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CONSTRUCTION OF NON-ALTERNATING KNOTS

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Abstract. We investigate the behaviour of Rasmussen's invariant s under the sharp operation on knots and obtain a lower bound for the sharp unknotting number. This bound leads us to an interesting move that transforms arbitrary knots into non-alternating knots.

1. Introduction

An unknotting operation is a local operation that allows us to untie every knot in finitely many steps. The most popular unknotting operation is a simple crossing change. Every unknotting operation gives rise to a measure of complexity for knots, called an unknotting number. An effective lower bound for the usual unknotting number was introduced by Rasmussen ([4]). His invariant led to an easy computation of the genera and unknotting numbers of torus knots. In this paper, we study the sharp unknotting operation via Rasmussen's invariant s.

The sharp unknotting operation is a local move that acts on link diagrams, as shown in Figure 1. It has been introduced by Murakami ([3]) and gives rise to the unknotting number $u_{\#}$. The usual unknotting number is denoted by u.

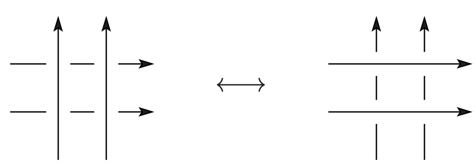

Figure 1. Sharp operation

Our main result involves a special sharp operation, called a positive sharp operation. A positive sharp operation introduces eight positive crossings to a link diagram, as shown in Figure 2.

For a diagram D of a knot K, we denote by $w(D)$, $O(D)$, and $g(K)$ the writhe of D (i.e., the algebraic crossing number of D), the number of Seifert circles of D, and the genus of K, respectively.

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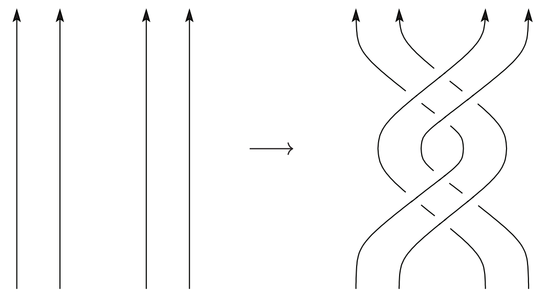

FIGURE 2. Positive sharp operation

Theorem 1.1. *Let D be any knot diagram, and suppose D' is obtained from D by the application of n positive sharp operations. If*

$$n > g(K) - \frac{1 + w(D) - O(D)}{2},$$

then D' represents a non-alternating knot.

The quantity on the right-hand side of the inequality in Theorem 1.1 is always positive, as follows from Bennequin's inequality ([1]). For positive knot diagrams, it is actually zero.

Corollary 1.2. *Let D be a positive knot diagram, and suppose D' is obtained from D by the application of one positive sharp operation. Then D' represents a non-alternating knot.*

Examples 1.3.

(i) The closure of the braid $\sigma_1^{-1}\sigma_2\sigma_1\sigma_3\sigma_2$ represents the trivial knot O. For the corresponding knot diagram D (see Figure 3, on the left-hand side), Bennequin's inequality is an equality:

$$g(O) - \frac{1}{2}(1 + w(D) - O(D)) = 0 - \frac{1}{2}(1 + 3 - 4) = 0.$$

Therefore, if we apply one positive sharp operation at the top of this braid diagram, we obtain a non-alternating knot. It is a 2-cable of the positive trefoil knot.

(ii) The closure of the braid $\sigma_1\sigma_2\sigma_3$ represents the trivial knot, too (see Figure 3, on the right-hand side). By Corollary 1.2, the application of one positive sharp operation at the top of that braid diagram yields a non-alternating knot. This time, we obtain the knot 10_{139}, in Rolfsen's notation ([5]).

A sharp unknotting operation changes 4 crossings of a diagram. Therefore, u cannot exceed $4u_\#$. Rasmussen's invariant allows us to detect knots with $u(K) = 4u_\#(K)$.

Theorem 1.4.

(1) $u_\#(K) \geqslant \frac{|s(K)|}{8}$.

(2) *If $u_\#(K) = \frac{|s(K)|}{8}$, then K is either trivial or non-alternating. In any case, $u(K) = 4u_\#(K)$ holds.*

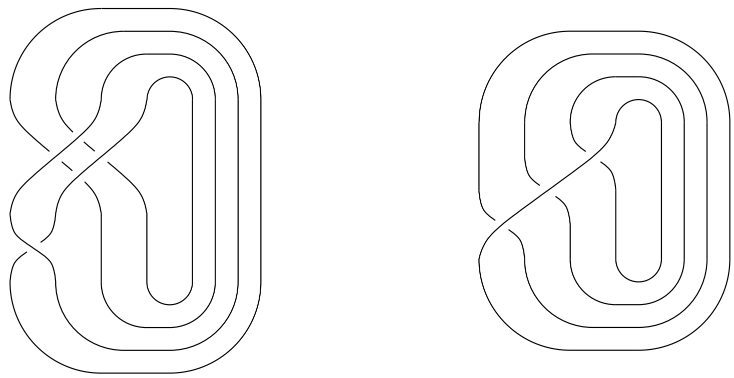

FIGURE 3. Two diagrams of the trivial knot

Examples 1.5. (i) The diagram D of the 2-cable knot K we constructed above has 4 Seifert circles and writhe 11, whence $1 + w(D) - O(D) = 8$. The latter quantity is a lower bound for the invariant $s(K)$ (see [7]). This proves $u_{\#}(K) = \frac{|s(K)|}{8} = 1$.

(ii) The diagram D of the knot 10_{139} we constructed above has 4 Seifert circles and writhe 11. Again, we conclude $u_{\#}(10_{139}) = \frac{|s(10_{139})|}{8} = 1$.

2. RASMUSSEN'S INVARIANT AND THE SHARP UNKNOTTING OPERATION

The proofs of Theorems 1.1 and 1.4 are based upon the following three properties of Rasmussen's invariant s:

(1) $|s(K)| \leqslant 2u(K)$,

(2) $s(K) = \sigma(K)$, for all alternating knots K (here $\sigma(K)$ is the signature of the knot K),

(3) $1 + w(D) - O(D) \leqslant s(K)$, where D is any diagram of a knot K.

The first two properties were proved by Rasmussen ([4]), whereas the third inequality was proved by Shumakovitch ([7]). The main argument in the proof of (3) is Rudolph's reduction to the case of positive diagrams ([6]).

As we remarked after Theorem 1.4, the usual unknotting number u cannot exceed $4u_{\#}$. Together with the inequality (1), this immediately proves the first statement of Theorem 1.4:

$$u_{\#}(K) \geqslant \frac{u(K)}{4} \geqslant \frac{|s(K)|}{8}.$$

In [3], Murakami proved the following estimate for $u_{\#}$, in terms of the signature σ of a knot:

$$u_{\#}(K) \geqslant \frac{|\sigma(K)|}{6}.$$

This implies the second statement of Theorem 1.4:

Let K be a knot with $u_{\#}(K) = \frac{|s(K)|}{8}$. Murakami's inequality tells us that

$$\frac{|s(K)|}{8} \geqslant \frac{|\sigma(K)|}{6}.$$

If, in addition, K is alternating, then $s(K) = \sigma(K)$, by (2). Therefore, $s(K) = \sigma(K) = 0$, $u_{\#}(K) = 0$, and K is the trivial knot. In any case, $4u_{\#}(K) = u(K)$ holds.

In order to prove Theorem 1.1, we have to study the behaviour of the numbers $w(D)$ and $O(D)$ under a positive sharp operation: a positive sharp operation increases the writhe by 8 and leaves the number of Seifert circles invariant. Now, let D be any knot diagram of a knot K. Furthermore, suppose D' is obtained from D by the application of n positive sharp operations. D' represents a knot K'. Using (3), we find the following lower bound for $s(K')$:

$$s(K') \geqslant 1 + w(D') - O(D') = 1 + w(D) + 8n - O(D).$$

On the other hand, we have the following upper bound for the signature $\sigma(K')$:

$$\sigma(K') \leqslant \sigma(K) + 6n \leqslant 2g(K) + 6n.$$

The first inequality is due to Murakami ([3]): the signature of a knot cannot increase by more than 6 under a sharp operation. The second inequality is obvious, since the signature of a knot K is the signature of a Seifert matrix of size $2g(K)$. Now, if

$$n > g(K) - \frac{1 + w(D) - O(D)}{2},$$

then

$$s(K') - \sigma(K') \geqslant 2n - 2g(K) + 1 + w(D) - O(D) > 0,$$

whence K' is non-alternating. This completes the proof of Theorem 1.1.

Remark 2.1. Throughout this paper, we could replace Rasmussen's invariant s by the concordance invariant 2τ coming from knot Floer homology, since the three properties (1), (2) and (3) are also valid for 2τ. A list of properties that are shared by s and 2τ is contained in [2]. In the same paper, M. Hedden and P. Ording show that the invariants s and 2τ are not equal.

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