AN INFINITE-DIMENSIONAL INTEGRAL IDENTITY
FOR THE SEGAL-BARGMANN TRANSFORM

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Abstract. We prove an infinite-dimensional integral identity equating the
integral of a function on a subspace of a linear space to the integral of its
Segal-Bargmann transform over the orthogonal complement.

1. Introduction

The purpose of this paper is to prove an integral identity for functions on infinite-
dimensional linear spaces which is an analog of the finite dimensional identity
\[(1.1) \int_V f(x) \, dx = \int_{V^\perp} \hat{f}(k) \, dk \]
where \(\hat{f}\) is the Fourier transform of a regular enough function \(f\) on \(\mathbb{R}^n\), and \(V\) is a
subspace of \(\mathbb{R}^n\). The simplest check on this finite-dimensional identity is obtained
by taking \(f(x) = e^{-2\pi i a \cdot x - \pi x^2}\), where \(a = (a_1, ..., a_n) \in \mathbb{R}^n\) and
\(a \cdot x = \sum_{j=1}^n a_j x_j\), for which \(\hat{f}(k) \overset{\text{def}}{=} \int f(x) e^{-2\pi i k \cdot x} \, dx = e^{-\pi (a + k)^2}\), and then both sides of \((1.1)\) work
out to the common value \(e^{-\pi a_V^2}\), with \(a_V\) being the orthogonal projection of \(a\) on
\(V\). This finite-dimensional identity appears in Hörmander [2] (Theorem 7.1.25),
where it is generalized to submanifolds of \(\mathbb{R}^n\) and several consequences developed.
Our objective in this paper is to prove an infinite-dimensional Gaussian version of
\((1.1)\); possibly, non-linear infinite-dimensional versions also exist.

In the infinite-dimensional setting the role of Lebesgue measure is played by
Gaussian measure, which is meaningful in infinite dimensions. Let \(V\) be a closed
subspace of the real, separable, Hilbert space \(H_0\). Let \(\mu_V\) be the standard Gaussian
measure on \(V\), and \(\mu_{V^\perp}\) the Gaussian measure on \(V^\perp\); technically, these measures
live on a certain space of distributions we denote \(\mathcal{H}'\). The main result of this paper
states that the identity
\[ \int_{\mathcal{H}'} F \, d\mu_V = \int_{\mathcal{H}'} SF(ik) \, d\mu_{V^\perp}(k) \]
holds for any test function \(F\) on \(\mathcal{H}'\). Here \(S\) is the Segal-Bargmann transform,
which takes over the role of Fourier transform in infinite-dimensional analysis.

Sections 2 and 3 summarize the essential notions and results necessary to for-
mulate and prove our main result, Theorem 4.1. Section 4 contains the formal
statement and proof of the identity, using some auxiliary results presented in Section 3.

2. Test functions over infinite-dimensional spaces

In this section we summarize the necessary notions concerning test functions on infinite-dimensional linear spaces. We also set up notation to be used in the rest of the paper.

2.1. Test functions and distributions. A distribution over a space $X$ is a continuous linear functional on a space $E$ of appropriately chosen ‘test functions’ over $X$. For analysis we would also have some measure $\mu$ on $X$ and $E = L^2(\mu)$.

The classical example is Schwartz space $S(\mathbb{R}) \subset L^2(\mathbb{R})$. The topology on $E$ is given by some family of norms. Thus, in abstract, the basic framework is a pair

\[(2.1) \quad H \subset H_0, \]

where $H_0$ is a separable real Hilbert space with norm $|\cdot|_0$ and inner-product $\langle \cdot, \cdot \rangle$ and $H$ is a nuclear space. To form $H$ we take an operator $A$ on $H_0$ such that there exists an orthonormal basis \{ $e_k$ : $k = 1, 2, 3, \ldots$ \} for $H_0$ satisfying

1. $Ae_k = \lambda_k e_k$, for $k = 1, 2, 3, \ldots$,
2. $1 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$,
3. $\sum_{k=1}^{\infty} \lambda_k^{-2} < \infty$.

For the Schwartz space $S(\mathbb{R})$, the operator $A$ is the harmonic oscillator Hamiltonian $-\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{1}{2}$.

Note that $A^{-1}$ is a bounded operator with norm given by

\[(2.2) \quad \rho = \|A^{-1}\| = \frac{1}{\lambda_1}. \]

The condition that $1 < \lambda_1$ is needed when proving continuity of test functions (Theorem 2.2).

Now for each $p \geq 0$ we define the norm

\[(2.3) \quad |x|_p = |A^p x|_0 = \sqrt{\sum_{k \geq 1} \lambda_k^{2p} |\langle x, e_k \rangle|^2} \]

and let

\[(2.4) \quad H_p = \{ x \in H_0 : |x|_p < \infty \}. \]

This is a Hilbert space under the obvious inner-product $\langle \cdot, \cdot \rangle_p$, and

$H_p \subset H_q$ for any $p \geq q$

and the inclusion map $I_{p,p-1} : H_p \hookrightarrow H_{p-1}$ is a Hilbert–Schmidt operator. We then define $H$ to be the projective limit of $\{H_p : p = 0, 1, 2, \ldots\}$, and this gives us

$H = \bigcap_{p \geq 0} H_p \subset \cdots \subset H_2 \subset H_1 \subset H_0$.

Below we describe in brief how a space of test functions is constructed over the dual space $H'$ using this framework.
The symmetric Fock space $F_s(V)$ over a Hilbert space $V$ is the subspace of symmetric tensors in the completion of the tensor algebra $T(V)$ under the inner-product given by

$$\langle a, b \rangle_{T(V)} = \sum_{n=0}^{\infty} n! \langle a_n, b_n \rangle_{V^\otimes n},$$

where $a = \{a_n\}_{n \geq 0}, b = \{b_n\}_{n \geq 0}$ are elements of $T(V)$ with $a_n, b_n$ in the tensor power $V^\otimes n$. Then we have

$$F_s(H) \overset{\text{def}}{=} \cap_{p \geq 0} F_s(H_p) \subset \cdots \subset F_s(H_2) \subset F_s(H_1) \subset F_s(H_0).$$

Thus, the pair (2.1) gives rise to a corresponding pair by taking symmetric Fock spaces:

$$F_s(H) \subset F_s(H_0).$$

The dual space $H'$ of continuous real linear functionals on $H$ is the union

$$H' = \bigcup_{p \geq 0} H_{-p}$$

where $H_{-p}$ is the set of real linear functionals on $H$ which are continuous in the $|\cdot|_p$ norm. Note that $H_{-p}$ is naturally isomorphic to $H'_p \simeq H_p$ and is thus a Hilbert space. We denote the norm on $H_{-p}$ by $|\cdot|_{-p}$. Thus the inner product on $H_0$ extends to a bilinear pairing, also denoted by $\langle \cdot, \cdot \rangle$, between $H_p$ and $H_{-p}$ with

$$|\langle x, f \rangle| \leq |x|_p |f|_{-p}$$

for all $p \geq 0, x \in H_p$ and $f \in H_{-p}$. We then have a chain of inclusions

$$H_0 \simeq H_{-0} \subset H_{-1} \subset \cdots \subset \bigcup_{p \geq 0} H_{-p} = H'$$

where the inner product on $H_0$ extends to a bilinear pairing between $H$ and $H'$. The dual space $H'$ may be equipped with the weak or the strong or the inductive limit topologies.

**Fact 2.1.** The Borel sigma algebras generated by the weak, strong, and inductive topologies on $H'$ are equivalent.

Although this result is known and has been used implicitly or explicitly in the literature, a complete readily accessible proof can be found in [1].

2.2. **The Gaussian measure $\mu$.** In infinite dimensions the role of Lebesgue measure is played by Gaussian measure. The standard Gaussian measure $\mu$ for the pair (2.1) is a Borel measure on $H'$, specified uniquely by

$$\int_{H'} e^{if\hat{\phi}} \, d\mu = e^{-\langle f, \hat{\phi} \rangle / 2}$$

for all $f \in H$, where

$$\hat{\phi} : H' \to \mathbb{R} : \phi \mapsto \phi(f).$$

There is a standard unitary isomorphism, the Wiener-Itô isomorphism or wave-particle duality map, which identifies the complexified Fock space $F_s(H_0)_c$ with $L^2(H', \mu)$. This is uniquely specified by

$$I : F_s(H_0)_c \to L^2(H', \mu) : \text{Exp}(x) \mapsto e^{x - \frac{1}{2} x^2}$$

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where \( x \in \mathcal{H}, \ x^2 = |x|^2_0, \) and
\[
\text{Exp}(x) = \sum_{n \geq 0} \frac{1}{n!} x^\otimes n.
\]
Indeed, it is readily checked that \( I \) preserves inner-products (the inner-product is as described in (2.5)). Using \( I, \) for each \( \mathcal{F}_s(H_p) \) with \( p \geq 0, \) we have the corresponding space \( [H]_p \subset L^2(\mathcal{H}', \mu) \) with the norm \( \| \cdot \|_p \) induced by the norm on the space \( \mathcal{F}_s(H_p). \) From this the chain of spaces (2.6) can be transferred into a chain of function spaces:
\[
[H] = \bigcap_{p \geq 0} [H]_p \subset \cdots \subset [H]_2 \subset [H]_1 \subset [H]_0 = L^2(\mathcal{H}', \mu).
\]
Observe that \( [H] \) is a nuclear space with topology induced by the norms \( \{ \| \cdot \|_p; p = 0, 1, 2, \ldots \}. \) Thus, starting with the pair
\[
[\mathcal{H}] \subset H_0
\]
one obtains a corresponding pair
\[
[H] \subset H_0
\]
(2.10) Note that the measure \( \mu \) uses \( H_0 \) as a real Hilbert space.

The identification of \( H_0 \) with \( H_0 \) leads to a complete chain
(2.11) \[
\mathcal{H} = \bigcap_{p \geq 0} H_p \subset \cdots \subset H_1 \subset H_0 \simeq H_\infty \subset H_{-1} \subset \cdots \subset \bigcup_{p \geq 0} H_{-p} = \mathcal{H}'.
\]
In the same way we have a chain for the \('\text{second quantized}' \) spaces \( \mathcal{F}_s(H_q)_c \simeq [H]_q. \)

The unitary isomorphism \( I \) extends to unitary isomorphisms
(2.12) \[
I : \mathcal{F}_s(H_{-p})_c \to [H]_{-p} \overset{\text{def}}{=} [H]_p' \subset [\mathcal{H}]',
\]
for all \( p \geq 0. \) In more detail, for \( a \in \mathcal{F}_s(H_{-p})_c \) the distribution \( I(a) \) is specified by
(2.13) \[
\langle I(a), f \rangle = \langle a, I^{-1}(f) \rangle,
\]
for all \( f \in [H]. \) On the right side here we have the pairing of \( \mathcal{F}_s(H_{-p})_c \) and \( \mathcal{F}_s(H_p)_c \)
induced by the duality pairing of \( H_{-p} \) and \( H_p; \) in particular, the pairings above are complex bilinear (not sesquilinear).

2.3. Properties of test functions. The following theorem summarizes the properties of \([H] \) we need. The results here are standard (see, for instance, Kuo’s monograph [K]), and we compile them for ease of reference.

**Theorem 2.2.** Every function in \([H] \) is \( \mu \)-almost-everywhere equal to a unique continuous function on \( \mathcal{H}'. \) Moreover, working with these continuous versions,

(1) \( [\mathcal{H}] \) is an algebra under pointwise operations;
(2) pointwise addition and multiplication are continuous operations \( ([\mathcal{H}] \times [\mathcal{H}]) \to [\mathcal{H}]; \)
(3) for any \( \phi \in \mathcal{H}', \) the evaluation map
\[
\delta_\phi : [\mathcal{H}] \to \mathbb{R} : F \mapsto F(\phi)
\]
is continuous;
(4) the exponentials \( e^{ij-\frac{1}{2}f^2_0}, \) with \( f \) running over \( \mathcal{H}, \) span a dense subspace
of \( [\mathcal{H}]. \)
Remark 2.3. Note that (4) immediately gives us that the span of \(e^{zf}\) with \(f\) running over \(\mathcal{H}\) is dense in \([\mathcal{H}]\). It also follows from (4) that the span of \(e^{z^2}\) with \(z\) running over \(\mathcal{H}_c\) is dense in \([\mathcal{H}]\).

A complete characterization of the space \([\mathcal{H}]\) was obtained by Y. J. Lee (see the account in [3] on page 89).

2.4. The Segal-Bargmann transform. The Segal–Bargmann transform takes a function \(F \in L^2(\mathcal{H}', \mu)\) to the function \(SF\) on the complexified space \(\mathcal{H}_c\) given by

\begin{equation}
SF(z) = \int_{\mathcal{H}'} e^{\bar{z} - z^2/2} F \, d\mu, \quad z \in \mathcal{H}_c,
\end{equation}

with notation as follows: if \(z = a + ib\), with \(a, b \in \mathcal{H}\), then

\begin{equation}
\bar{z}(x) \overset{\text{def}}{=} zx \overset{\text{def}}{=} \langle x, a \rangle + i\langle x, b \rangle, \quad \text{for } x \in \mathcal{H}',
\end{equation}

and \(z^2 = zz\), where the product \(zu\) is specified through

\begin{equation}
z u \overset{\text{def}}{=} \langle a, s \rangle - \langle b, t \rangle + i\left(\langle a, t \rangle + \langle b, s \rangle\right)
\end{equation}

if \(z = a + ib\) and \(u = s + it\), where \(a, b, s, t \in \mathcal{H}\).

Let \(\mu_c\) be the Gaussian measure \(\mathcal{H}_c'\) specified by the requirement that

\begin{equation}
\int_{\mathcal{H}_c'} e^{ax + by} \, d\mu_c(x + iy) = e^{(a^2 + b^2)/4}
\end{equation}

for every \(a, b \in \mathcal{H}\). For convenience, let us introduce the renormalized exponential function

\begin{equation}
c_w = e^{\bar{w} - w^2/2}
\end{equation}

for all \(w \in \mathcal{H}_c\). Thankfully, \(c_w\) lies in \(L^2(\mathcal{H}', \mu)\), which means the integrand in (2.14) exists for all \(z \in \mathcal{H}_c\). It is readily checked that for any \(w \in \mathcal{H}_c\)

\begin{equation}
[S c_w](z) = e^{wz}, \quad \text{for all } z \in \mathcal{H}_c.
\end{equation}

Thus we may take \(S c_w\) as a function on \(\mathcal{H}_c'\) given by

\begin{equation}
S c_w = e^{\tilde{w}},
\end{equation}

where now \(\tilde{w}\) is a function on \(\mathcal{H}_c'\) in the natural way. Then \(S c_w \in L^2(\mathcal{H}_c', \mu_c)\) and one has

\(\langle S c_w, S c_u \rangle_{L^2(\mu_c)} = \langle c_w, c_u \rangle_{L^2(\mu)} = e^{w\overline{u}}\).

This shows that \(S\) provides an isometry from the linear span of the exponentials \(c_w\) in \(L^2(\mathcal{H}', \mu)\) onto the linear span of the complex exponentials \(e^{\bar{w}}\) in \(L^2(\mathcal{H}_c', \mu_c)\). Passing to the closure one obtains the Segal-Bargmann unitary isomorphism

\(S : L^2(\mathcal{H}', \mu) \rightarrow Hol^2(\mathcal{H}_c', \mu_c)\)

where \(Hol^2(\mathcal{H}_c', \mu_c)\) is the closed linear span of the complex exponential functions \(e^{\bar{w}}\) in \(L^2(\mathcal{H}_c', \mu_c)\).

An explicit expression for \(SF(z)\) is suggested by (2.14). For any \(F \in [\mathcal{H}]\) and \(z \in \mathcal{H}_c\), we have

\begin{equation}
\langle SF \rangle(z) = \langle I(\text{Exp}(z)), F \rangle
\end{equation}

where the right side is the evaluation of the distribution \(I(\text{Exp}(z))\) on the test function \(F\). Indeed it may be readily checked that if \(SF(z)\) is defined in this way, then \([S c_w](z) = e^{wz}\).
In view of (2.21), it is natural to extend the Segal-Bargmann transform to distributions: for \( \Phi \in [H]' \), define \( S\Phi \) to be the function on \( H_c \) given by
\[
S\Phi(z) \overset{\text{def}}{=} \langle \Phi, I(\text{Exp}(z)) \rangle.
\]

2.5. Images of test functions and distributions under \( S \). The following result (Theorem 8.2 on page 79 in [3]) describes the image \( S([H]') \).

**Theorem 2.4** (Potthoff–Streit). Suppose a function \( G \) on \( H_c \) satisfies:
1. for any \( z,w \in H_c \), the function \( G(\alpha z + w) \) is an entire function of \( \alpha \in \mathbb{C} \),
2. there exist nonnegative constants \( A, p, C \) such that
\[
|G(z)| \leq Ce^{A|z|^2_p} \quad \text{for all } z \in H_c.
\]
Then there is a unique distribution \( \Phi \in [H]' \) for which \( G = S\Phi \). Conversely, for any \( \Phi \in [H]' \), the function \( G = S\Phi \) satisfies (1) and (2) above.

Let us note the following bound on \( SF \), for any test function \( F \):

**Lemma 2.5.** If \( F \in [H]^p \subset L^2(\mu) \), with \( p \in \{0, 1, 2, \ldots\} \) and \( z \in H_c' \), then
\[
|SF(z)| \leq |F|_p e^{1/2|z|^2_p/2}.
\]

**Proof.** For any \( F \in [H] \) and \( z \in H_c' \):
\[
SF(z) = \langle I(\text{Exp}(z)), F \rangle.
\]
Then letting \( \| \cdot \|_{-p} \) denote the norm in \( [H]_{-p} \), we have
\[
|SF(z)| \leq |I(\text{Exp}(z))|_{-p}|F|_p = e^{1/2|z|^2_p/2}|F|_p
\]
where we have used the easily checked expression for the \( p \)-norm of \( \text{Exp}(z) \). \( \square \)

3. Measures corresponding to distributions

A Borel measure \( \nu \) for which \( [H] \subset L^1(\nu) \) and the mapping \( F \mapsto \int_{H'} F \, d\nu \) is continuous on \( [H] \) is called a Hida measure. A Hida measure induces a distribution \( \tilde{\nu} \) given by
\[
\langle \tilde{\nu}, F \rangle = \int_{H'} F \, d\nu \quad \text{for all } F \in [H].
\]
The following result characterizes Hida measures (see Theorem 15.17, page 333, in [3] for a proof of a more general form of this result):

**Theorem 3.1.** A measure \( \nu \) on \( H' \) is a Hida measure if and only if for some \( p \geq 1, H_{-p} \) is of full measure and
\[
\int_{H_{-p}} e^{1/2|z|^2_p} \, d\nu(x) < \infty.
\]

Finally we record an observation for use in the next section:

**Proposition 3.2.** Suppose \( \nu \) is a finite Borel measure on \( H' \) such that
\[
|\int_{H'} e^{z \cdot -z/2} \, d\nu| \leq e^{A|z|^2_p} \quad \text{for every } z \in H_c,
\]
for some constants \( p \geq 0 \) and \( A > 0 \). Then
\[
F \mapsto \int_{H'} F \, d\nu
\]
is a continuous linear functional on $\mathcal{H}$. In other words, there is a unique distribution $\Phi \in \mathcal{H}'$ for which

$$\langle \Phi, F \rangle = \int_{\mathcal{H}'} F \, d\nu$$

holds for every test function $F \in [\mathcal{H}]$.

Proof. Taking $G(z) = \int_{\mathcal{H}'} e^{z - z^2/2} \, d\nu$ as in Theorem 2.4, the growth bound (3.1) on the $S$-transform of the finite measure $\nu$ implies that there is a unique distribution $\Phi \in \mathcal{H}'$ such that

$$\langle \Phi, e^{i\hat{x}} \rangle = \int_{\mathcal{H}'} e^{i\hat{x}} \, d\nu$$

for every $x \in \mathcal{H}$.

Let $F \in [\mathcal{H}]$, a test function. By Theorem 2.2 there is a sequence of functions $F_n$, each a linear combination of exponentials of the form $e^{i\hat{x}}$ with $x \in \mathcal{H}$, such that $F_n \to F$ in the topology of $[\mathcal{H}]$. Again, by Theorem 2.2, with the limits being in $[\mathcal{H}]$, and hence also pointwise,

$$\lim_{n \to \infty} |F_n - F|^2 = \lim_{n \to \infty} (F_n - F)(\bar{F}_n - \bar{F}) = 0$$

and

$$\lim_{m \to \infty} |F_n - F_m|^2 = |F_n - F|^2.$$

Also, $\langle \Phi, \psi \rangle = \int \psi \, d\nu$ for every $\psi \in L$, the linear span of the exponentials $e^{i\hat{x}}$ for $x \in \mathcal{H}$. Using these facts and further applications of continuity of multiplication of test functions, we have

$$\int |F_n - F|^2 \, d\nu \leq \liminf_{m \to \infty} \int |F_n - F_m|^2 \, d\nu \quad \text{by Fatou's lemma and (3.3)}$$

$$= \liminf_{m \to \infty} \langle \Phi, |F_n - F_m|^2 \rangle \quad \text{because } |F_n - F_m|^2 \in L$$

$$= \langle \Phi, |F_n - F|^2 \rangle.$$

Letting $n \to \infty$ and using continuity of $\Phi \in [\mathcal{H}]'$ we conclude that

$$F_n \to F \text{ in } L^2(\nu), \text{ as } n \to \infty.$$

Since $\nu$ is a finite measure it follows that $\int F_n \, d\nu \to \int F \, d\nu$ and, again by continuity of $\Phi$, we also have $\langle \Phi, F_n \rangle \to \langle \Phi, F \rangle$. This proves that $\int F \, d\nu = \langle \Phi, F \rangle$. 

4. The identity

We continue to work with a pair

$$\mathcal{H} \subset H_0,$$

where $H_0$ is a real separable Hilbert space and $\mathcal{H}$ a dense subspace which is a nuclear space as described earlier. We then have the Gaussian measure $\mu$ on $\mathcal{H}'$, and the corresponding space $[\mathcal{H}]_0 = L^2(\mu)$. There is the dense subspace $[\mathcal{H}] \subset L^2(\mu)$, whose elements are our continuous test functions, and $[\mathcal{H}]$ is closed under pointwise addition and multiplication, and contains the linear span of $\{e^{i\hat{x}} : x \in \mathcal{H}\}$ as a dense subspace.
Recall that for \( f \in \mathcal{H} \) we have the evaluation function \( \hat{f} : \mathcal{H}' \to \mathbb{R} : \phi \mapsto \langle \phi, f \rangle \).
If \( V \) is a closed subspace of \( H_0 \), then there is a unique probability measure \( \mu_V \) on \( \mathcal{H}' \) specified through
\[
\int_{\mathcal{H}'} e^{ik \cdot \hat{f}} d\mu_V = e^{-\frac{1}{2}||f||_V^2} \tag{4.1}
\]
for every \( f \in \mathcal{H} \), where \( f_V \) denotes the orthogonal projection of \( f \) onto a closed subspace \( V \) of \( H_0 \). Note that if \( f \in V^\perp \), then the above Fourier transform implies that \( \hat{f} \) is zero \( \mu_V \)-a.e. Thus, in this sense, \( \mu_V \) is a Gaussian measure concentrated on the subspace \( V \). Observe that the characteristic function of \( \mu_V \) immediately implies \( \mu_V \) is a Hida measure by Proposition 3.2.

Our main result is:

**Theorem 4.1.** Let \( V \) be a closed subspace of the real, separable, Hilbert space \( H_0 \).
Let \( \mu_V \) be the Gaussian measure for \( V \), and \( \mu_V^\perp \) the Gaussian measure for \( V^\perp \), constructed as before. Then, for any test function \( F \in \mathcal{H} \) on \( \mathcal{H}' \), we have
\[
\int_{\mathcal{H}'} F d\mu_V = \int_{\mathcal{H}'} SF(ik) d\mu_V^\perp(k). \tag{4.2}
\]

**Proof:** Consider first the function \( c_z \) for \( F \), where \( c_z = e^{\frac{z}{2} + \frac{1}{2}z^2} \), where \( z \in \mathcal{H}_c \). Then the left side of (4.2) is
\[
\int_{\mathcal{H}'} c_z d\mu_V = e^{\frac{z}{2} + \frac{1}{2}z^2} e^{-\frac{1}{2}z^2} = e^{-\frac{1}{2}z^2} \phi_k \tag{4.3}
\]
for every \( f \in \mathcal{H} \), where \( \phi_k \) is a closed subspace of \( \mathcal{H}' \) and \( K \) a closed subspace of \( H_0 \), we denote by \( z_K \) the element \( a_K + ib_K \), where \( a_K \) and \( b_K \) are the respective orthogonal projections onto \( K \). The right side of (4.2) is
\[
\int_{\mathcal{H}'} SF_z(ik) d\mu_V^\perp = \int_{\mathcal{H}'} SF_z d\mu_V^\perp(k) = e^{-\frac{1}{2}z^2} \phi_k \tag{4.4}
\]
Thus our identity (4.3) holds for the functions \( c_z \), and hence for \( F \) in the linear span \( L \) of these functions. Now take \( F \) to be any test function in \([\mathcal{H}]\). Then, by Theorem 2.2, there is a sequence of functions \( F_n \in L \) which converges in the topology of \([\mathcal{H}]\) to \( F \) and hence also pointwise. Moreover, for every \( p \geq 0 \)
\[
\lim_{n \to \infty} F_n = F \quad \text{in} \quad \| \cdot \|_p \text{-norm.}
\]
From the identity
\[
\int_{\mathcal{H}'} F_n d\mu_V = \int_{\mathcal{H}'} SF_n(ik) d\mu_V^\perp(k) \tag{4.5}
\]
Proposition 3.2 implies that as \( n \to \infty \) the left side converges to \( \int_{\mathcal{H}'} F d\mu_V \). On the other hand, by the bound in Lemma 2.3, the integrands on the right are bounded by \( \|F_n\|_p e^{\frac{|k|^2}{2}} \), for every \( t \geq 0 \). As noted before \( \mu_V^\perp \) is a Hida measure, and so by Theorem 4.1 there is a \( p \geq 1 \) for which
\[
\int_{\mathcal{H}'} e^{\frac{|k|^2}{2}} d\mu_V^\perp(k) < \infty. \tag{4.6}
\]
Combining these facts we see that the dominated convergence applies to the right side of (4.5) and the latter converges to \( \int_{\mathcal{H}'} SF(ik) d\mu_V^\perp(k) \). This completes the proof of the identity. \( \square \)
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