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A SMOOTH COUNTEREXAMPLE TO NORI'S CONJECTURE ON THE FUNDAMENTAL GROUP SCHEME

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Abstract. We show that Nori's fundamental group scheme $\pi(X,x)$ does not base change correctly under extension of the base field for certain smooth projective ordinary curves X of genus 2 defined over a field of characteristic 2.

1. Introduction

In the paper [N] Madhav Nori introduced the fundamental group scheme $\pi(X,x)$ for a reduced and connected scheme X defined over an algebraically closed field k as the Tannaka dual group of the Tannakian category of essentially finite vector bundles over X. In characteristic zero $\pi(X,x)$ coincides with the étale fundamental group, but in positive characteristic it does not (see, e.g., [MS]). By analogy with the étale fundamental group, Nori conjectured that $\pi(X,x)$ base changes correctly under extension of the base field. More precisely:

Nori's conjecture (see [MS]**, page 144 or** [N]**, page 89).** If K is an algebraically closed extension of k, then the canonical homomorphism

$$h_{X,K} : \pi(X_K, x) \longrightarrow \pi(X,x) \times_k \mathrm{Spec}(K) \tag{1.1}$$

is an isomorphism.

In [MS] V.B. Mehta and S. Subramanian show that Nori's conjecture is false for a projective curve with a cuspidal singularity. In this note (Corollary 4.2) we show that certain *smooth* projective ordinary curves of genus 2 defined over a field of characteristic 2 also provide counterexamples to Nori's conjecture.

The proof has two ingredients: the first is an equivalent statement of Nori's conjecture in terms of F-trivial bundles due to V.B. Mehta and S. Subramanian (see section 2), and the second is the description of the action of the Frobenius map on rank-2 vector bundles over a smooth ordinary curve X of genus 2 defined over a field of characteristic 2 (see section 3). In section 4 we explicitly determine the set of F-trivial bundles over X.

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2. Nori's conjecture and F-trivial bundles

Let X be a smooth projective curve defined over an algebraically closed field k of characteristic $p > 0$. Let $F : X \to X$ denote the absolute Frobenius of X and F^n its n-th iterate for some positive integer n.

2.1. **Definition.** A rank-r vector bundle E over X is said to be F^n-trivial if

$$E \text{ is stable} \qquad \text{and} \qquad F^{n*}E \cong \mathcal{O}_X^r.$$

2.2. **Proposition** ([MS] Proposition 3.1)**.** *If the canonical morphism $h_{X,K}$ (see (1.1)) is an isomorphism, then any F^n-trivial vector bundle E_K over $X_K := X \times_k \mathrm{Spec}(K)$ is isomorphic to $E_k \otimes_k K$ for some F^n-trivial vector bundle E_k over X.*

3. The action of the Frobenius map on rank-2 vector bundles

We briefly recall some results from [LP1] and [LP2].

Let X be a smooth projective ordinary curve of genus 2 defined over an algebraically closed field k of characteristic 2. By [LP2, section 2.3] the curve X equipped with a level-2 structure can be uniquely represented by an affine equation of the form

$$y^2 + x(x+1)y = x(x+1)(ax^3 + (a+b)x^2 + cx + c), \tag{3.1}$$

for some scalars $a, b, c \in k$. Let $\mathcal{M}_X$ denote the moduli space of S-equivalence classes of semistable rank-2 vector bundles with trivial determinant over X; see, e.g., [LeP]. We identify $\mathcal{M}_X$ with the projective space $\mathbb{P}^3$ (see [LP1] Proposition 5.1). We denote by $V : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ the rational map induced by pull-back under the absolute Frobenius $F : X \to X$. There are homogeneous coordinates $(x_{00} : x_{01} : x_{10} : x_{11})$ on $\mathbb{P}^3$ such that the equations of V are given as follows (see [LP2, section 5]):

$$V(x_{00} : x_{01} : x_{10} : x_{11}) = (\sqrt{abc}P_{00}^2(x) : \sqrt{b}P_{01}^2(x) : \sqrt{c}P_{10}^2(x) : \sqrt{a}P_{11}^2(x)), \tag{3.2}$$

with

$$P_{00}(x) = x_{00}^2 + x_{01}^2 + x_{10}^2 + x_{11}^2, \qquad P_{10}(x) = x_{00}x_{10} + x_{01}x_{11},$$
$$P_{01}(x) = x_{00}x_{01} + x_{10}x_{11}, \qquad P_{11}(x) = x_{00}x_{11} + x_{10}x_{01}.$$

Given a semistable rank-2 vector bundle E with trivial determinant, we denote by $[E] \in \mathcal{M}_X = \mathbb{P}^3$ its S-equivalence class. The semistable boundary of $\mathcal{M}_X$ equals the Kummer surface Kum_X of X. Given a degree 0 line bundle N on X, we also denote the point $[N \oplus N^{-1}] \in \mathbb{P}^3$ by N.

3.1. **Proposition** ([LP1] Proposition 6.1 (4))**.** *The preimage $V^{-1}(N)$ of the point $N \in \mathrm{Kum}_X \subset \mathcal{M}_X = \mathbb{P}^3$ with coordinates $(x_{00} : x_{01} : x_{10} : x_{11})$*

- *is a projective line if $x_{00} = 0$;*
- *consists of the 4 square roots of N if $x_{00} \neq 0$.*

4. Computations

In this section we prove the following.

4.1. **Proposition.** *Let $X = X_{a,b,c}$ be the smooth projective ordinary curve of genus 2 given by the affine model (3.1). Suppose that*

$$a^2 + b^2 + c^2 + a + c = 0. \tag{4.1}$$

Then there exists a nontrivial family $\mathcal{E} \to X \times S$ parametrized by a 1-dimensional variety S (defined over k) of F^4-trivial rank-2 vector bundles with trivial determinant over X. Moreover any F^4-trivial rank-2 vector bundle E with trivial determinant appears in the family $\mathcal{E}$, i.e., is of the form $(\mathrm{id}_X \times s)^*\mathcal{E}$ for some k-valued point $s : \mathrm{Spec}(k) \to S$.

We therefore obtain a counterexample to Nori's conjecture.

4.2. **Corollary.** *Let $X = X_{a,b,c}$ be a curve satisfying* (4.1). *Then for any algebraically closed extension K, the morphism $h_{X,K}$ is not an isomorphism.*

Proof. Since S is 1-dimensional, there exists a K-valued point $s : \mathrm{Spec}(K) \to S$ that is not a k-valued point. Then the bundle $E_K = (\mathrm{id}_X \times s)^*\mathcal{E}$ over X_K is not of the form $E_k \otimes_k K$. Now apply Proposition 2.2. □

Proof of Proposition 4.1. The method of the proof is to determine explicitly all F^n-trivial rank-2 vector bundles E over X for $n = 1, 2, 3, 4$. Taking tensor products of E with 2^{n+1}-torsion line bundles allows us to restrict attention to F^n-trivial vector bundles with trivial determinant.

We first compute the preimage under iterates of V of the point $A_0 \in \mathbb{P}^3$ determined by the trivial rank-2 vector bundle over X. We recall (see, e.g., [LP1], Lemma 2.11 (i)) that the coordinates of $A_0 \in \mathbb{P}^3$ in the coordinate system $(x_{00} : x_{01} : x_{10} : x_{11})$ are $(1 : 0 : 0 : 0)$. It follows from Proposition 3.1 and equations (3.2) that $V^{-1}(A_0)$ consists of the 4 points

$$(1:0:0:0), \quad (0:1:0:0), \quad (0:0:1:0) \text{ and } (0:0:0:1), \tag{4.2}$$

which correspond to the 2-torsion points of the Jacobian of X. Abusing notation we denote by A_1 both the 2-torsion line bundle on X and the point $(0:1:0:0) \in \mathbb{P}^3$.

Both points A_0 and A_1 correspond to S-equivalence classes of semistable rank-2 vector bundles. The set of isomorphism classes represented by the two S-equivalence classes A_0 and A_1 equal $\mathbb{P}\mathrm{Ext}^1(A_1, A_1) \cup \{0\}$ and $\mathbb{P}\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \cup \{0\}$, respectively, where 0 denotes the trivial extensions $A_1 \oplus A_1$ and $\mathcal{O}_X \oplus \mathcal{O}_X$. Note that the two cohomology spaces $\mathrm{Ext}^1(A_1, A_1)$ and $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$ are canonically isomorphic to $\mathrm{H}^1(\mathcal{O}_X)$. The pull-back by the absolute Frobenius F of X induces a rational map

$$F^* : \mathbb{P}\mathrm{Ext}^1(A_1, A_1) \longrightarrow \mathbb{P}\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X),$$

which coincides with the projectivized p-linear map on the cohomology $\mathrm{H}^1(\mathcal{O}_X) \to \mathrm{H}^1(\mathcal{O}_X)$ induced by the Frobenius map F. Since we have assumed X ordinary, this p-linear map is bijective. Hence we obtain that there is only one (strictly) semistable bundle E such that $[E] = A_1$ and $F^*E \cong \mathcal{O}_X^2$, namely $E = A_1 \oplus A_1$. In particular there are no F^1-trivial rank-2 vector bundles over X.

By Proposition 3.1 and using the equations (3.2), we easily obtain that the preimage $V^{-1}(A_1)$ is a projective line $\mathbb{L} \cong \mathbb{P}^1$, which passes through the two points

$$(1:1:1:1) \text{ and } (0:0:1:1).$$

We now determine the bundles E satisfying $F^*E \cong A_1 \oplus A_1$. Given E with $[F^*E] = A_1 \in \mathbb{P}^3$ we easily establish the equivalence

$$F^*E \cong A_1 \oplus A_1 \qquad \Longleftrightarrow \qquad \dim \mathrm{Hom}(F^*E, A_1) = \dim \mathrm{Hom}(E, F_*A_1) = 2.$$

Suppose that E is stable and $F^*E \cong A_1 \oplus A_1$. The quadratic map

$$\det : \mathrm{Hom}(E, F_*A_1) \longrightarrow \mathrm{Hom}(\det E, \det F_*A_1) = H^0(\mathcal{O}_X(w))$$

has nontrivial fibre over 0, since $\dim \mathrm{Hom}(E, F_*A_1) = 2$. Hence there exists a nonzero $f \in \mathrm{Hom}(E, F_*A_1)$ not of maximal rank. We consider the line bundle $N = \mathrm{im}\, f \subset F_*A_1$. Since F_*A_1 is stable (see [LaP], Proposition 1.2), we obtain the inequalities

$$0 = \mu(E) < \deg N < \frac{1}{2} = \mu(F_*A_1),$$

a contradiction. Therefore E is strictly semistable and $[E] = [A_2 \oplus A_2^{-1}]$ for some 4-torsion line bundle A_2 with $A_2^{\otimes 2} = A_1$. The S-equivalence class $[A_2 \oplus A_2^{-1}]$ contains three isomorphism classes, and a standard computation shows that only the decomposable bundle $A_2 \oplus A_2^{-1}$ is mapped by F^* to $A_1 \oplus A_1$. In particular there are no F^2-trivial rank-2 bundles.

We now determine the coordinates of A_2 by intersecting the line $\mathbb{L}$, which can be parametrized by $(r : r : s : s)$ with $r, s \in k$, with the Kummer surface, whose equation is (see [LP2], Proposition 3.1)

$$c(x_{00}^2 x_{10}^2 + x_{01}^2 x_{11}^2) + b(x_{00}^2 x_{01}^2 + x_{10}^2 x_{11}^2) + a(x_{00}^2 x_{11}^2 + x_{10}^2 x_{01}^2) + x_{00}x_{01}x_{10}x_{11} = 0.$$

The computations are straightforward and will be omitted. Let $u \in k$ be a root of the equation

$$u^2 + u = b. \tag{4.3}$$

Then $u + 1$ is the other root. The coordinates of the two 4-torsion line bundles (modulo the canonical involution of the Jacobian of X) A_2 such that $A_2^{\otimes 2} = A_1$ are

$$(u : u : \sqrt{b} : \sqrt{b}) \qquad \text{and} \qquad (u + 1 : u + 1 : \sqrt{b} : \sqrt{b}).$$

Now the equation $u = 0$ (resp. $u + 1 = 0$) implies by (4.3) that $b = 0$, which is excluded because we have assumed X smooth. So by Proposition 3.1 the preimage $V^{-1}(A_2)$ consists of the 4 line bundles A_3 such that $A_3^{\otimes 2} = A_2$. In particular there are no F^3-trivial rank-2 bundles.

One easily verifies that the image under the rational map V given by (3.2) of the hyperplane $x_{00} = 0$ is the quartic surface given by the equation

$$b x_{11}^2 x_{10}^2 + c x_{11}^2 x_{01}^2 + a x_{10}^2 x_{01}^2 + x_{00}x_{10}x_{01}x_{11} = 0. \tag{4.4}$$

When we replace $(x_{00} : x_{01} : x_{10} : x_{11})$ with $(u : u : \sqrt{b} : \sqrt{b})$ in (4.4) we obtain the equation

$$b^2 + u^2(1 + a + c) = 0. \tag{4.5}$$

Similarly replacing $(x_{00} : x_{01} : x_{10} : x_{11})$ with $(u + 1 : u + 1 : \sqrt{b} : \sqrt{b})$ in (4.4) we obtain the equation

$$b^2 + (u^2 + 1)(1 + a + c) = 0. \tag{4.6}$$

Finally the product of (4.5) with (4.6) equals (here one uses (4.3)) equation (4.1) up to a factor b^2, which we can drop since $b \neq 0$ — note that we have assumed X smooth, hence $b \neq 0$ by [LP2, Lemma 2.1]. To summarize we have shown that if (4.1) holds, then by Proposition 3.1 there exists an 8-torsion line bundle A_3 with $A_3^{\otimes 4} = A_1$ and such that the preimage $V^{-1}(A_3)$ is a projective line $\Delta \subset \mathbb{P}^3$.

Consider a point $[E] \in \Delta$ away from the Kummer surface — note that Δ is not contained in the Kummer surface Kum_X because its intersection is contained in the set of 16-torsion points. Then E is stable and $[F^*E] = [A_3 \oplus A_3^{-1}]$. There are three isomorphism classes represented by the S-equivalence class $[A_3 \oplus A_3^{-1}]$, namely the trivial extension $A_3 \oplus A_3^{-1}$ and two nontrivial extensions (for the details see [LP1, Remark 6.2]). Since E is invariant under the hyperelliptic involution we obtain $F^*E = A_3 \oplus A_3^{-1}$ and finally that E is F^4-trivial. Hence any stable point on Δ is F^4-trivial.

Therefore, assuming (4.1), there exists a 1-dimensional subvariety $\Delta_0 \subset \mathcal{M}_X \setminus \mathrm{Kum}_X$ parametrizing all F^4-trivial rank-2 bundles. Passing to an étale cover $S \to \Delta_0$ ensures existence of a "universal" family $\mathcal{E} \to X \times S$ and we are done. □

Remark. Note that equation (4.1) depends on the choice of a nontrivial 2-torsion line bundle A_1. If one chooses the 2-torsion line bundle $(0:0:1:0)$ or $(0:0:0:1)$ (see (4.2)), then the correponding equations are

$$a^2 + b^2 + c^2 + a + b = 0 \qquad \text{or} \qquad a^2 + b^2 + c^2 + b + c = 0.$$

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