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# CONSTRUCTION OF NON-ALTERNATING KNOTS

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ABSTRACT. We investigate the behaviour of Rasmussen's invariant $s$ under the sharp operation on knots and obtain a lower bound for the sharp unknotting number. This bound leads us to an interesting move that transforms arbitrary knots into non-alternating knots.

## 1. Introduction

An unknotting operation is a local operation that allows us to untie every knot in finitely many steps. The most popular unknotting operation is a simple crossing change. Every unknotting operation gives rise to a measure of complexity for knots, called an unknotting number. An effective lower bound for the usual unknotting number was introduced by Rasmussen ([4]). His invariant led to an easy computation of the genera and unknotting numbers of torus knots. In this paper, we study the sharp unknotting operation via Rasmussen's invariant $s$.

The sharp unknotting operation is a local move that acts on link diagrams, as shown in Figure 1. It has been introduced by Murakami ([3]) and gives rise to the unknotting number $u_{\#}$. The usual unknotting number is denoted by $u$.

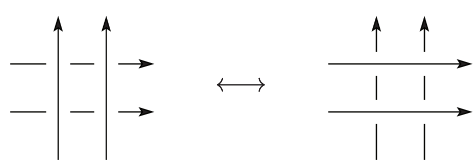

FIGURE 1. Sharp operation

Our main result involves a special sharp operation, called a positive sharp operation. A positive sharp operation introduces eight positive crossings to a link diagram, as shown in Figure 2.

For a diagram $D$ of a knot $K$, we denote by $w(D)$, $O(D)$, and $g(K)$ the writhe of $D$ (i.e., the algebraic crossing number of $D$), the number of Seifert circles of $D$, and the genus of $K$, respectively.

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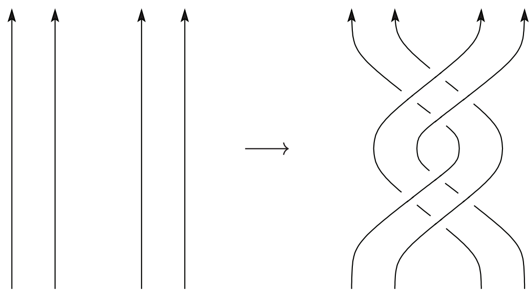

FIGURE 2. Positive sharp operation

**Theorem 1.1.** *Let $D$ be any knot diagram, and suppose $D'$ is obtained from $D$ by the application of $n$ positive sharp operations. If*

$$n > g(K) - \frac{1 + w(D) - O(D)}{2},$$

*then $D'$ represents a non-alternating knot.*

The quantity on the right-hand side of the inequality in Theorem 1.1 is always positive, as follows from Bennequin's inequality ([1]). For positive knot diagrams, it is actually zero.

**Corollary 1.2.** *Let $D$ be a positive knot diagram, and suppose $D'$ is obtained from $D$ by the application of one positive sharp operation. Then $D'$ represents a non-alternating knot.*

**Examples 1.3.**

(i) The closure of the braid $\sigma_1^{-1}\sigma_2\sigma_1\sigma_3\sigma_2$ represents the trivial knot $O$. For the corresponding knot diagram $D$ (see Figure 3, on the left-hand side), Bennequin's inequality is an equality:
$$g(O) - \frac{1}{2}(1 + w(D) - O(D)) = 0 - \frac{1}{2}(1 + 3 - 4) = 0.$$
Therefore, if we apply one positive sharp operation at the top of this braid diagram, we obtain a non-alternating knot. It is a 2-cable of the positive trefoil knot.

(ii) The closure of the braid $\sigma_1\sigma_2\sigma_3$ represents the trivial knot, too (see Figure 3, on the right-hand side). By Corollary 1.2, the application of one positive sharp operation at the top of that braid diagram yields a non-alternating knot. This time, we obtain the knot $10_{139}$, in Rolfsen's notation ([5]).

A sharp unknotting operation changes 4 crossings of a diagram. Therefore, $u$ cannot exceed $4u_\#$. Rasmussen's invariant allows us to detect knots with $u(K) = 4u_\#(K)$.

**Theorem 1.4.**

(1) $u_\#(K) \geqslant \frac{|s(K)|}{8}$.

(2) *If $u_\#(K) = \frac{|s(K)|}{8}$, then $K$ is either trivial or non-alternating. In any case, $u(K) = 4u_\#(K)$ holds.*

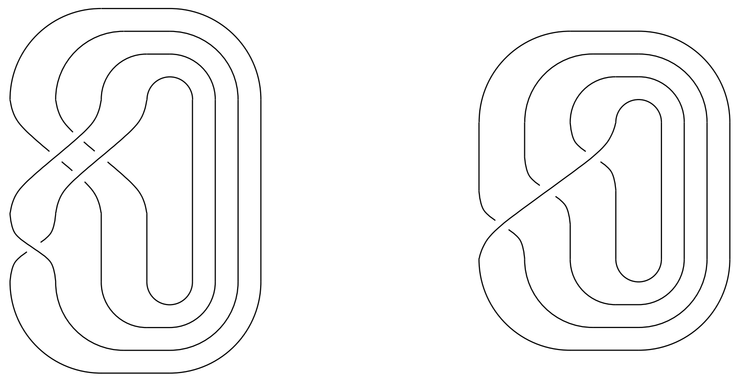

FIGURE 3. Two diagrams of the trivial knot

**Examples 1.5.** (i) The diagram $D$ of the 2-cable knot $K$ we constructed above has 4 Seifert circles and writhe 11, whence $1 + w(D) - O(D) = 8$. The latter quantity is a lower bound for the invariant $s(K)$ (see [7]). This proves $u_{\#}(K) = \frac{|s(K)|}{8} = 1$.

(ii) The diagram $D$ of the knot $10_{139}$ we constructed above has 4 Seifert circles and writhe 11. Again, we conclude $u_{\#}(10_{139}) = \frac{|s(10_{139})|}{8} = 1$.

## 2. Rasmussen's invariant and the sharp unknotting operation

The proofs of Theorems 1.1 and 1.4 are based upon the following three properties of Rasmussen's invariant $s$:

(1) $|s(K)| \leqslant 2u(K)$,

(2) $s(K) = \sigma(K)$, for all alternating knots $K$ (here $\sigma(K)$ is the signature of the knot $K$),

(3) $1 + w(D) - O(D) \leqslant s(K)$, where $D$ is any diagram of a knot $K$.

The first two properties were proved by Rasmussen ([4]), whereas the third inequality was proved by Shumakovitch ([7]). The main argument in the proof of (3) is Rudolph's reduction to the case of positive diagrams ([6]).

As we remarked after Theorem 1.4, the usual unknotting number $u$ cannot exceed $4u_{\#}$. Together with the inequality (1), this immediately proves the first statement of Theorem 1.4:

$$u_{\#}(K) \geqslant \frac{u(K)}{4} \geqslant \frac{|s(K)|}{8}.$$

In [3], Murakami proved the following estimate for $u_{\#}$, in terms of the signature $\sigma$ of a knot:

$$u_{\#}(K) \geqslant \frac{|\sigma(K)|}{6}.$$

This implies the second statement of Theorem 1.4:

Let $K$ be a knot with $u_{\#}(K) = \frac{|s(K)|}{8}$. Murakami's inequality tells us that

$$\frac{|s(K)|}{8} \geqslant \frac{|\sigma(K)|}{6}.$$

If, in addition, $K$ is alternating, then $s(K) = \sigma(K)$, by (2). Therefore, $s(K) = \sigma(K) = 0$, $u_{\#}(K) = 0$, and $K$ is the trivial knot. In any case, $4u_{\#}(K) = u(K)$ holds.

In order to prove Theorem 1.1, we have to study the behaviour of the numbers $w(D)$ and $O(D)$ under a positive sharp operation: a positive sharp operation increases the writhe by 8 and leaves the number of Seifert circles invariant. Now, let $D$ be any knot diagram of a knot $K$. Furthermore, suppose $D'$ is obtained from $D$ by the application of $n$ positive sharp operations. $D'$ represents a knot $K'$. Using (3), we find the following lower bound for $s(K')$:

$$s(K') \geqslant 1 + w(D') - O(D') = 1 + w(D) + 8n - O(D).$$

On the other hand, we have the following upper bound for the signature $\sigma(K')$:

$$\sigma(K') \leqslant \sigma(K) + 6n \leqslant 2g(K) + 6n.$$

The first inequality is due to Murakami ([3]): the signature of a knot cannot increase by more than 6 under a sharp operation. The second inequality is obvious, since the signature of a knot $K$ is the signature of a Seifert matrix of size $2g(K)$. Now, if

$$n > g(K) - \frac{1 + w(D) - O(D)}{2},$$

then

$$s(K') - \sigma(K') \geqslant 2n - 2g(K) + 1 + w(D) - O(D) > 0,$$

whence $K'$ is non-alternating. This completes the proof of Theorem 1.1.

*Remark* 2.1. Throughout this paper, we could replace Rasmussen's invariant $s$ by the concordance invariant $2\tau$ coming from knot Floer homology, since the three properties (1), (2) and (3) are also valid for $2\tau$. A list of properties that are shared by $s$ and $2\tau$ is contained in [2]. In the same paper, M. Hedden and P. Ording show that the invariants $s$ and $2\tau$ are not equal.

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