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THE NOETHER MAP II

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ABSTRACT. Let $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a faithful representation of a finite group G. In this paper we proceed with the study of the image of the associated Noether map

$$\eta_G^G : \mathbb{F}[V(G)]^G \to \mathbb{F}[V]^G.$$

In our 2005 paper it has been shown that the Noether map is surjective if V is a projective $\mathbb{F}G$-module. This paper deals with the converse. The converse is in general not true: we illustrate this with an example. However, for p-groups (where p is the characteristic of the ground field $\mathbb{F}$) as well as for permutation representations of any group the surjectivity of the Noether map implies the projectivity of V.

Let $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a faithful representation of a finite group G of order d over a field $\mathbb{F}$. The representation ρ naturally induces an action of G on the vector space $V = \mathbb{F}^n$ of dimension n and hence on the ring of polynomial functions $\mathbb{F}[V] = \mathbb{F}[x_1, \ldots, x_n]$. Our interest is focused on the subring of invariants

$$\mathbb{F}[V]^G = \{f \in \mathbb{F}[V]^G \text{ such that } gf = f \forall g \in G\},$$

which is a graded connected Noetherian commutative algebra. Denote by $\mathbb{F}G$ the group algebra. Let

$$V(G) = \mathbb{F}G \otimes_{\mathbb{F}} V$$

be the induced module. The group G acts on $V(G)$ by left multiplication on the first component. We obtain a G-equivariant surjection

$$(\star) \qquad V(G) \to V, (g, v) \mapsto gv.$$

Let us choose a basis $e_1, \ldots, e_n$ for V. Let $x_1, \ldots, x_n$ be the standard dual basis for V^*, and set $G = \{g_1, \ldots, g_d\}$. Then $V(G)$ can be written as

$$V(G) = \mathrm{span}_{\mathbb{F}}\{e_{ij} | i = 1, \ldots, n,\ j = 1, \ldots, d\},$$

and the map $(\star)$ translates into

$$V(G) \to V,\ e_{ij} \mapsto g_j e_i.$$

Similarly, we have

$$V(G)^* = \mathrm{span}_{\mathbb{F}}\{x_{ij} | i = 1, \ldots, n,\ j = 1, \ldots, d\}$$

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with

$$V(G)^* \to V^*,\ x_{ij} \mapsto g_j x_i.$$

We obtain a surjective G-equivariant map between the rings of polynomial functions

$$\eta_G : \mathbb{F}[V(G)] \to \mathbb{F}[V].$$

The group G acts on $\mathbb{F}[V(G)]$ by permuting the basis elements x_{ij}. By restriction to the induced ring of invariants, we obtain the classical Noether map, cf. Section 4.2 in [9],

$$\eta_G^G : \mathbb{F}[V(G)]^G \to \mathbb{F}[V]^G.$$

We note that $V(G)$ is the n-fold regular representation of G. Thus $\mathbb{F}[V(G)]^G$ are the n-fold vector invariants of the regular representation of G.

In the classical nonmodular case, where $p \nmid d$, the map η_G^G is surjective; see Proposition 4.2.2 in [9]. This has been generalized in the sense that the Noether map is surjective if V is a projective $\mathbb{F}G$-module; see Proposition 3.1 in [8]. The converse may fail as we illustrate with the next example.

Example. Let $\mathrm{GL}(2,\ \mathbb{F}_3)$ be the general linear group of 2×2 matrices with entries from the field with three elements. By Corollary 9.14 in [4] the top Dickson class $\mathbf{d}_{2,0}$ is in the image of the transfer. Hence it is in the image of the Noether map. In order to also see that the other Dickson class $\mathbf{d}_{2,1}$ is in the image of the Noether map, we note that $\mathrm{GL}(2,\ \mathbb{F}_3)$ contains a supgroup H of order 6 generated by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda & 0 \\ 0 & \lambda, \end{bmatrix}$$

where $\lambda \in \mathbb{F}^\times$. Denote these six elements by $h_1, \dots, h_6$. Then the stabilizer subgroup of the monomial

$$(h_1 \otimes x_1) \cdots (h_6 \otimes x_1) \in \mathbb{F}[V(\mathrm{GL}(2, \mathbb{F}_3))]$$

is H. Direct computation yields

$$\eta_{\mathrm{GL}(2,\mathbb{F}_3)}^{\mathrm{GL}(2,\mathbb{F}_3)}(o((h_1 \otimes x_1) \cdots (h_6 \otimes x_1))) = -\mathbf{d}_{2,1}.$$

Thus $\eta_{\mathrm{GL}(2,\mathbb{F}_3)}^{\mathrm{GL}(2,\mathbb{F}_3)}$ is surjective, but the tautological representation of $\mathrm{GL}(2, \mathbb{F}_3)$ is not projective.

In the next section we prove that whenever G is a p-group or ρ is a permutation representation, the Noether map is surjective if and only if V is a projective $\mathbb{F}G$-module.

Before we proceed we present a general characterization:

Proposition. *V is projective if and only if*

$$\eta_G^G : \mathbb{F}[\mathrm{End}(V)(G)]^G \to \mathbb{F}[\mathrm{End}(V)]^G$$

is surjective.

Proof. V is projective if and only if $\mathrm{End}(V)$ is projective by [2]. Thus the Noether map on that vector space is surjective by Proposition 3.1 in [8]. Conversely, if the above Noether map is surjective, then it is surjective in degree one. Hence the transfer map is surjective in degree one by Corollary 1.2 below. In particular, the identity on V is in the image of the transfer. Thus V is projective by the Higman criterion; see, e.g., Proposition 3.6.4 in [3]. □

1. p-Groups and permutation representations

In this section we want to show that the converse Proposition 3.1 in [8] is true in the case of p-groups P and in the case of permutation representations.

Lemma 1.1. *Let P be a cyclic p-group, and let $\mathbb{F}$ have characteristic p. Then*

$$\operatorname{Im}(\operatorname{Tr}^P)_{(1)} = \mathbb{F}[V]^P_{(1)}$$

if and only if V is the k-fold regular representation of P for some $k \in \mathsf{N}$.

Proof. Since the transfer is additive it suffices to consider indecomposable modules only.

Let the order of the group be p^s. Then up to isomorphism there are exactly p^s indecomposable $\mathbb{F}P$-modules $V_1, \ldots, V_{p^s}$ with $\dim_{\mathbb{F}} V_i = i$. The action of P on V_i is afforded by the matrix consisting of one Jordan block with 1's on the diagonal and superdiagonal. Note that $V_i^P = V_1$ for all i.

Set $\Delta = g - 1$ where $g \in P$ is a generator. Then

$$\Delta(V_i^*) = \begin{cases} V_{i-1}^* & \text{for } i = 2, \ldots, p^s, \\ 0 & \text{for } i = 1. \end{cases}$$

Since, $\operatorname{Tr}^P = \Delta^{p^s - 1}$, we obtain

$$\operatorname{Tr}^P(V_i^*) = \Delta^{p^s-1}(V_i^*) = \begin{cases} 0 & \text{for } i = 1, \ldots, p^s - 1, \\ V_1^* & \text{for } i = p^s, \end{cases}$$

as desired. □

We obtain the following corollary that we note here for later reference.

Corollary 1.2. *Let $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a faithful representation of a finite group. Let $i \in \mathbb{F}^\times$. Then*

$$\operatorname{Im}(\eta_G^G|_{(i)}) = \operatorname{Im}(\operatorname{Tr}^G|_{(i)}).$$

Proof. By construction we obtain a commutative diagram as follows:

$$\begin{array}{ccc} \mathbb{F}[V(G)]^G|_{(i)} & \xrightarrow{\eta_G^G|_{(i)}} & \mathbb{F}[V]^G|_{(i)} \\ \uparrow_{\operatorname{Tr}^G|_{(i)}} & & \uparrow_{\operatorname{Tr}^G|_{(i)}} \\ \mathbb{F}[V(G)]|_{(i)} & \xrightarrow{\eta_G|_{(i)}} & \mathbb{F}[V]|_{(i)}. \end{array}$$

By Theorem 3.2 in [7] and the remark following it the transfer map on the left

$$\operatorname{Tr}^G|_{(i)} : \mathbb{F}[V(G)]|_{(i)} \to \mathbb{F}[V(G)]^G|_{(i)}$$

is surjective. By construction the lower map $\eta_G|_{(i)}$ is surjective. Thus the result follows. □

Theorem 1.3. *Let $\rho : P \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a p-group over a field $\mathbb{F}$ of characteristic p. Then the following are equivalent:*

1. *The Noether map is surjective.*
2. *The Noether map is surjective in degree one.*
3. *V is a projective $\mathbb{F}P$-module.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. The implication (3) $\Rightarrow$ (1) was proven in Proposition 3.1 in [8]. Thus we need to show that V is projective if $\eta_P^P|_{(1)}$ is surjective.

Consider the short exact sequence of $\mathbb{F}P$-modules

$$(*) \qquad 0 \to K^* \to V(P)^* \stackrel{\eta_P|_{(1)}}{\to} V^* \to 0.$$

The module $V(P)$ is free and therefore cohomologically trivial. Thus the long exact cohomology sequence breaks up into

$$0 \to (K^*)^P \to (V(P)^*)^P \stackrel{\eta_P^P|_{(1)}}{\to} (V^*)^P \to \mathrm{H}^1(P, K^*) \to 0$$

and

$$\mathrm{H}^i(P, V^*) \cong \mathrm{H}^{i+1}(P, K^*) \quad \forall i \geq 1.$$

Since $\eta_P^P|_{(1)}$ is surjective by assumption, we obtain

$$\mathrm{H}^1(P, K^*) = 0.$$

Thus K^* is a projective $\mathbb{F}P$-module (see, e.g., Proposition 4.4.11 in [10]). Since P is finite and K^* finitely generated, this implies that K^* is injective; see Corollary 2.7 in [5]. Thus the sequence $(*)$ splits and V^* is projective as desired. □

We illustrate this result with an example.

Example 1. Let $\mathbb{F}$ be the field with q elements of characteristic p. Let $P \leq \mathrm{GL}(n,\ \mathbb{F})$ be a p-Sylow subgroup of the general linear group. We assume without loss of generality that P consists of all upper triangular matrices with 1's on the diagonal. Then

$$\mathbb{F}[V(P)]_{(1)}^P = \mathrm{span}_{\mathbb{F}}\{o(x_{i1}) = \sum_{j=1}^{|P|} x_{ij} \text{ such that } i = 1, \ldots, n\}.$$

Thus

$$\begin{aligned}
\eta_P^P(o(x_{i1})) = \sum_{j=1}^{|P|} g_j x_i &= \sum_{(a_{i+1},\ldots,a_n)\in\mathbb{F}^{n-i}} (x_i + a_{i+1}x_{i+1} + \cdots + a_n x_n) \\
&= q^{\frac{n(n-1)}{2}-(n-i)}\big(q^{n-i}x_i + q^{n-i-1}\left(\sum_{a_{i+1}\in\mathbb{F}} a_{i+1}x_{i+1} + \cdots + \sum_{a_n\in\mathbb{F}} a_n x_n\right)\big) \\
&= q^{\frac{n(n-1)}{2}} x_i + q^{\frac{n(n-1)}{2}-1}\left(\sum_{a_{i+1}\in\mathbb{F}} a_{i+1}x_{i+1} + \cdots + \sum_{a_n\in\mathbb{F}} a_n x_n\right).
\end{aligned}$$

If $n \leq 1$, then P is the trivial group. Therefore V is $\mathbb{F}P$-projective and the Noether map is surjective.

If $n \geq 2$, then the factor $q^{\frac{n(n-1)}{2}}$ vanishes. The factor $q^{\frac{n(n-1)}{2}-1}$ is nonzero if and only if $n = 2$. Thus we proceed by having a closer look at the two-dimensional case: We have by the above calculations

$$\eta_P^P(o(x_{11})) = \sum_{j=1}^{|P|} g_j x_1 = \sum_{a_2 \in \mathbb{F}} (x_1 + a_2 x_2) = \Big(\sum_{a_2 \in \mathbb{F}} a_2\Big) x_2,$$
$$\eta_P^P(o(x_{21})) = \sum_{j=1}^{|P|} g_j x_2 = 0.$$

If p is odd, then for every nonzero $a_2 \in \mathbb{F}$ there exists a negative $-a_2 \neq a_2$. Therefore

$$\sum_{a_2 \in \mathbb{F}} a_2 = 0.$$

If $p = 2$, then

$$\Big(\sum_{a_2 \in \mathbb{F}} a_2\Big) x_2 = \begin{cases} x_2 & \text{if } q = 2, \\ 0 & \text{if } q > 2. \end{cases}$$

Thus we have that the Noether map is surjective if and only if $n = 2 = p = q$. Explicitely we find

$$\eta_P^P(o(x_{11})) = x_2 \quad \text{and} \quad \eta_P^P(o(x_{11}x_{12})) = x_1^2 + x_1 x_2.$$

Note that in this case

$$\mathrm{Syl}_2(\mathrm{GL}(2,\ \mathbb{F}_2)) \cong \mathbb{Z}/2$$

and our representation is projective.

Before proceeding to permutation representations, we want to mention two corollaries.

Corollary 1.4. *Let $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a faithful representation of a finite group. Assume that the rings of invariants of G and its p-Sylow subgroup coincide in degree one. Then the Noether map is surjective if and only if V is $\mathbb{F}G$-projective.*

Proof. Denote by P the p-Sylow subgroup of G. We consider the relative Noether map given by the following commutative diagram:

$$\begin{array}{ccc} \mathbb{F}[V(G)]^G & \xrightarrow{\eta_G^G} & \mathbb{F}[V]^G \\ \big\downarrow & & \big\downarrow \\ \mathbb{F}[V(G)]^P & \xrightarrow{\eta_G^P} & \mathbb{F}[V]^P \\ \big\uparrow & & \big\| \\ \mathbb{F}[V(P)]^P & \xrightarrow{\eta_P^P} & \mathbb{F}[V]^P; \end{array}$$

cf. [8]. If η_G^G is surjective, then it is surjective in degree one. Hence η_G^P is surjective in degree one by assumption. Therefore η_P^P is surjective in degree one by Proposition 2.1 in [8]. Thus V is projective by Theorem 1.3. The converse was shown in Proposition 3.1 in [8]. □

Corollary 1.5. *Let $G = H \times P$ be a direct product of a p-group P and a p'-group H. Assume that P is a cyclic p-group. Consider a faithful representation ρ of G over a field $\mathbb{F}$ of characteristic p such that V is indecomposable as an $\mathbb{F}P$-module. Then the Noether map is surjective if and only if V is $\mathbb{F}G$-projective.*

Proof. If V is $\mathbb{F}G$-projective then the Noether map η_G^G is surjective by Proposition 3.1 in [8].

To prove the converse, let η_G^G be surjective. By Proposition 2.1 in [8] it is enough to show that the relative Noether map η_G^P is surjective. We proceed by contradiction and assume that η_G^P is not surjective. Then, by Proposition 2.1 in [8], the map η_P^P is not surjective. Hence V is not a projective $\mathbb{F}P$-module by Theorem 1.3.

Let σ be a generator for P. The isomorphism type of a P-module is determined by the Jordan canonical form of σ. Up to isomorphism there are $|P|$ indecomposable P modules $V_1, V_2, \dots V_{|P|}$, where $\dim V_i = i$ and σ acts on V_i by an $i \times i$ matrix consisting of a single Jordan block with ones on the diagonal and superdiagonal. Moreover $V_{|P|}$ is the only indecomposable module which is projective. Thus by assumption we have that $V = V_n$ for $1 \leq n < |P|$.

Let $x_1, x_2, \dots, x_n$ be the basis of V such that

$$\sigma x_i = \begin{cases} x_1 & \text{if } i = 1, \\ x_{i-1} + x_i & \text{otherwise.} \end{cases}$$

Since the action of P commutes with the action of H and the action of H is nonmodular, it follows that $V = V_n$ is a direct sum of copies of isomorphic eigenspaces for H, and the variables $x_1, x_2, \dots, x_n$ may be taken as eigenvectors. Let $\mathsf{N} = \prod_{g \in P} g(x_n)$ be the norm of x_n. Since p and $|H|$ are relatively prime, there exists a positive integer m such that $m|P| \equiv -1 \mod |H|$. Consider the polynomial $x_1\mathsf{N}^m$. This polynomial is P-invariant since both x_1 and N are. Let $h \in H$. Then

$$h(x_1\mathsf{N}^m) = \lambda_h x_1 \lambda_h^{m|P|} \mathsf{N}^m = x_1\mathsf{N}^m.$$

It follows that $x_1\mathsf{N}^m$ is G-invariant.

Next we want to see that $x_1\mathsf{N}^m$ is not in the image of Tr^P. Since V is not projective, the fixed point x_1 is not in the image of Tr^P. The degree-one-component $\mathbb{F}[V]_{(1)}$ is a direct summand in $\mathbb{F}[V]_{m|P|+1}$ by multiplication by N, [6]. Thus the invariant $x_1\mathsf{N}^m$ is not in the image of Tr^P either. However, if a G-invariant polynomial is not in the image of Tr^P, then it is not in the image of Tr^G.

Since the degree of the polynomial $x_1\mathsf{N}^m$ is relatively prime to p, we have that it is not in the image of η_G^G by Corollary 1.2. This is a contradiction. □

Corollary 1.6. *Let $P \cong \mathbb{Z}/p$ and let V be an indecomposable $\mathbb{F}P$-module. Then the Noether map η_P^P is surjective in degrees divisible by p.*

Proof. As above denote by $V = V_n$ the indecomposable $\mathbb{F}\mathbb{Z}/p$-modules and $x_1, x_2, \dots, x_n$ be the basis for V on which $\mathbb{Z}/p$ acts through a single Jordan block of dimension n. We note that

$$\mathbb{F}[V] = B \oplus \mathsf{N}\mathbb{F}[V]$$

as $\mathbb{F}P$-modules, where B consists of the polynomials of x_n-degree less than p, [6].

We proceed by induction on the degree. The decomposition

$$\mathbb{F}[V]^P_{(p)} = B^P_{(p)} \oplus \mathsf{N}\mathbb{F}[V]^P$$

yields that any invariant in degree p is a direct summand of a fixed point of a free module and the polynomial N. Since fixed points of free modules and N are in the image of η_P^P, the result follows for degree p.

Using the decomposition for degree kp we have that

$$\mathbb{F}[V]^P_{(kp)} = B^P_{(kp)} \oplus \mathsf{N}\mathbb{F}[V]^P_{((k-1)p)}.$$

Since η_P^P is an algebra map, and $\mathbb{F}[V]^P_{((k-1)p)}$ is in the image of η_P^P by induction, the result follows. □

We turn to permutation representations.

Theorem 1.7. *Let $\rho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$ be a permutation representation of a finite group of order d. Then the Noether map η_G^G is surjective if and only if $V = \mathbb{F}^n$ is projective.*

Proof. By Proposition 3.1 in [8] we know that η_G^G is surjective if V is projective as an $\mathbb{F}G$-module.

We show that the converse is also true as follows:

Let η_G^G be surjective. Then its restriction to degree one, η_G^G such that $_{(1)}$, is also surjective:

$$\eta_G^G \text{ such that }_{(1)} : (V(G)^*)^G \to (V^*)^G.$$

We note that $(V(G)^*)^G$ has an $\mathbb{F}$-basis consisting of

$$o(x_{ij}) = \sum_{j=1}^{d} x_{ij} \quad \text{for } i = 1, \ldots, n.$$

Therefore, the image under the Noether map is spanned by

$$\eta_G^G \left(\sum_{j=1}^{d} x_{ij} \right) = k_i o(x_i) = |\mathrm{Stab}_G(x_i)| \mathrm{Tr}^G(x_i) \quad \text{for } i = 1, \ldots, n,$$

where

$$k_i = |\mathrm{Stab}_G(x_i)|$$

is the order of the stabilizer of x_i in G. Since ρ is a permutation representation, $(V^*)^G$ is spanned by the orbit sums of $x_1, \ldots, x_n$. It follows that k_i's are not zero, since the Noether map is surjective. Hence

$$|\mathrm{Stab}_G(x_i)| \not\equiv 0 \bmod \ p.$$

In other words, no element in a p-Sylow subgroup P of G fixes x_i, $i = 1, \ldots, n$. Therefore

$$(✠) \qquad o^P(x_i) = \mathrm{Tr}^P(x_i) = \eta_P^P \text{ such that }_{(1)}(x_{i1}),$$

where $o^P(-)$ denotes the orbit sum under the action of P, and g_1 is the identity element. Since $(V^*)^P$ is also spanned by the orbit sums of the x_i's, we found in (✠) that η_P^P such that $_{(1)}$ is surjective. Therefore, η_P^P is surjective by Proposition 1.3. Hence V^* is a projective $\mathbb{F}P$-module, by the same Propositon 1.3. Since P is a p-Sylow subgroup of G, the module V^* is projective as a $\mathbb{F}G$-module; see Corollary 3 on Page 66 of [1]. □

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