by χ_E the characteristic function of E.

2. The proof of the theorems

To prove Theorem 1.1, we need the following lemmas.

Lemma 2.1. *$f \in BLO_{\mathcal{F}}$ if and only if $Mf - f \in L^\infty$. Furthermore,*

$$\|Mf - f\|_{L^\infty} \sim \|f\|_{BLO_{\mathcal{F}}}.$$

Proof. Suppose first that $f \in BLO_{\mathcal{F}}$. Observe that, by the Lebesgue differentiation theorem, for μ-a.e. $x \in \mathbb{R}^n$, we have

$$\lim_{r \to 0} \frac{1}{\mu(S(x,r))} \int_{S(x,r)} f(y) d\mu(y) = f(x).$$

Let x be any such point, and let $S \in \mathcal{F}$ be any section containing x. Then $f(x) \ge \inf_{x \in S} f(x)$ and so

$$m_S(f) - f(x) \le m_S(f) - \inf_{x \in S} f(x) \le \|f\|_{BLO_{\mathcal{F}}}.$$

Taking the supremum over all sections containing x, we get

$$Mf(x) - f(x) \leq \|f\|_{BLO_{\mathcal{F}}}.$$

Therefore $Mf - f \in L^\infty$ and $\|Mf - f\|_{L^\infty} \leq \|f\|_{BLO_{\mathcal{F}}}$.

Conversely, suppose $Mf - f \in L^\infty$ and let $S \in \mathcal{F}$. If any $x \in S$ such that

$$f(x) < m_S(f) - \|Mf - f\|_{L^\infty},$$

then

$$Mf(x) - f(x) \geq m_S(f) - f(x) > \|Mf - f\|_{L^\infty}.$$

So, for μ-a.e. $x \in S$,

$$f(x) \geq m_s(f) - \|Mf - f\|_{L^\infty};$$

consequently,

$$m_S(f) - \inf_{x \in S} f(x) < \|Mf - f\|_{L^\infty}.$$

Then we get that $f \in BLO_{\mathcal{F}}$ and

$$\|f\|_{BLO_{\mathcal{F}}} \leq C\|Mf - f\|_{L^\infty}.$$

Thus, the proof of Lemma 2.1 is complete. □

Lemma 2.2. *We define $k(x, y)$ as* (1.4)*. Then there exists a positive constant C such that*

$$|k(x, y_0) - k(x, y)| + |k(y_0, x) - k(y, x)| \leq \frac{C}{\mu(S(y_0, 2^k d(y_0, y)))} 2^{-\epsilon_1 k},$$

if $d(y_0, x) \geq 2^k 4\theta^3 d(y_0, y)$ and k is a nonnegative integer.

The proof of Lemma 2.2 is similar to that of Lemma 1 in [8]. We omit the details here.

Lemma 2.3. *If $f \in L^p$ for $1 \leq p < \infty$, then for any $r > 0$,*

$$H^* f(x) \leq A_r (M(|Hf|^r)(x))^{1/r} + M(|f|)(x) \quad \forall x \in \mathbb{R}^n,$$

where A_r denotes a positive constant depending only on r.

Subsequently, the maximal operator H^* is bounded on L^p for $1 < p < \infty$.

Proof. Fixing $\bar{x} \in \mathbb{R}^n$ and an $\epsilon > 0$, let $B = B_d(\bar{x}, \epsilon)$. Decompose $f = f_1 + f_2$, where $f_1 = f\chi_B$. The truncated operator H_ϵ is defined by

$$H_\epsilon f(x) = \int_{d(x,y) > \epsilon} k(x, y) f(y) d\mu(y);$$

then $H_\epsilon f(\bar{x}) = Hf_2(\bar{x})$. We claim that

$$|Hf_2(\bar{x}) - Hf_2(x)| \leq CM(|f|)(\bar{x}) \quad \text{if} \quad d(\bar{x}, x) < \epsilon/(4\theta^3). \tag{2.1}$$

In fact, applying Lemma 2.2 and the properties (D), (E) and (F) of the sections, then

$$
\begin{aligned}
|Hf_2(\bar{x}) - Hf_2(x)| &\le \int_{d(\bar{x},y)>\epsilon} |k(x,y) - k(\bar{x},y)||f(y)|d\mu(y) \\
&= \sum_{k=0}^{\infty} \int_{2^k\epsilon \le d(\bar{x},y) < 2^{k+1}\epsilon} |k(x,y) - k(\bar{x},y)||f(y)|d\mu(y) \\
&\le C \sum_{k=0}^{\infty} 2^{-k\epsilon_1} \mu(S(\bar{x}, 2^k\epsilon))^{-1} \int_{S(\bar{x},2^k\epsilon)} |f(y)|d\mu(y) \\
&\le CM(|f|)(\bar{x}).
\end{aligned}
$$

Thus, (2.1) holds. Hence

$$
|H_\epsilon f(\bar{x})| \le |Hf(x)| + |Hf_1(x)| + CM(|f|)(\bar{x}) \quad \text{if } x \in B_d(\bar{x}, \epsilon/(4\theta^3)). \tag{2.2}
$$

On the other hand, we have

$$
\begin{aligned}
\mu(\{x \in B_d(\bar{x}, \epsilon/(4\theta^3)) : \ |Hf(x)| > \lambda\}) &\le \lambda^{-r} \int_{B_d(\bar{x},\epsilon/(4\theta^3))} |Hf(x)|^r \, d\mu(x) \\
&\le \lambda^{-r} \mu(B_d(\bar{x}, \epsilon/(4\theta^3))) M(|Hf|^r)(\bar{x})
\end{aligned}
$$

for any $r > 0$; see also [9]. Thus, if $\lambda \ge 4^{1/r}[M(|Hf|^r)(\bar{x})]^{1/r}$, then

$$
\mu(\{x \in B_d(\bar{x}, \epsilon/(4\theta^3)) : \ |Hf(x)| > \lambda\}) \le \frac{1}{4}\mu(B_d(\bar{x}, \epsilon/(4\theta^3))). \tag{2.3}
$$

By the weak type (1,1) of H, we have

$$
\begin{aligned}
\mu(\{x \in B_d(\bar{x}, \epsilon/(4\theta^3)) : \ |Hf_1(x)| > \lambda\}) &\le \frac{A}{\lambda} \int_{\mathbb{R}^n} |f_1(x)|d\mu(x) \\
&= \frac{A}{\lambda} \int_{B_d(\bar{x},\epsilon/(4\theta^3))} |f(x)|d\mu(x) \\
&\le \frac{A}{\lambda} \mu(x \in B_d(\bar{x}, \epsilon/(4\theta^3))) M(|f|)(\bar{x}).
\end{aligned}
$$

So if $\lambda \ge 4AM(|f|)(\bar{x})$, then

$$
\mu(\{x \in B_d(\bar{x}, \epsilon/(4\theta^3)) : \ |Hf_1(x)| > \lambda\}) \le \frac{1}{4}\mu(B_d(\bar{x}, \epsilon/(4\theta^3))). \tag{2.4}
$$

Therefore if $\lambda \ge 4^{1/r}[M(|Hf|^r)(\bar{x})]^{1/r} + 4AM(|f|)(\bar{x})$, from (2.3) and (2.4), then there exists an $x \in B_d(\bar{x}, \epsilon/(4\theta^2))$ so that $|Hf(x)| \le \lambda$ and $|Hf_1(x)| \le \lambda$. Substituting this in (2.2) yields

$$
H_\epsilon f(\bar{x}) \le A_r \left([M(|Hf|^r)(\bar{x})]^{1/r} + M(|f|)(\bar{x}) \right).
$$

Taking the supremum for $\epsilon > 0$, we obtain Lemma 2.3. □

Proof of Theorem 1.1. Let $x \in \mathbb{R}^n$ and let $S(x_0, r)$ be any section containing x. For any fixed section $S(x_0, r)$, let $B = B_d(x, 4\theta^3 r)$ contain $S(x_0, r)$. Since $f \in L^p \cap L^\infty$

for some $p \in [1,\infty)$, by L^2 boundedness of H^* (see Lemma 2.3), we have

$$\begin{aligned}
\frac{1}{\mu(S)}\int_S H^*(f\chi_B)\,d\mu &\le \frac{1}{\mu(S)^{1/2}}\left(\int_S [H^*(f\chi_B)]^2\,d\mu\right)^{1/2}\\
&\le \frac{C}{\mu(S)^{1/2}}\left(\int_{\mathbb{R}^n} |f\chi_B|^2\,d\mu\right)^{1/2}\\
&\le C\frac{\mu(B)^{1/2}}{\mu(S)^{1/2}}\|f\|_{L^\infty}\\
&\le C\|f\|_{L^\infty}.
\end{aligned} \tag{2.5}$$

From the definition of H^*, we have

$$H^*(f\chi_{\mathbb{R}^n\setminus B})(x) \le H^*f(x).$$

Now, we claim that

$$|H^*(f_B)(y) - H^*(f_B)(x)| \le C\|f\|_{L^\infty}$$

holds for any $y \in S$, where $f_B = f\chi_{\mathbb{R}^n\setminus B}$.

In fact,

$$\begin{aligned}
|H^*(f_B)(y) - H^*(f_B)(x)| \le &\int_{\mathbb{R}^n\setminus B} |k(x,z)-k(y,z)||f(z)|\,dz\\
&+\sup_{\epsilon>0}\int_{d(y,z)>\epsilon,d(x,z)<\epsilon} |k(y,z)||f_B(z)|\,dz\\
&+\sup_{\epsilon>0}\int_{d(x,z)>\epsilon,d(y,z)<\epsilon} |k(x,z)||f_B(z)|\,dz\\
= &\ I + II + III.
\end{aligned}$$

For I, since $d(x,y) < r$, similar to the proof of (2.1) in Lemma 2.1, we have

$$I \le C\|f\|_{L^\infty}.$$

For II, if $\epsilon < 4\theta^3 r$, then $II = 0$.

Thus, we only need to consider the case $\epsilon \ge 4\theta^3 r$. We observe that

$$\begin{aligned}
\int_{d(y,z)>\epsilon,d(x,z)<\epsilon} |k(y,z)||f_B(z)|\,dz &\le \int_{\epsilon<d(y,z)\le\theta(d(x,z)+d(y,x))} |k(y,z)||f_B(z)|\,dz\\
&\le \int_{\epsilon<d(y,z)\le\theta(\epsilon+\theta r)} |k(y,z)|\,dz\|f\|_{L^\infty}\\
&\le \int_{\epsilon<d(y,z)\le 2\theta\epsilon} |k(y,z)|\,dz\|f\|_{L^\infty}.
\end{aligned}$$

From (1.6) and (1.8), it is easy to see that

$$\int_{\epsilon<d(y,z)\le 2\theta\epsilon} |k(y,z)|\,dz \le C.$$

Therefore,

$$II \le C\|f\|_{L^\infty}.$$

Similar to the proof of II, for III we have

$$III \le C\|f\|_{L^\infty}.$$

Thus, the claim is proved.

By these, for any $y \in S$, we have

$$H^*(f\chi_{\mathbb{R}^n\setminus B})(y) \le |H^*(f\chi_{\mathbb{R}^n\setminus B})(y) - H^*(f\chi_{\mathbb{R}^n\setminus B})(x)| + H^*(f\chi_{\mathbb{R}^n\setminus B})(x)$$
$$\le C\|f\|_{L^\infty} + H^*(f)(x).$$

Hence,

$$\frac{1}{\mu(S)}\int_S H^*(f\chi_{\mathbb{R}^n\setminus B})d\mu \le C\|f\|_{L^\infty} + H^*(f)(x).$$

From this and (2.5), we obtain

$$\frac{1}{\mu(S)}\int_S H^* f d\mu \le C\|f\|_{L^\infty} + H^*(f)(x).$$

So,

$$\|M(H^*f) - H^*f\|_{L^\infty} \le C\|f\|_{L^\infty}.$$

Using Lemma 2.1, we can obtain the desired result. □

Proof of Theorem 1.2. We fix a section $S = S(x_0, r)$, and write $\bar{S} = S(x_0, \theta^2 r)$. For $f \in \mathrm{BMO}_{\mathcal{F}}$ we write

$$f = (f - m_S(f))\chi_{\bar{S}} + m_{\bar{S}}(f)\chi_{\bar{S}} + f\chi_{\mathbb{R}^n\setminus\bar{S}}.$$

By the strong type (2,2) of M and the Nirenberg inequality for $BMO_{\mathcal{F}}$ (see [4] and [6]), we have

$$\int_S M((f - m_S(f))\chi_{\bar{S}})d\mu \le \mu(S)^{1/2}\left(\int_{\mathbb{R}^n} |M((f - m_S(f))\chi_{\bar{S}})|^2 d\mu\right)^{1/2}$$
$$\le C\mu(S)^{1/2}\left(\int_{\bar{S}} |f - m_S(f)|^2 d\mu\right)^{1/2}$$
$$\le C\mu(S)\|f\|_{BMO_{\mathcal{F}}}.$$

Next, we shall show that

$$\frac{1}{\mu(S)}\int_S M(m_{\bar{S}}(f)\chi_{\bar{S}} + f\chi_{\mathbb{R}^n\setminus\bar{S}})d\mu \le C\|f\|_{BMO_{\mathcal{F}}} + \inf_{x\in S} Mf(x).$$

It suffices to show that

$$M(m_{\bar{S}}(f)\chi_{\bar{S}} + f\chi_{\mathbb{R}^n\setminus\bar{S}})(x) \le C\|f\|_{BMO_{\mathcal{F}}} + \inf_{x\in S} Mf(x),\ \mu-\text{a.e. } x \in S.$$

Hence, we only need to prove that

$$\frac{1}{\mu(R)}\int_R (m_{\bar{S}}(f)\chi_{\bar{S}} + f\chi_{\mathbb{R}^n\setminus\bar{S}})d\mu \le C\|f\|_{BMO_{\mathcal{F}}} + \inf_{x\in S} Mf(x) \tag{2.6}$$

for any section $R \ni x \in S$. If $R \subset \bar{S}$, the result follows immediately:

$$\frac{1}{\mu(R)}\int_R (m_{\bar{S}}(f)\chi_{\bar{S}} + f\chi_{\mathbb{R}^n\setminus\bar{S}})d\mu = m_{\bar{S}}(f) \le \inf_{x\in S} Mf(x).$$

We now assume that $R \cap (\mathbb{R}^n \setminus \bar{S}) \ne \emptyset$. Set $R = S(x', r')$; since $x \in R$, using the property (D) of the sections, then

$$S(x', r') \subset S(x, \theta r') \subset S(x', \theta^2 r') \subset S(x, \theta^3 r'). \tag{2.7}$$

On the other hand, since $x \in S(x_0, r)$, we have

$$S(x_0, r) \subset S(x, \theta r) \subset S(x_0, \theta^2 r) \subset S(x, \theta^3 r). \tag{2.8}$$

It is clear that $r' \geq r$. Otherwise, if $r' < r$, we have

$$R = S(x', r') \subset S(x, \theta r') \subset S(x, \theta r) \subset S(x_0, \theta^2 r) = \bar{S},$$

which contradicts the assumption $R \cap (\mathbb{R}^n \setminus \bar{S}) \neq \emptyset$.

Letting $R' = S(x, \theta^3 r')$, from (2.7) and (2.8) we know that

$$R \cup \bar{S} \subset R' \quad \textit{and} \quad \mu(R') \leq C\mu(R).$$

From this, we then have

$$\begin{aligned}
&\int_R \big(m_{\bar{S}}(f)\chi_{\bar{S}} + f\chi_{\mathbb{R}^n\setminus\bar{S}} - m_{R'}(f)\big)\,d\mu \\
&\quad \leq \int_R |m_{\bar{S}}(f)\chi_{\bar{S}} + f\chi_{\mathbb{R}^n\setminus\bar{S}} - m_{R'}(f)|d\mu \\
&\quad = \int_{\bar{S}} |m_{\bar{S}}(f) - m_{R'}(f)|d\mu + \int_{R'\cap(\mathbb{R}^n\setminus\bar{S})} |f - m_{R'}(f)|d\mu \\
&\quad = \left(\int_{\bar{S}} + \int_{R'\cap(\mathbb{R}^n\setminus\bar{S})}\right) |f - m_{R'}(f)|d\mu \\
&\quad \leq \int_{R'} |f - m_{R'}(f)|d\mu.
\end{aligned}$$

So, we get

$$m_R(m_{\bar{S}}(f)\chi_{\bar{S}} + f\chi_{\mathbb{R}^n\setminus\bar{S}} - m_{R'}(f)) \leq C\|f\|_{BMO_{\mathcal{F}}}. \tag{2.9}$$

Since $S \subset R'$, from (2.9), we obtain

$$\begin{aligned}
&m_R(m_{\bar{S}}(f)\chi_{\bar{S}} + f\chi_{\mathbb{R}^n\setminus\bar{S}}) \\
&\qquad = m_R(m_{\bar{S}}(f)\chi_{\bar{S}} + f\chi_{\mathbb{R}^n\setminus\bar{S}} - m_{R'}(f)) + m_{R'}(f) \\
&\qquad \leq \|f\|_{BMO_{\mathcal{F}}} + \inf_S Mf.
\end{aligned}$$

This establishes (2.6). Thus, Theorem 1.2 is proved. □

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