CHARACTERS OF ASSOCIATION SCHEMES CONTAINING A STRONGLY NORMAL CLOSED SUBSET OF PRIME INDEX

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ABSTRACT. We consider a relation between characters of an association scheme and its strongly normal closed subsets with prime index. As an application of our result, we show that an association scheme of prime square order with a proper strongly normal closed subset is commutative.

1. INTRODUCTION

It seems to be natural to consider relations between characters of an association scheme and its closed subsets. In [3], the author showed that Clifford theory works for commutative schemes and their strongly normal closed subsets very well. For non-commutative schemes, Clifford theory does not work so well. But for characters of association schemes and their strongly normal closed subsets, we can apply the theory of group-graded algebras by Dade [2].

Let (X, S) be an association scheme in the sense of [11], and T a strongly normal closed subset of S. For $x \in X$, the adjacency algebra of the subscheme $(X, S)_{xT}$, defined in [11, §1.5], is isomorphic to $\mathbb{C}T$. So we may regard $\mathbb{C}T$ as an adjacency algebra. In this article, we consider a relation between irreducible characters of S and T, under the assumption that the factor scheme $(X, S)^T$, defined in [11, §1.5], is thin of prime order. Our result is a generalization of [9, Corollary 6.9] for group characters. But there are interesting cases which do not occur for group characters. We will construct such concrete examples. As an application, we will also show that (X, S) is commutative if it has prime square order and there exists a proper strongly normal closed subset of S.

2. PRELIMINARIES

Let (X, S) be an association scheme. For $s \in S$, put $s^* = \{(y, x) \mid (x, y) \in s\}$. The valency of $s \in S$ will be denoted by n_s. Also, for a subset U of S, put $n_U = \sum_{u \in U} n_u$. Especially, $n_S = |X|$, and this number will be called the *order* of S. For a closed subset T of S, n_S/n_T will be called the *index* of T in S. The adjacency matrix of $s \in S$ is denoted by σ_s. For a subset U of S, we put $\mathbb{C}U = \bigoplus_{u \in U} \mathbb{C}\sigma_u$

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as a subset of the adjacency algebra $\mathbb{C}S$. It is known that the adjacency algebra over the complex number field $\mathbb{C}$ is a semisimple algebra. The set of irreducible complex characters of S will be denoted by $\mathrm{Irr}(S)$. The map $\sigma_s \mapsto n_s$ is always an irreducible character of S. We call this the *trivial character* of S and denote it by 1_S. The central primitive idempotent corresponding to the trivial character is $n_S{}^{-1} \sum_{s \in S} \sigma_s$. So $\sum_{s \in S} \sigma_s$ annihilates the irreducible $\mathbb{C}S$-module affording any non-trivial character.

Let (X, S) be an association scheme, and T a strongly normal closed subset of S. Put $G = S//T$ and regard it as a finite group. We use Dade's notation in [2]. Then the adjacency algebra $\mathbb{C}S$ is a G-graded algebra with the grading

$$\mathbb{C}S = \bigoplus_{s^T \in S//T} \mathbb{C}(TsT).$$

Let φ be an irreducible character of T, and L an irreducible right $\mathbb{C}T$-module affording φ. The induction of L to S is defined by

$$L^S = L \otimes_{\mathbb{C}T} \mathbb{C}S = \bigoplus_{s^T \in S//T} L \otimes \mathbb{C}(TsT).$$

Now $L \otimes \mathbb{C}(TsT)$ is a $\mathbb{C}T$-module for any $s \in S$. Define the *support* $\mathrm{Supp}(L^S)$ of L^S by

$$\mathrm{Supp}(L^S) = \{s^T \in S//T \mid L \otimes \mathbb{C}(TsT) \neq 0\},$$

and define the *stabilizer* $G\{L\}$ of L in $G = S//T$ by

$$G\{L\} = \{s^T \in S//T \mid L \otimes \mathbb{C}(TsT) \cong L \ \ (\text{as } \mathbb{C}T\text{-modules})\}.$$

Then $G\{L\}$ is a subgroup of G, but $\mathrm{Supp}(L^S)$ need not be a subgroup of G. In general, $\mathrm{Supp}(L^S)$ is a union of left cosets of $G\{L\}$ in G. It is shown in [2, Proposition 7.8] that $L \otimes \mathbb{C}(TsT)$ is an irreducible $\mathbb{C}T$-module for any $s^T \in \mathrm{Supp}(L^S)$.

We put $\mathrm{Supp}(\varphi^S) = \mathrm{Supp}(L^S)$ and $G\{\varphi\} = G\{L\}$. We write φ^{s^T} for the character afforded by $L \otimes \mathbb{C}(TsT)$ for $s^T \in \mathrm{Supp}(\varphi^S)$.

For a character φ of T we write φ^S for the induced character of φ to S, and for a character χ of S we write χ_T for the restriction of χ to T. We will use the following Frobenius reciprocity for characters of an association scheme and its closed subset. Let $\chi \in \mathrm{Irr}(S)$ and $\varphi \in \mathrm{Irr}(T)$. Then the multiplicity of χ in φ^S is equal to the multiplicity of φ in χ_T. For example, this is shown by [10, Theorem 1.11.1] directly.

3. Strongly normal closed subsets of prime index

Let (X, S) be an association scheme, and T a strongly normal closed subset of S. Suppose $G = S//T$ is isomorphic to the cyclic group of prime order p. Write $\mathrm{Irr}(G) = \{\xi_i \mid i = 1, 2, \cdots, p\}$. For $\chi \in \mathrm{Irr}(S)$, we define the product $\chi\xi_i$ by $\chi\xi_i(\sigma_s) = \chi(\sigma_s)\xi_i(\sigma_{s^T})$. Then $\chi\xi_i \in \mathrm{Irr}(S)$ by [5].

Lemma 3.1. *Let φ be an irreducible character of T, and let χ be an irreducible constituent of φ^S. Suppose $\chi(\sigma_s) \neq 0$ for some $s \in S - T$. Then $\varphi^S = \sum_{i=1}^{p} \chi\xi_i$.*

Proof. It is easy to see that $\chi\xi_i$ $(i = 1, 2, \cdots, p)$ are all distinct. For every i, $(\chi\xi_i)_T = \chi_T$ contains φ as an irreducible constituent. By the Frobenius reciprocity, φ^S contains $\chi\xi_i$ for all i. Since $(\varphi^S)_T$ is a sum of at most p irreducible characters, so is φ^S. This shows the lemma. $\square$

Lemma 3.2. *For $\varphi \in \mathrm{Irr}(T)$, suppose $G\{\varphi\} = G$. Then $\varphi^S = \sum_{i=1}^p \chi \xi_i$ for some $\chi \in \mathrm{Irr}(S)$ with $\chi_T = \varphi$.*

Proof. Let $\{\chi_j \mid j = 1, 2, \cdots, r\}$ be the set of irreducible constituents of φ^S. If $\chi_j(\sigma_s) \neq 0$ for some j and some $s \in S - T$, then Lemma 3.1 shows the lemma. So we may assume $\chi_j(\sigma_s) = 0$ for every j and every $s \in S - T$. Then $(\chi_j)_T = (\chi_j(1)/\varphi(1))\varphi$ shows that $\{\chi_j \mid j = 1, 2, \cdots, r\}$ is linearly dependent if $r > 1$. But any subset of $\mathrm{Irr}(S)$ is linearly independent, so we have $r = 1$. If $\varphi^S = a\chi_1$, then by Frobenius reciprocity we have $(\varphi^S)_T = a^2\varphi$. But we know that $(\varphi^S)_T = p\varphi$ by the assumption $G\{\varphi\} = G$. This is a contradiction. $\qquad\square$

Lemma 3.3. *For $\varphi \in \mathrm{Irr}(T)$, suppose $G\{\varphi\} = 1$. Then $\varphi^S \in \mathrm{Irr}(S)$. Put $\chi = \varphi^S$. Then $\chi(\sigma_s) = 0$ for any $s \in S-T$ and χ_T is a sum of $|\mathrm{Supp}(\varphi^S)|$ distinct irreducible characters of T. For every irreducible constituent ψ of χ_T, we have $\psi^S = \chi$.*

Proof. We have $(\varphi^S)_T = \sum_{s^T \in \mathrm{Supp}(\varphi^S)} \varphi^{s^T}$ and all φ^{s^T} are distinct. If there exist an irreducible constituent χ of φ^S and an element $s \in S - T$ such that $\chi(\sigma_s) \neq 0$, then Lemma 3.1 shows that $(\varphi^S)_T = p\varphi$. This is impossible, so we can say that $\chi(\sigma_s) = 0$ for every $s \in S - T$ and every irreducible constituent χ of φ^S. By the same argument in the proof of Lemma 3.2, there exists only one irreducible constituent χ of φ^S. Since the multiplicity of φ in $(\varphi^S)_T$ is one, we have $\varphi^S = \chi$. The rest of the statements in the lemma are clear by [2, §7]. $\qquad\square$

Since $G = S//T$ is a cyclic group of prime order, $G\{\varphi\} = 1$ or $G\{\varphi\} = G$ holds. Combining Lemmas 3.2 and 3.3, we have the following.

Theorem 3.4. *Let (X, S) be an association scheme, and T a strongly normal closed subset of S. Suppose $G = S//T$ is isomorphic to the cyclic group of prime order p. Write $\mathrm{Irr}(G) = \{\xi_i \mid i = 1, 2, \cdots, p\}$. Then, for $\chi \in \mathrm{Irr}(S)$, one of the following statements holds:*

   (1) *$\chi_T \in \mathrm{Irr}(T)$ and $(\chi_T)^S = \sum_{i=1}^p \chi \xi_i$.*
   (2) *$\chi(\sigma_s) = 0$ for any $s \in S-T$ and χ_T is a sum of at most p distinct irreducible characters. If ψ is an irreducible constituent of χ_T, then $\psi^S = \chi$.*

Now we give examples of Theorem 3.4. The cases Theorem 3.4 (1) and (2) with $\mathrm{Supp}(\varphi^S) = G$ appear in the usual Clifford theory for finite groups. Theorem 3.4 (2) with $\mathrm{Supp}(\varphi^S) = 1$ occurs for non-trivial characters of wreath products of association schemes [7]. We are interested in the case $G\{\varphi\} = 1$ and $1 \subsetneq \mathrm{Supp}(\varphi^S) \subsetneq G$. In this case, $\mathrm{Supp}(\varphi^S)$ is not a subgroup of G since we are assuming that G is a cyclic group of order p.

Example 3.5. Let $\mathbb{F}$ be the field with 2^q elements. To simplify our argument, we suppose $p = 2^q - 1$ is a prime number. The multiplicative group $\mathbb{F}^\times = \mathbb{F} - \{0\}$ acts on $\mathbb{F}$ by multiplication. Let H be the semidirect product of the additive group $\mathbb{F}$ by $\mathbb{F}^\times$. Then H is a Frobenius group with kernel $\mathbb{F}$. Let K be a subgroup of $\mathbb{F}$ of order 2^r. We can define a Schurian scheme (X, S) by H and its subgroup K. Namely (X, S) is the factor scheme of the thin scheme defined by the group H by its closed subset K. Let T be the image of $\mathbb{F}$ in S. Then T is a strongly normal closed subset of prime index p. Let χ be the unique nonlinear character of H. By [1, Theorem 11.25] or [6, Theorem 3.8], χ defines an irreducible character χ' of S. Now we have $|\mathrm{Supp}(\varphi^S)| = \chi'(1) = 2^{q-r} - 1$ for any irreducible constituent φ of χ'_T. So, if $1 < q - r < q$, then $1 \subsetneq \mathrm{Supp}(\varphi^S) \subsetneq S//T$.

4. Association schemes of prime square order

In [8, Theorem 3.3], it is shown that the following theorem holds.

Theorem 4.1 (Hanaki-Uno). *Let (X, S) be an association scheme. If the order of (X, S) is a prime number, then (X, S) is commutative. Moreover, all non-trivial irreducible characters of S are algebraically conjugate.*

It is natural to ask whether an association scheme (X, S) of prime square order is commutative. No counterexample to the question is known. In this section, we will give a partial answer to this question as an application of the results in the previous section. Namely the following theorem holds.

Theorem 4.2. *Let p be a prime number, (X, S) an association scheme of order p^2 which has a proper strongly normal closed subset T. Then (X, S) is commutative. Moreover, if $n_T = p$, then $|S| = p|T|$ or $|S| = |T| + p - 1$.*

Proof. If $n_T = 1$, then S is thin and commutative.

We suppose $n_T = p$. Then we can apply Theorem 3.4. Note that $\mathbb{C}T$ is a commutative algebra by Theorem 4.1, so every irreducible character of T is linear. For any $\chi \in \mathrm{Irr}(S)$, there exists $\varphi \in \mathrm{Irr}(T)$ such that φ^S contains χ as an irreducible constituent. So it is enough to show that every irreducible constituent of φ^S is linear for any $\varphi \in \mathrm{Irr}(T)$.

For the trivial character 1_T of T, every irreducible constituent of $(1_T)^S$ is linear since it is essentially a character of the cyclic group of order p.

Let φ be a non-trivial irreducible character of T. We say that φ is extendible to S if there exists a character of S whose restriction to T is φ. If φ is extendible to S, then every irreducible constituent of φ^S is linear. Assume that φ is not extendible to S. Put $\chi = \varphi^S$. Then $\chi \in \mathrm{Irr}(S)$ and $\chi(1) > 1$. The number of non-trivial irreducible characters of T is at most $p - 1$, so we have $\chi(1) < p$. By Theorem 3.4, we have $\chi(\sigma_s) = 0$ for any $s \in S - T$.

For $s \in S - T$, we have that the valency n_s of s is at most p by our assumption. In [4], it is show that n_s is the unique eigenvalue of σ_s in an arbitrary field of characteristic p. So we have

$$0 = \chi(\sigma_s) \equiv \chi(1)n_s \pmod{p}.$$

Therefore $n_s = p$ holds for any $s \in S - T$. This means that $TsT = \{s\}$. Note that the inverse of the group element $s^T \in S//T$ is $(s^*)^T$. Then $\mathbb{C}(TsT)\mathbb{C}(Ts^*T) = (\mathbb{C}\sigma_s)(\mathbb{C}\sigma_{s^*}) = \mathbb{C}(\sum_{t \in T} \sigma_t)$, and this annihilates the irreducible $\mathbb{C}T$-module affording φ. Now [2, Theorem 8.6] shows that $s^T \notin \mathrm{Supp}(\varphi^S)$. So we have $\mathrm{Supp}(\varphi^S) = 1$. This contradicts $\chi(1) > 1$.

The trivial character 1_T of T has exactly p extensions to S. Any non-trivial irreducible character φ of T is extendible to S. Let χ be one of the extensions of φ. If $\chi(\sigma_s) \neq 0$ for some $s \in S - T$, then φ has exactly p extensions. Then every non-trivial irreducible character of T has exactly p extensions since they are algebraically conjugate. In this case, we have $|S| = p|T|$. If $\chi(\sigma_s) = 0$ for any $s \in S - T$, then φ has exactly one extension. Then every non-trivial irreducible character of T has exactly one extension, and we have $|S| = |T| + p - 1$. $\square$

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