SVEP FOR MULTIPLIERS
ON A FAITHFUL COMMUTATIVE BANACH ALGEBRA

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Abstract. We give some sufficient conditions that each multiplier on a faithful commutative Banach algebra has SVEP. On the other hand, we show that there exist a faithful commutative Banach algebra and a multiplier on it without SVEP. Such examples of multipliers can actually be found within the class of multiplication operators on unital commutative Banach algebras. This answers in negative a question that is stated as Open problem 6.2.1 by Laursen and Neumann, 2000.

1. Introduction

A mapping $T : \mathcal{A} \to \mathcal{A}$ on a commutative complex Banach algebra $\mathcal{A}$ is a multiplier if

$$a(Tb) = (Ta)b \quad \text{for all} \quad a, b \in \mathcal{A}. $$

For every element $a \in \mathcal{A}$, the multiplication operator $L_a : \mathcal{A} \to \mathcal{A}$ that is given by $L_ab = ab$ ($b \in \mathcal{A}$) is clearly a multiplier. When $\mathcal{A}$ has a unit, say $e$, the converse holds. Namely, if $T$ is a multiplier on $\mathcal{A}$, then $Tb = e(Tb) = (Te)b$, for all $b \in \mathcal{A}$, which means that $T$ is a multiplication operator induced by $Te$ on $\mathcal{A}$.

Let $L(\mathcal{A})$ be the set of all multiplication operators on $\mathcal{A}$. For a non-unital commutative Banach algebra $\mathcal{A}$, the set $M(\mathcal{A})$ of all multipliers on $\mathcal{A}$ may be larger than $L(\mathcal{A})$. However, if $\mathcal{A}$ is faithful in the sense that, for $a \in \mathcal{A}$, the products $ab$ ($b \in \mathcal{A}$) are all zero if and only if $a = 0$, then $M(\mathcal{A})$ is a closed commutative Banach subalgebra of $B(\mathcal{A})$, the algebra of all bounded linear operators on $\mathcal{A}$, that contains $I$, the identity operator on $\mathcal{A}$, and all multiplication operators. Every multiplier $T$ on a faithful commutative Banach algebra satisfies $T(ab) = aT(b)$ for all $a, b \in \mathcal{A}$. Note that any unital algebra is faithful. Moreover, any semi-prime commutative Banach algebra, i.e. a commutative Banach algebra without non-zero nilpotent elements, is faithful. The reader is referred to [2] for details about multipliers.

Let $X$ be a complex Banach space and let $T \in B(X)$. Then $T$ is an operator with SVEP (single-valued extension property) if, for every open set $U \subseteq \mathbb{C}$, the only analytic function $f : U \to X$ that satisfies $(T - \lambda)f(\lambda) = 0$ ($\lambda \in U$) is the zero function on $U$ ([3], Definition 1.2.9). We are concerned with the question whether all multipliers on a faithful commutative Banach algebra do have SVEP. The question is stated as an open problem in [3] (Problem 6.2.1). In Section 3 we...
shall answer the question in the negative. Before that, in Section 2, we give some sufficient conditions for a multiplier on a faithful commutative Banach algebra to have SVEP.

2. SUFFICIENT CONDITIONS FOR SVEP

During this section, $\mathcal{A}$ will always mean a faithful commutative Banach algebra and $M(\mathcal{A})$ will be the algebra of all multipliers on $\mathcal{A}$. We shall give some sufficient conditions for $T \in M(\mathcal{A})$ to have SVEP.

The dual space $\mathcal{A}^\ast$ of the algebra $\mathcal{A}$ has a natural structure of a left Banach $\mathcal{A}$-module. Namely, for $a \in \mathcal{A}$ and $\xi \in \mathcal{A}^\ast$, the product $a \cdot \xi \in \mathcal{A}^\ast$ is given by $\langle a \cdot \xi, b \rangle = (\xi, ab)$, for all $b \in \mathcal{A}$. For $T \in M(\mathcal{A})$, we shall denote by $T^\ast$ the adjoint operator of $T$ on $\mathcal{A}^\ast$.

**Proposition 2.1.** If, for a multiplier $T \in M(\mathcal{A})$, the adjoint operator $T^\ast \in B(\mathcal{A}^\ast)$ has SVEP, then $T$ has SVEP, as well.

**Proof.** Let $U \subseteq \mathbb{C}$ be an arbitrary non-empty open set and let $f : U \to \mathcal{A}$ be an analytic function such that $(T - \lambda)f(\lambda) = 0$, for all $\lambda \in U$. Choose an arbitrary $\xi \in \mathcal{A}^\ast$ and denote $F_\xi(\lambda) = f(\lambda) \cdot \xi$. It is obvious that $F_\xi : U \to \mathcal{A}^\ast$ is an analytic function. Since

$$
\langle (T^\ast - \lambda)F_\xi(\lambda), a \rangle = \langle \xi, f(\lambda)((T - \lambda)a) \rangle = \langle \xi, ((T - \lambda)f(\lambda))a \rangle = 0 \quad (\lambda \in U)
$$

holds for all $a \in \mathcal{A}$ we conclude that $(T^\ast - \lambda)F_\xi(\lambda) = 0$, for $\lambda \in U$. By the assumption, $T^\ast$ has SVEP, therefore $F_\xi \equiv 0$ on $U$. Thus, $f(\lambda) \cdot \xi = 0$, for all $\xi \in \mathcal{A}^\ast$ and all $\lambda \in U$. This gives $\langle \xi, f(\lambda)a \rangle = 0$, for all $a \in \mathcal{A}$, $\xi \in \mathcal{A}^\ast$, and $\lambda \in U$. By the faithfulness of $\mathcal{A}$, we conclude $f \equiv 0$ on $U$. □

Let $\mathcal{X}$ be a Banach space, $T \in B(\mathcal{X})$, and let $\mathcal{B} \subseteq B(\mathcal{X})$ be a closed subspace such that $TB \subseteq \mathcal{B}$, for any $B \in \mathcal{B}$. It is not hard to see that the multiplication operator $L_T$ that is induced by $T$ on $\mathcal{B}$ has SVEP whenever $T$ has SVEP. The following proposition asserts that the converse holds if $\mathcal{X} = \mathcal{A}$, $T \in M(\mathcal{A})$, and $\mathcal{B}$ contains $L(\mathcal{A})$, the space of all multiplication operators.

**Proposition 2.2.** Let $T \in M(\mathcal{A})$ and assume that $\mathcal{B} \subseteq B(\mathcal{A})$ is a closed subspace such that $L(\mathcal{A}) \subseteq \mathcal{B}$ and $TB \subseteq \mathcal{B}$, for any $B \in \mathcal{B}$. Then $T$ has SVEP whenever the multiplication operator $L_T$ that is induced by $T$ on $\mathcal{B}$ has SVEP.

**Proof.** If $T$ does not have SVEP, then there exist a non-empty open set $U \subseteq \mathbb{C}$ and a non-zero analytic function $f : U \to \mathcal{A}$ such that $(T - \lambda)f(\lambda) = 0$, for all $\lambda \in U$. Define $F : U \to \mathcal{B}$ by $F(\lambda) = L_f(\lambda)$. Since $\mathcal{A}$ is faithful $F$ is a non-zero analytic function. Of course, $(L_T - \lambda)L(\mathcal{A}) = 0$, for all $\lambda \in U$, which shows that $L_T$ is without SVEP. □

Let $T$ be a multiplier on a faithful commutative Banach algebra $\mathcal{A}$. If $\alpha$ and $\beta$ are distinct eigenvalues of $T$ and $a, b \in \mathcal{A}$ are corresponding eigenvectors, then $ab = 0$. This follows from $\alpha ab = (Ta)b = a(Tb) = \beta ab$. Similarly, if $\alpha$ is an eigenvalue of $T$, $\beta \neq \alpha$ is an eigenvalue of $T^\ast$, and $a \in \mathcal{A}$, $\xi \in \mathcal{A}^\ast$ are corresponding eigenvectors, then $a \cdot \xi = 0$. Namely, it is easily seen that $(Ta) \cdot \xi = a \cdot (T^\ast \xi)$ holds for all $a \in \mathcal{A}$ and $\xi \in \mathcal{A}^\ast$. Thus, the assertion follows from the equality $\alpha a \cdot \xi = (Ta) \cdot \xi = a \cdot (T^\ast \xi) = \beta a \cdot \xi$. Also, a similar assertion holds if $T^\ast$ is replaced by multiplication operator $L_T$ on $\mathcal{B}$, where $\mathcal{B} \subseteq B(\mathcal{A})$ is a closed subspace.
that contains \( L(A) \) and is invariant for multiplication by \( T \) (i.e. \( TB \in B \), for any \( B \in B \)).

**Proposition 2.3.** Let \( T \in M(A) \). Let \( U, V \subseteq \mathbb{C} \) be open sets and let \( f : U \to A \) and \( g : V \to A \) be analytic functions such that \( (T - \lambda)f(\lambda) = 0 \) (\( \lambda \in U \)) and \( (T - \mu)g(\mu) = 0 \) (\( \mu \in V \)). Then \( f^{(m)}(\lambda)g^{(n)}(\mu) = 0 \), for all \( \lambda \in U \) and \( \mu \in V \), where \( f^{(m)} \) and \( g^{(n)} \) are derivatives of \( f \) and \( g \) of orders \( m \geq 0 \) and \( n \geq 0 \), respectively.

**Proof.** If \( \lambda \in U \) and \( \mu \in V \) are distinct, then \( f(\lambda)g(\mu) = 0 \), by the observation above. Since \( f \) and \( g \) are continuous we conclude that \( f(\lambda)g(\lambda) = 0 \) holds as well (for \( \lambda \in U \cap V \)). By double induction, \( f^{(m)}(\lambda)g^{(n)}(\mu) = 0 \). □

The following corollary was obtained, with a different proof, as Proposition 4.2.1 in [3].

**Corollary 2.4.** Any multiplier on a semi-prime commutative Banach algebra has SVEP.

**Proof.** Let \( T \in M(A) \). If \( U \subseteq \mathbb{C} \) is a non-empty open subset and \( f : U \to A \) is an analytic function such that \( (T - \lambda)f(\lambda) = 0 \), for all \( \lambda \in U \), then \( f(\lambda)^2 = 0 \) (\( \lambda \in U \)), by Proposition 2.3. It follows that \( f \equiv 0 \) on \( U \), by semi-primeness. □

**Remark 2.5.** Let \( T \) be a multiplier on \( A \) and let \( T^* \in B(A) \) be its adjoint operator. If \( U, V \subseteq \mathbb{C} \) are non-empty open sets and \( f : U \to A, \ G : V \to A^* \) are analytic functions such that \( (T - \lambda)f(\lambda) = 0 \), for all \( \lambda \in U \), and \( (T^* - \mu)G(\mu) = 0 \), for all \( \mu \in V \), then \( f^{(m)}(\lambda) \cdot G^{(n)}(\mu) = 0 \), for all \( \lambda \in U \), \( \mu \in V \) and all non-negative integers \( m \) and \( n \). The proof of this assertion is a simple modification of the proof of Proposition 2.3. A similar assertion holds if \( T^* \) is replaced by \( LT \), where \( LT \) is the multiplication operator induced by \( T \) on \( B \) (a closed subspace of \( B(A) \) that contains \( L(A) \) and is invariant for multiplication by \( T \)).

Let \( B \subseteq A \) be a subalgebra. We shall say that the multiplication on \( B \) is trivial if \( ab = 0 \), for all \( a, b \in B \).

**Proposition 2.6.** If each subalgebra \( B \) of \( A \) on which the multiplication is trivial is of finite dimension or of finite co-dimension, then every multiplier on \( A \) has SVEP.

**Proof.** Assume towards a contradiction that there is a multiplier \( T \in M(A) \) without SVEP. Then there exist a non-empty open set \( U \subseteq \mathbb{C} \) and a non-zero analytic function \( f : U \to A \) such that \( (T - \lambda)f(\lambda) = 0 \), for all \( \lambda \in U \). There is no loss of generality if we assume that \( U = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \} \). Thus, there exist elements \( a_n \in A \), not all zero, such that

\[
f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \quad (\lambda \in U).
\]

Of course, we may assume that \( a_0 \neq 0 \). Since \( Tf(\lambda) = \lambda f(\lambda) \), for all \( \lambda \in U \), we get, by comparing the coefficients, \( Ta_0 = 0 \) and \( Ta_n = a_{n-1} \), for \( n \geq 1 \). It follows that

\[
a_n a_m = (T^{m+1}a_{n+m+1})a_m = a_{n+m+1}(T^{m+1}a_m) = 0,
\]

for all non-negative integers \( m \) and \( n \). Denote by \( B \) the subalgebra of \( A \) that is generated by \( \{a_n : n \geq 0 \} \). It is obvious that the multiplication on \( B \) is trivial. For every \( n \geq 0 \), the set \( \{a_0, \ldots, a_n\} \) is linearly independent. Namely, assume that this is not the case. Then there exists a minimal \( n \geq 1 \) such that \( a_n =

Proposition 3.1. The direct sum \( A \oplus X \) is a Banach algebra through the norm and multiplication that are given by
\[
\|a \oplus x\| = \|a\| + \|x\|, \quad (a \oplus x)(b \oplus y) = ab \oplus (a \cdot y + x \cdot b) \quad (a \oplus x, b \oplus y \in A \oplus X).
\]
If \( A \) has unit, say \( e \), then \( e \oplus 0 \) is unit for \( A \oplus X \).

Proof. Straightforward; see the proof of Theorem 5 in [4]. \( \square \)

Corollary 3.2. Let \( X \) be a Banach space and let \( A \subset B(X) \) be a closed commutative subalgebra; then \( A \oplus X \) is a commutative Banach algebra for the multiplication
\[
(A \oplus x)(B \oplus y) = AB \oplus (Ay + Bx) \quad (A \oplus x, B \oplus y \in A \oplus X).
\]
If \( A \) contains the identity operator \( I \), then \( A \oplus X \) is unital with unit \( I \oplus 0 \).

Proof. If \( A \) is a commutative Banach algebra and \( X \) is a left Banach \( A \)-module, then \( X \) can be considered as a Banach \( A \)-bimodule if the right module structure on \( X \) is given by \( x \cdot a := a \cdot x \). \( \square \)

The following proposition may be deduced from the results in [1]. However, we shall include a more transparent and direct proof of it.

Proposition 3.3. Let \( X \) be a Banach space and let \( A \subset B(X) \) be a closed commutative subalgebra that contains \( I \). Then, for an operator \( S \in A \), the following assertions are equivalent:
\begin{enumerate}
\item \( S \) has SVEP;
\item for every \( x \in X \), the multiplication operator \( L_{S \oplus x} \) on \( A \oplus X \) has SVEP;
\item there exists \( x \in X \) such that \( L_{S \oplus x} \) has SVEP.
\end{enumerate}
Proof. Assume that there exists \( x \in \mathcal{X} \) such that \( L_{S \oplus x} \) is without SVEP. Let \( U \subseteq \mathbb{C} \) be a non-empty open set and \( F : U \to A \oplus \mathcal{X} \) a non-zero analytic function such that \( (L_{S \oplus x} - \lambda)F(\lambda) = 0 \), for \( \lambda \in U \). Of course, \( F(\lambda) = \varphi(\lambda) \oplus f(\lambda) \), for analytic functions \( \varphi : U \to A \) and \( f : U \to \mathcal{X} \). It follows from \( (L_{S \oplus x} - \lambda)F(\lambda) = 0 \) that \( (S - \lambda)\varphi(\lambda) + (S - \lambda)f(\lambda) = 0 \) and therefore

\[
(L_{S} - \lambda)\varphi(\lambda) = 0 \quad \text{and} \quad \varphi(\lambda)x + (S - \lambda)f(\lambda) = 0 \quad (\lambda \in U),
\]

where \( L_{S} \) is the multiplication operator induced by \( S \) on \( A \). If \( \varphi \) is non-zero, then \( L_{S} \) does not have SVEP and consequently \( S \) does not have SVEP. On the other hand, if \( \varphi \equiv 0 \) on \( U \), then \( (S - \lambda)f(\lambda) = 0 \), for \( \lambda \in U \), and \( f \) is a non-zero function. Thus, again, \( S \) does not have SVEP. This proves the implication \( (a) \Rightarrow (b) \). The implication \( (b) \Rightarrow (c) \) is obvious.

Assume that \( S \in A \) does not have SVEP and let \( x \in \mathcal{X} \) be an arbitrary vector. Then there exists a non-empty open subset \( U \subseteq \mathbb{C} \) and a non-zero analytic function \( f : U \to \mathcal{X} \) such that \( (S - \lambda)f(\lambda) = 0 \), for all \( \lambda \in U \). Define \( F(\lambda) = 0 \oplus f(\lambda) \), for \( \lambda \in U \). Then \( F : U \to A \oplus \mathcal{X} \) is a non-zero analytic function such that \( (L_{S \oplus x} - \lambda)F(\lambda) = 0 \), for all \( \lambda \in U \). Thus (a) follows from (c).

Let \( S \) be an operator without SVEP on a Banach space \( \mathcal{X} \) and let \( A \subseteq B(\mathcal{X}) \) be the closed subalgebra that is generated by \( S \) and \( I \). Then \( A \oplus \mathcal{X} \) is a faithful commutative Banach algebra such that the multiplication operator \( L_{S \oplus 0} \) on \( A \oplus \mathcal{X} \) does not have SVEP, by Proposition 3.3.

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References


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