ORDER-WEAKLY COMPACT OPERATORS FROM VECTOR-VALUED FUNCTION SPACES TO BANACH SPACES

MARIAN NOWAK

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ABSTRACT. Let $E$ be an ideal of $L^0$ over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$, and let $E^*$ stand for the order dual of $E$. For a real Banach space $(X, \| \cdot \|_X)$ let $E(X)$ be a subspace of the space $L^0(X)$ of $\mu$-equivalence classes of strongly $\Sigma$-measurable functions $f: \Omega \to X$ and consisting of all those $f \in L^0(X)$ for which the scalar function $\|f(\cdot)\|_X$ belongs to $E$. For a real Banach space $(Y, \| \cdot \|_Y)$ a linear operator $T: E(X) \to Y$ is said to be order-weakly compact whenever for each $u \in E^+$ the set $T(\{f \in E(X) : \|f(\cdot)\|_X \leq u\})$ is relatively weakly compact in $Y$. In this paper we examine order-weakly compact operators $T: E(X) \to Y$. We give a characterization of an order-weakly compact operator $T$ in terms of the continuity of the conjugate operator of $T$ with respect to some weak topologies. It is shown that if $(E, \| \cdot \|_E)$ is an order continuous Banach function space, $X$ is a Banach space containing no isomorphic copy of $l^1$ and $Y$ is a weakly sequentially complete Banach space, then every continuous linear operator $T: E(X) \to Y$ is order-weakly compact. Moreover, it is proved that if $(E, \| \cdot \|_E)$ is a Banach function space, then for every Banach space $Y$ any continuous linear operator $T: E(X) \to Y$ is order-weakly compact if the norm $\| \cdot \|_E$ is order continuous and $X$ is reflexive. In particular, for every Banach space $Y$ any continuous linear operator $T: L^1(X) \to Y$ is order-weakly compact if $X$ is reflexive.

1. INTRODUCTION AND PRELIMINARIES

P. G. Dodds [D] considered order-weakly compact operators from a vector lattice $E$ to a Banach space $Y$. Recall that a linear operator $T: E \to Y$ is called order-weakly compact if the set $T([-u, u])$ is relatively-weakly compact in $Y$ for every $u \in E^+$. Some further properties of these operators can be found in [M] Section 3.4]. M. Duhoux [Du] extended Dodd’s results to the setting $Y$ being a locally convex space. Z. Ercan [E] examined some properties of order-weakly compact operators from a vector lattice $E$ to a topological vector space $Y$.

In this paper, we consider order-weakly compact operators from a vector-valued function space $E(X)$ to a Banach space $Y$.
For terminology concerning Riesz spaces and function spaces we refer to [AB1], [AB2] and [KÄ]. Given a topological vector space \((L, \tau)\), by \((L, \tau)^*\) we will denote its topological dual. We denote by \(\sigma(L, K)\), \(\beta(L, K)\) and \(\tau(L, K)\) the weak topology, the strong topology and the Mackey topology, respectively, for a dual system \((L, K)\). By \(\mathbb{N}\) and \(\mathbb{R}\) we will denote the sets of all natural and real numbers, respectively.

Throughout the paper we assume that \((\Omega, \Sigma, \mu)\) is a complete \(\sigma\)-finite measure space and \(L^0\) denotes the corresponding space of \(\mu\)-equivalence classes of all \(\Sigma\)-measurable real-valued functions. Let \(E\) be an ideal of \(L^0\) with \(\text{supp} E = \Omega\) and let \(E^-\) stand for the order dual of \(E\).

Let \((X, \| \cdot \|_X)\) be a real Banach space, and let \(S_X\) stand for the unit sphere of \(X\). By \(L^0(X)\) we denote the set of \(\mu\)-equivalence classes of all strongly \(\Sigma\)-measurable functions \(f: \Omega \rightarrow X\). For \(f \in L^0(X)\) let us set \(\hat{f}(\omega) := \|f(\omega)\|_X\) for \(\omega \in \Omega\). Let
\[
E(X) = \{ f \in L^0(X) : \hat{f} \in E \}.
\]

For each \(u \in E^+\) the set \(D_u = \{ f \in E(X) : \hat{f} \leq u \}\) will be called an order interval in \(E(X)\).

Following [D] we are now ready to define two classes of linear operators.

**Definition 1.1.** Let \(E\) be an ideal of \(L^0\), and let \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) be real Banach spaces. A linear operator \(T: E(X) \rightarrow Y\) is said to be **order-weakly compact** (resp. **order-bounded**) whenever for each \(u \in E^+\) the set \(T(D_u)\) is relatively-weakly compact (resp. norm-bounded) in \(Y\).

Clearly each order-weakly compact operator \(T: E(X) \rightarrow Y\) is order-bounded.

Order-bounded operators \(T: E(X) \rightarrow Y\) have been considered in [N5].

Now we recall some terminology and results concerning the duality theory of the function spaces \(E(X)\) as set out in [B1], [BL], [N1], [N2].

For a linear functional \(F\) on \(E(X)\) let us put
\[
|F|(f) = \sup \{ |F(h)| : h \in E(X), \tilde{h} \leq \tilde{f} \} \quad \text{for } f \in E(X).
\]

The set
\[
E(X)^\sim = \{ F \in E(X)^\# : |F|(f) < \infty \quad \text{for all } f \in E(X) \}
\]
will be called the order dual of \(E(X)\) (here \(E(X)^\#\) denotes the algebraic dual of \(E(X)\)).

For \(F_1, F_2 \in E(X)^\sim\) we will write \(|F_1| \leq |F_2|\) whenever \(|F_1|(f) \leq |F_2|(f)\) for all \(f \in E(X)\). A subset \(A\) of \(E(X)^\sim\) is said to be **solid** whenever \(|F_1| \leq |F_2|\) with \(F_1 \in E(X)^\sim\) and \(F_2 \in A\) imply \(F_1 \in A\).

In particular, for a Banach function space \((E, \| \cdot \|_E)\) the space \(E(X)^\sim\) provided with the norm \(\|f\|_{E(X)^\sim} := \|\tilde{f}\|_E\) is a Banach space, and it is usually called a **Köthe-Bochner space**. It is well known that \((E(X), \| \cdot \|_{E(X)^\sim}) = E(X)^\sim\) (see [BL] §3, Lemma 12).

For each \(f \in E(X)\) let
\[
\rho_f(F) = |F|(f) \quad \text{for } F \in E(X)^\sim.
\]
We define the **absolute weak topology** \(\sigma(E(X)^\sim, E(X))\) on \(E(X)^\sim\) as a locally convex topology generated by the family \(\{ \rho_f : f \in E(X) \}\) of seminorms. Clearly, \(\sigma(E(X)^\sim, E(X))\) is the topology of uniform convergence on the family of all order intervals \(D_u\), where \(u \in E^+\).
Now let $\mathcal{B}_0$ be the family of all absolutely convex subsets of $E(X)$ that absorb every order interval in $E(X)$. Then $\mathcal{B}_0$ is a local base at zero for a locally convex topology $\tau_0$ on $E(X)$ (called an order-bounded topology), which is the finest locally convex topology on $E(X)$ for which every order interval is a bounded set (see [Na], [N5]).

The following characterization of $\tau_0$ and order-bounded operators $T : E(X) \to Y$ will be needed (see [N5, Corollary 2.2] and Theorem 2.3).

**Theorem 1.2.** The order-bounded topology $\tau_0$ on $E(X)$ coincides with the Mackey topology $\tau(E(X), E(X)^\sim)$, i.e., $\tau_0 = \tau(E(X), E(X)^\sim)$. For a linear operator $T : E(X) \to Y$ the following statements are equivalent:

(i) $T$ is order-bounded;

(ii) $T$ is $(\tau(E(X), E(X)^\sim), \|\cdot\|_Y)$-continuous;

(iii) $T$ is $(\sigma(E(X), E(X)^\sim), \sigma(Y,Y^*))$-continuous.

Moreover, for a Banach function space $(E, \|\cdot\|_E)$, the statements (i)–(iii) are equivalent to the following:

(iv) $T$ is $(\|\cdot\|_{E(X)}^\sim, \|\cdot\|_Y)$-continuous.

2. Characterization of order–weakly compact operators

Let $\tau$ be a linear topology on $E(X)$. Recall that a linear operator $T : E(X) \to Y$ is $\tau$-weakly compact whenever there exists a neighbourhood $U$ of 0 for $\tau$ such that the set $T(U)$ is relatively-weakly compact in $Y$.

In view of the definition of the order-bounded topology $\tau_0$ on $E(X)$ we can easily observe that every $\tau_0$-weakly compact operator $T : E(X) \to Y$ is order-weakly compact. Hence, if $(E, \|\cdot\|_E)$ is a Banach function space, then $\tau_0$ coincides with the $\|\cdot\|_{E(X)}$-topology, so every weakly compact operator $T : E(X) \to Y$ is order-weakly compact. In particular, a linear operator $T : L^\infty(X) \to Y$ is order-weakly compact if and only if $T$ is weakly compact (because the unit ball in $L^\infty(X)$ coincides with the order interval $D_{10}$).

In this section we characterize order-weakly operators $T : E(X) \to Y$ in terms of the continuity of the conjugate operator $T^*$ with respect to appropriate weak topologies. We start by recalling some concepts and results of the duality theory of the spaces $E(X)$ (see [BL1], [BL2], [N1], [N2]).

For a linear functional $V$ on $E(X)^\sim$ let us put:

$$|V|(F) = \sup \{ |V(G)| : G \in E(X)^\sim, |G| \leq |F| \} \quad \text{for} \quad F \in E(X)^\sim .$$

The set

$$(E(X)^\sim)^\# = \{ V \in (E(X)^\sim)^\# : |V|(F) < \infty \quad \text{for all} \quad F \in E(X)^\sim \}$$

will be called the order dual of $E(X)^\sim$ (here $(E(X)^\sim)^\#$ denotes the algebraic dual of $E(X)^\sim$).

For $V_1, V_2 \in (E(X)^\sim)^\#$ we will write $|V_1| \leq |V_2|$ whenever $|V_1|(F) \leq |V_2|(F)$ for all $F \in E(X)^\sim$. A subset $K$ of $(E(X)^\sim)^\#$ is said to be solid whenever $|V_1| \leq |V_2|$ with $V_1 \in (E(X)^\sim)^\#$, $V_2 \in K$ imply $V_1 \in K$. A linear subspace $L$ of $(E(X)^\sim)^\#$ is called an ideal of $(E(X)^\sim)^\#$ if $L$ is a solid subset of $(E(X)^\sim)^\#$.

For each $f \in E(X)$ let us put

$$\pi_f(F) = F(f) \quad \text{for all} \quad F \in E(X)^\sim.$$
One can show (see [N2]) that for \( f \in E(X) \),
\[
|\pi_f| (F) = |F| (f) \quad \text{for} \quad F \in (E(X))^\sim \quad \text{and} \quad \pi_f \in (E(X))^\sim.
\]

Thus we have a natural embedding \( \pi : E(X) \ni f \mapsto \pi_f \in (E(X))^\sim \). Denote by \( E(X)_0 \) the ideal of \((E(X))^\sim\) generated by the set \( \pi(E(X)) \) (i.e., \( E(X)_0 \) is the smallest ideal of \((E(X))^\sim\) containing \( \pi(E(X)) \)).

**Theorem 2.1** (see [N2] Theorem 3.2). We have
\[
(E(X))^\sim, |\sigma|(E(X)^\sim, E(X)))^* = E(X)_0
\]
\[
= \{ V \in (E(X))^\sim : |V| \leq |\pi_f| \quad \text{for some} \ f \in E(X) \}.
\]

For \( u \in E^+ \) and \( f \in E(X) \) let:
\[
C_u := \pi(D_u) = \{ \pi_y : h \in E(X), \tilde{h} \leq u \} \quad (\text{an interval in} \ (E(X))) ,
\]
\[
I_f := \{ V \in E(X)_0 : |V| \leq |\pi_f| \} \quad (\text{an interval in} \ (E(X)_0) .
\]

The following properties of \( C_u \) and \( I_f \) will be needed.

**Theorem 2.2** ([N2] Theorem 2.4). Let \( f \in E(X) \). Then

(i) \( I_f \) is \( \sigma(E(X)_0, E(X)^\sim)\)-compact in \( E(X)_0 \);

(ii) \( C_f \) is \( \sigma(E(X)_0, E(X)^\sim)\)-dense in \( I_f \).

Assume now that a linear operator \( T : E(X) \to Y \) is order-bounded. Then in view of Theorem 1.1, \( T \) is \( (\sigma(E(X), E(X)^\sim), \sigma(Y,Y^*))\)-continuous. Hence, we can consider the linear mapping
\[
T^\sim : Y^* \to (E(X))^\sim
\]
defined by
\[
T^\sim(y^*)(f) = y^*(T(f)) \quad \text{for} \quad y^* \in Y \quad \text{and all} \quad f \in E(X).
\]

Then \( T^\sim \) is \( (\beta(Y^*, Y), \beta(E(X)^\sim, E(X)))\)-continuous; hence \( T^\sim \) is also \( (\beta(Y^*, Y), |\sigma|(E(X)^\sim, E(X)))\)-continuous (because \( |\sigma|(E(X)^\sim, E(X)) \subset \beta(E(X)^\sim, E(X)) \); see [N2] §4). Since \( (E(X)^\sim, |\sigma|(E(X)^\sim, E(X)))^* = E(X)_0 \) (see Theorem 2.1) and \( (Y^*, \beta(Y^*, Y))^* = Y^{**} \), we see that \( T^\sim \) is also \( (\sigma(Y^*, Y^{**}), \sigma(E(X)^\sim, E(X)_0))\)-continuous. Finally, we consider the linear mapping
\[
T^\sim\sim : E(X)_0 \to Y^{**}
\]
defined by
\[
T^\sim\sim(V)(y^*) = V(T^\sim(y^*)) \quad \text{for} \quad V \in E(X)_0 \quad \text{and all} \quad y^* \in Y^*,
\]
which is \( (\sigma(E(X)_0, E(X)^\sim), \sigma(Y^{**}, Y^*))\)-continuous.

Let \( i : Y \ni y \mapsto i_y \in Y^{**} \) stand for the canonical isometry, i.e.,
\[
i_y(y^*) = y^*(y) \quad \text{for} \quad y^* \in Y^*.
\]

It is seen that \( T^\sim\sim \circ \pi = i \circ T \), where \( \pi : E(X) \to (E(X)^\sim) \) is a natural embedding.

Now we are ready to state a “vector-valued version” of a characterization of order-weakly compact operators \( T : E \to Y \) (see [D] Theorem 4.2).

**Theorem 2.3.** For an order-bounded operator \( T : E(X) \to Y \) the following statements are equivalent:

(i) \( T \) is order-weakly compact;
Theorem 3.1. Let $\sigma(E(X)_0, E(X))$ be a Banach space containing no isomorphic copy of $\ell^1$. Then for each $u \in E^+$ the order interval $D_u$ is relatively-weakly compact in $E(X)$.

As an application of Theorem 3.1 we get:

$$
\text{Order-weakly compact operators from K"othe-Bochner spaces}
$$

3. ORDER-WEAKLY COMPACT OPERATORS FROM KÖTHE-BOCHNER SPACES TO BANACH SPACES

In this section we assume that $(E, \| \cdot \|_E)$ is a Banach function space. Then $E(X)\sim = (E(X), \| \cdot \|_{E(X)})$. In view of Theorem 1.1 for a Banach space $Y$ a linear operator $T : E(X) \rightarrow Y$ is order-bounded if and only if $T$ is continuous (i.e., $(\| \cdot \|_{E(X)}, \| \cdot \|_Y)$-continuous).

Recall that a subset $A$ of the Banach space is said to be conditionally-weakly compact whenever each sequence in $A$ contains a weakly Cauchy subsequence.

Making use of [N3, Corollary 2.8] we get:

**Theorem 3.1.** Let $(E, \| \cdot \|_E)$ be an order continuous Banach function space and $X$ a Banach space containing no isomorphic copy of $\ell^1$. Then for each $u \in E^+$ the order interval $D_u$ is conditionally-weakly compact in $E(X)$.
Theorem 3.2. Let \((E, \| \cdot \|_E)\) be an order continuous Banach function space, \(X\) a Banach space containing no isomorphic copy of \(l^1\) and \(Y\) a weakly sequentially complete Banach space. Then every continuous linear operator \(T : E(X) \to Y\) is order-weakly compact.

Proof. Let \(u \in E^+\). Then by Theorem 3.1, \(D_u\) is a conditionally-weakly compact set in \(E(X)\). It follows that the set \(T(D_u)\) is conditionally-weakly compact in \(Y\). Since \(Y\) is supposed to be weakly sequentially complete, we conclude that \(T(D_u)\) is relatively-weakly sequentially compact in \(Y\). Hence \(T(D_u)\) is relatively-weakly compact in \(Y\). □

In particular, we have:

Corollary 3.3. Let \(X\) be a Banach space containing no isomorphic copy of \(l^1\) and let \(Y\) be a weakly sequentially complete Banach space. Then every continuous linear operator \(T : L^p(X) \to Y\) \((1 \leq p < \infty)\) is order-weakly compact.

It is well known that a Lebesgue-Bochner space \(L^q(Y)\) \((1 \leq q < \infty)\) is weakly sequentially complete whenever a Banach space \(Y\) is weakly sequentially complete (see [T, Theorem 11]). Hence, as a consequence of Corollary 3.3 we get:

Corollary 3.4. Let \(X\) be a Banach space containing no isomorphic copy of \(l^1\) and let \(Y\) be a weakly sequentially complete Banach space. Then every continuous linear operator \(T : L^p(X) \to L^q(Y)\) \((1 \leq p < \infty, 1 \leq q < \infty)\) is order-weakly compact.

Now, we present necessary and sufficient conditions for the order-weak compactness of any continuous operator \(T : E(X) \to Y\) with any Banach space \(Y\). For this purpose we shall need the following characterization of weak compactness of order intervals in \(E(X)\).

Theorem 3.5 (see [BL, §4, Corollary 1], [B2, Proposition 2], [N4, Theorem 2.4]). Let \((E, \| \cdot \|_E)\) be a Banach function space and \(X\) a Banach space. Then the following statements are equivalent:

(i) the norm \(\| \cdot \|_E\) is order continuous and \(X\) is reflexive;
(ii) for each \(u \in E^+\) the order interval \(D_u\) is weakly compact.

Theorem 3.6. Let \((E, \| \cdot \|_E)\) be a Banach function space and \(X\) a Banach space. Then the following statements are equivalent:

(i) the norm \(\| \cdot \|_E\) is order continuous and \(X\) is reflexive;
(ii) for every Banach space \(Y\), any continuous linear operator \(T : E(X) \to Y\) is order-weakly compact.

Proof. (i) \(\Rightarrow\) (ii) This follows from Theorem 3.5.

(ii) \(\Rightarrow\) (i) Assume that (ii) holds and let \((Y, \| \cdot \|_Y) = (E(X), \| \cdot \|_{E(X)})\). Then the identity operator \(id : E(X) \to E(X)\) is order-weakly compact. It follows that for each \(u \in E^+\) the set \(D_u\) is weakly compact in \(E(X)\), so by Theorem 3.5 the norm \(\| \cdot \|_E\) is order continuous and \(X\) is reflexive. □

In particular, as an application of Theorem 3.6 we get:

Corollary 3.7. For a Banach space \(X\) the following statements are equivalent:

(i) \(X\) is reflexive;
(ii) for every Banach space \(Y\), any continuous linear operator \(T : L^p(X) \to Y\) \((1 \leq p < \infty)\) is order-weakly compact.
References


