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# ON THE CHARACTERISTIC POLYNOMIAL OF THE ALMOST MATHIEU OPERATOR

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ABSTRACT. Let $A_\theta$ be the rotation C*-algebra for angle $\theta$. For $\theta = p/q$ with $p$ and $q$ relatively prime, $A_\theta$ is the sub-C*-algebra of $M_q(C(\mathbb{T}^2))$ generated by a pair of unitaries $u$ and $v$ satisfying $uv = e^{2\pi i\theta}vu$. Let

$$h_{\theta,\lambda} = u + u^{-1} + \lambda/2(v + v^{-1})$$

be the almost Mathieu operator. By proving an identity of rational functions we show that for $q$ even, the constant term in the characteristic polynomial of $h_{\theta,\lambda}$ is $(-1)^{q/2}(1 + (\lambda/2)^q) - (z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q + z_2^{-q}))$.

## 1. Introduction

Let $\theta$, $\lambda$, and $\psi$ be real numbers with $\lambda$ positive. The second order difference operator $H_{\theta,\lambda,\psi}$ on $\ell^2(\mathbb{Z})$ given by

$$H_{\theta,\lambda,\psi}(\xi)(n) = \xi(n+1) + \xi(n-1) + \lambda\cos(2\pi n\theta + \psi)\xi(n)$$

for $\xi \in \ell^2(\mathbb{Z})$ is called the almost Mathieu operator. $H_{\theta,\lambda,\psi}$ is a discrete Schrödinger operator which models an electron moving in a crystal lattice in a plane perpendicular to a magnetic field.

An object of much study has been the spectrum $\sigma(\theta,\lambda) = \bigcup_\psi \sigma(H_{\theta,\lambda,\psi})$. In [H], Hofstadter calculated $\sigma(\theta, 2)$ for $\theta = p/q$ and $1 \le p < q \le 50$. The remarkable pattern he found is called Hofstadter's butterfly. For irrational $\theta$, a longstanding concern has been the connectedness and Lebesgue measure of $\sigma(\theta,\lambda)$ and the labelling of the gaps, about which quite a bit is now known (see [AJ], [AK], and [P] for spectacular recent advances as well as [AVMS], [BS], [B], [CEY], [LT] for earlier work). In addition there has been numerical work on computing the spectrum to high accuracy for large $q$ [$A_1$, $A_2$, L].

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Let $A_\theta$ be the rotation C*-algebra (see [B]). For $\theta = p/q$ with $p$ and $q$ relatively prime and $\rho = e^{2\pi\theta}$ let

$$u_\theta = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} \text{ and } v_\theta = \begin{pmatrix} \rho & & & & \\ & \rho^2 & & & \\ & & \ddots & & \\ & & & \rho^{q-1} & \\ & & & & 1 \end{pmatrix},$$

i.e. $u_\theta$ cyclically permutes the elements of the standard basis and $v_\theta$ is a diagonal operator. Then define $u, v : \mathbb{T}^2 \to M_q(\mathbb{C})$ by $u(z_1, z_2) = z_1 u_\theta$ and $v(z_1, z_2) = z_2 v_\theta$. Then $u\,v = \rho v\,u$ and $A_\theta$ is the C*-algebra generated by $u$ and $v$ (see [B]). The operator $h_{\theta,\lambda} = u + u^{-1} + \lambda/2(v + v^{-1})$ contains all the spectral information of $H_{\theta,\lambda,\psi}$ in that $\mathrm{Sp}(h_{\theta,\lambda}) = \sigma_{\theta,\lambda} := \bigcup_\psi \mathrm{Sp}(H_{\theta,\lambda,\psi})$.

The main tool in the analysis of $\sigma_{\theta,\lambda}$ is $\Delta_{\theta,\lambda}$, the discrete analogue of the discriminant. For $\theta = p/q$, $\Delta_{\theta,\lambda}(x) = \mathrm{Tr}(A_1(x) \cdots A_q(x))$ where

$$A_k(x) = \begin{pmatrix} x - \lambda\cos(2\pi k p/q + \pi/(2q)) & -1 \\ 1 & 0 \end{pmatrix}.$$

Below are the first few values of this polynomial. Note that the form of $\Delta_{\theta,\lambda}$ so displayed depends only on the denominator $q$; however, $\xi_\theta = 2\cos(2\pi p/q)$ depends on the numerator $p$.

| $q$ | $\Delta_{\theta,2}(x)$ for $\theta = p/q$ and $\xi_\theta = 2\cos(2\pi\theta)$ |
|---|---|
| 2 | $x^2 - 4$ |
| 3 | $x^3 - 6x$ |
| 4 | $x^4 - 8x^2 + 4$ |
| 5 | $x^5 - 10x^3 + 5(3 - \xi_\theta)x$ |
| 6 | $x^6 - 12x^4 + 6(5 - \xi_\theta)x^2 - 4$ |
| 7 | $x^7 - 14x^5 + 7(7 - \xi_\theta)x^3 - 7(6 - 2\xi_\theta + 2\xi_{2\theta})x$ |
| 8 | $x^8 - 16x^6 + 8(9 - \xi_\theta)x^4 - 8(12 - 4\xi_\theta + 2\xi_{2\theta})x^2 + 4$ |
| 9 | $x^9 - 18x^7 + 9(11 - \xi_\theta)x^5 - 9(31/3 - 6\xi_\theta + 2\xi_{2\theta})x^3 + 9(14 - 8\xi_\theta + 3\xi_{2\theta})x$ |

One can calculate for $k = 1, 2$ the coefficient of $x^{q-2k}$; for $k = 3$ the formula is conjectural (from numerical evidence). A deeper understanding of the structure of $\Delta_{\theta.\lambda}$ would be quite interesting.

| $k$ | coefficient of $x^{q-2k}$ in $\Delta_{\theta,\lambda}$ $(\mu = \lambda/2)$ |
|---|---|
| 1 | $-q(1 + \mu^q)$ |
| 2 | $q\left(\frac{1}{q-2}\binom{q-2}{2}\mu^4 + (q - 4 - \xi_\theta)\mu^2 + \frac{1}{q-2}\binom{q-2}{2}\right)$ |
| 3 | $-q(\frac{1}{q-3}\binom{q-3}{3}\mu^6 + (1 + \binom{q-5}{2} - (q-6)\xi_\theta + \xi_{2\theta})\mu^4$ |
| | $+ (1 + \binom{q-5}{2} - (q-6)\xi_\theta + \xi_{2\theta})\mu^2 + \frac{1}{q-3}\binom{q-3}{3})$ |

The connection with the characteristic polynomial of $h_{\theta,\lambda}$ is given by

$$\det(xI_q - h_{\theta,\lambda}(z_1, z_2)) = \Delta_{\theta,\lambda}(x) + z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q + z_2^{-q}) \tag{1}$$

and thus $\sigma_{\theta,\lambda} = \Delta_{\theta,\lambda}^{-1}[-2(1+(\lambda/2)^q),\ 2(1+(\lambda/2)^q)]$. Indeed, $\Delta_{\theta,\lambda}(x)$ can be written as a determinant (cf. Toda [T, §4])

$$\Delta_{p/q,\lambda}(x) = \det \begin{pmatrix} \alpha_1 & 1 & & & 1 \\ 1 & \alpha_2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & & & 1 \\ 1 & & & 1 & \alpha_q \end{pmatrix} + 2\Big\{(-1)^q + (\lambda/2)^q\Big\} \tag{2}$$

where all the other entries are 0 and $\alpha_k = x - \lambda\cos(2\pi kp/q + \pi/(2q))$. Since

$$\Delta_{p/q,\lambda}(-x) = (-1)^q \Delta_{p/q,\lambda}(x) \tag{3}$$

the coefficient of $x^{q-(2k+1)}$ is 0 for $0 \le k < q/2$.

The main result of the paper asserts that for $a_l = 2\cos(2\pi lp/q)$ and $1 \le k < q/2$ we have

$$\sum_{i_1,i_2,\ldots,i_{q-2k}} a_{i_1}a_{i_2}\cdots a_{i_{q-2k}} = 0$$

where the summation is over all subsets of $\{1,2,3,\ldots,q\}$ obtained by deleting $k$ pairs of adjacent elements – counting 1 and $k$ as adjacent. This is proved by establishing the following identity for $k \ge 3$ and $q \ge 2k-1$:

$$\sum_{i_1=1}^{q-2(k-1)} \cdots \sum_{i_k=i_{k-1}+2}^{q} \prod_{j=1}^{k} \frac{(x^{-i_j}+x^{i_j})^{-1}}{(x^{-i_j-1}+x^{i_j+1})} = \frac{(x^{-q}-x^{q})\prod_{i=k+1}^{2k-2}(x^{-q+i}-x^{q-i})}{\prod_{i=1}^{k}(x^{-2i}-x^{2i})\prod_{i=-1}^{k-2}(x^{-q+i}+x^{q-i})}$$

$$+ \frac{(x^{-1}+x^{1})^{-1}(x^{-q}+x^{q})^{-1}}{(x^{-2}+x^{2})(x^{-q-1}+x^{q+1})} \sum_{i_1=3}^{q-2(k-2)} \cdots \sum_{i_{k-2}=i_{k-3}+2}^{q-2} \prod_{j=1}^{k-2} \frac{(x^{-i_j}+x^{i_j})^{-1}}{(x^{-i_j-1}+x^{i_j+1})}.$$

We then use this to show that for $a_l = 2\cos(2\pi lp/q)$

$$\det \begin{pmatrix} a_1 & 1 & & & 1 \\ 1 & a_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & a_q \end{pmatrix} = \begin{cases} 0 & q \equiv 0 \pmod 4, \\ 4 & q \equiv 1,3 \pmod 4, \\ -8 & q \equiv 2 \pmod 4. \end{cases}$$

From this we show that the constant term (i.e. the coefficient of $x^0$) in $\det(xI_q - h_{\theta,\lambda}(z_1,z_2))$ is

$$(-1)^{q/2}2(1+(\lambda/2)^q)) - (z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q + z_2^{-q}))$$

when $q$ is even. When $q$ is odd it follows from (3) that the coefficient of $x^0$ is $-(z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q + z_2^{-q}))$.

Similar, though simpler, reasoning shows that the coefficient of $x^{q-2}$ is $-q(1+\lambda/2)$ and that the coefficient of $x^{q-4}$ is

$$(\lambda/2)^4 q(q-3)/2 + (\lambda/2)^2 q(q-4-2\cos(2\pi\theta)) + q(q-3)/2.$$

## 2. The main theorem

Let us use the following notation: let $a_1, \dots, a_n$ be elements of a commutative ring and let

$$(a_1, a_2, \dots, a_n) = \begin{vmatrix} a_1 & 1 & & & 0 \\ 1 & a_2 & 1 & & \\ & \ddots & a_3 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & a_n \end{vmatrix}$$

and

$$[\![a_1, a_2, \dots, a_n]\!] = \begin{vmatrix} a_1 & 1 & & & 1 \\ 1 & a_2 & 1 & & \\ & \ddots & a_3 & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & a_n \end{vmatrix}.$$

The first matrix is a tridiagonal matrix with 1's on the sub and super-diagonal and 0's elsewhere. The second matrix is the same tridiagonal matrix with in addition 1's in the upper right and lower left corners; all other entries are 0. Expanding along the bottom row we have

$$[\![a_1, a_2, \dots, a_n]\!] = (a_1, a_2, \dots, a_n) - (a_2, a_3, \dots, a_{n-1}) + 2(-1)^{n-1} \tag{4}$$

and

$$[\![-a_1, -a_2, \dots, -a_n]\!] = (-1)^n [\![a_1, a_2, \dots, a_n]\!] + 2(-1)^{n-1}. \tag{5}$$

Rewriting equation (2) we have

$$\Delta_{p/q,\lambda}(x) = [\![a_1, \dots, a_q]\!] + 2((-1)^q + (\lambda/2)^q). \tag{6}$$

*Notation* 2.1. (i) For $0 \le k \le n/2$, let $S\left[{n \atop k}\right] = \{I \subset \{1, 2, \dots, n\} \mid |I| = n - 2k$ and $I$ is obtained from $\{1, 2, \dots, n\}$ by deleting $k$ pairs of adjacent elements$\}$. $S\left[{2k \atop k}\right] = \{\emptyset\}$, $S\left[{2k+1 \atop k}\right] = \big\{\{1\}, \{3\}, \{5\}, \dots, \{2k+1\}\big\}$, $\dots$, $S\left[{n \atop 0}\right] = \Big\{\{1, 2, 3, \dots, n\}\Big\}$.

(ii) For $0 \le k \le (n-1)/2$, let $S'\left[{n \atop k}\right] = \{I \subset \{2, 3, \dots, n\} \mid |I| = n - 2k - 1$ and $I$ is obtained from $\{2, 3, \dots, n\}$ by deleting $k$ pairs of adjacent elements$\}$. $S'\left[{2k+1 \atop k}\right] = \emptyset$, $S'\left[{2k+2 \atop k}\right] = \big\{\{2\}, \{4\}, \{6\}, \dots, \{2k+2\}\big\}$, $\dots$, $S'\left[{n \atop 0}\right] = \{2, 3, \dots, n\}$.

(iii) For $\mathcal{S}$ a collection of subsets of $\{1, 2, \dots, n-1\}$ let $\mathcal{S} \vee \{n\} = \{I \cup \{n\} \mid I \in \mathcal{S}\}$.

(iv) For $0 \le k \le n/2$, let $\widetilde{S}\left[{n \atop k}\right] = \{I \subset \{1, 2, 3, \dots, n\} \mid |I| = n - 2k$ and $I$ is obtained from $\{1, 2, \dots, n\}$ by deleting $k$ pairs of adjacent elements, counting $\{n, 1\}$ as an adjacent pair$\}$. $\widetilde{S}\left[{2k \atop k}\right] = \emptyset$, $\widetilde{S}\left[{2k+1 \atop k}\right] = \big\{\{1\}, \{2\}, \dots, \{n\}\big\}$, $\dots$, $\widetilde{S}\left[{n \atop 0}\right] = \{1, 2, 3, \dots, n\}$.

(v) For $a_1, a_2, a_3, \dots, a_n$ elements of a commutative ring, and $I = \{i_1, i_2, i_3, \dots, i_k\} \subset \{1, 2, 3, \dots, n\}$, let $a_I = a_{i_1} a_{i_2} a_{i_3} \cdots a_{i_n}$. We shall adopt the convention that $a_\emptyset = 1$.

Part (*ii*) of the next proposition goes back to Sylvester's original paper on continuants [S]; part (*iv*) is a straightforward extension of this. For the reader's convenience we present a proof.

**Proposition 2.2.** (i) *Suppose* $1 \le k < n/2$; *then* $S\begin{bmatrix} n \\ k \end{bmatrix} = \left(S\begin{bmatrix} n-1 \\ k \end{bmatrix} \vee \{n\}\right) \cup S\begin{bmatrix} n-2 \\ k-1 \end{bmatrix}$.

(ii)

$$\sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in S\begin{bmatrix} n \\ k \end{bmatrix}} a_I = a_n \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{I \in S\begin{bmatrix} n-1 \\ k \end{bmatrix}} a_I - \sum_{k=0}^{[(n-2)/2]} (-1)^k \sum_{I \in S\begin{bmatrix} n-2 \\ k \end{bmatrix}} a_I.$$

(iii) $\widetilde{S}\begin{bmatrix} n \\ k \end{bmatrix} = S\begin{bmatrix} n \\ k \end{bmatrix} \cup S'\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ *for* $1 \le k \le n/2$.

(iv) *When* $n$ *is odd,*

$$\sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in \widetilde{S}\begin{bmatrix} n \\ k \end{bmatrix}} a_I = \sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in S\begin{bmatrix} n \\ k \end{bmatrix}} a_I - \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{I \in S'\begin{bmatrix} n-1 \\ k \end{bmatrix}} a_I.$$

*When* $n$ *is even,*

$$\sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in \widetilde{S}\begin{bmatrix} n \\ k \end{bmatrix}} a_I = \sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in S\begin{bmatrix} n \\ k \end{bmatrix}} a_I + (-1)^{n/2} - \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{I \in S'\begin{bmatrix} n-1 \\ k \end{bmatrix}} a_I.$$

*Proof.* (i) Let $I \in S\begin{bmatrix} n \\ k \end{bmatrix}$. If $n \notin I$, then $n-1 \notin I$ and so $I \in S\begin{bmatrix} n-2 \\ k-1 \end{bmatrix}$. Suppose $n \in I$. Let $K = \{1,2,3,\ldots,n\} \setminus I$ and $\dot{I} = I \setminus \{n\}$. Then $\dot{I} = \{1,2,3,\ldots n-1\} \setminus K$; so $\dot{I} \in S\begin{bmatrix} n-1 \\ k \end{bmatrix}$. Hence $I = \dot{I} \cup \{n\} \in S\begin{bmatrix} n-1 \\ k \end{bmatrix} \vee \{n\}$.

(ii) Let us assume that $n = 2m$ is even. The same idea works for odd $n$, but the proof is slightly simpler. Observe

$$\begin{aligned}
\sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in S\begin{bmatrix} n \\ k \end{bmatrix}} a_I &= \sum_{I \in S\begin{bmatrix} n \\ 0 \end{bmatrix}} a_I + \sum_{k=1}^{m-1} (-1)^k \sum_{I \in S\begin{bmatrix} n \\ k \end{bmatrix}} a_I + (-1)^m \\
&= \sum_{I \in S\begin{bmatrix} n \\ 0 \end{bmatrix}} a_I + a_n \sum_{k=1}^{m-1} (-1)^k \sum_{I \in S\begin{bmatrix} n-1 \\ k \end{bmatrix}} a_I + \sum_{k=1}^{m-1} (-1)^k \sum_{I \in S\begin{bmatrix} n-2 \\ k-1 \end{bmatrix}} a_I + (-1)^m \\
&= a_n \left\{ \sum_{I \in S\begin{bmatrix} n-1 \\ 0 \end{bmatrix}} a_I + \sum_{k=1}^{m-1} (-1)^k \sum_{I \in S\begin{bmatrix} n-1 \\ k \end{bmatrix}} a_I \right\} + \sum_{k=1}^{m-1} (-1)^k \sum_{I \in S\begin{bmatrix} n-2 \\ k-1 \end{bmatrix}} a_I + (-1)^m \\
&= a_n \sum_{k=0}^{m-1} (-1)^k \sum_{I \in S\begin{bmatrix} n-1 \\ k \end{bmatrix}} a_I - \sum_{k=0}^{m-1} (-1)^k \sum_{I \in S\begin{bmatrix} n-2 \\ k \end{bmatrix}} a_I \\
&= a_n \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{I \in S\begin{bmatrix} n-1 \\ k \end{bmatrix}} a_I - \sum_{k=0}^{[(n-2)/2]} (-1)^k \sum_{I \in S\begin{bmatrix} n-2 \\ k \end{bmatrix}} a_I.
\end{aligned}$$

(iii) For $I \in \widetilde{S}\begin{bmatrix} n \\ k \end{bmatrix}$ let $K_1 = \{1,2,3,\ldots,n\} \setminus I$ and $K_2 = \{2,3,\ldots,n\} \setminus I$. Also, $\min\{i \mid i \in K_1\}$ is odd if and only if $I \in S\begin{bmatrix} n \\ k \end{bmatrix}$ and $\min\{i \mid i \in K_2\}$ is even if and only if $I \in S'\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$.

(iv) Suppose $n = 2m$. Then

$$\begin{aligned}\sum_{k=0}^{[n/2]}(-1)^k \sum_{I\in\widetilde{S}\left[{n \atop k}\right]} a_I &= \sum_{I\in\widetilde{S}\left[{n \atop 0}\right]} a_I + \sum_{k=1}^{m-1}(-1)^k \sum_{I\in\widetilde{S}\left[{n \atop k}\right]} a_I + (-1)^m \sum_{I\in\widetilde{S}\left[{n \atop m}\right]} a_I \\ &= \left(\sum_{I\in S\left[{n \atop 0}\right]} a_I + \sum_{k=1}^{m}(-1)^k \sum_{I\in S\left[{n \atop k}\right]} a_I\right) + \sum_{k=1}^{m}(-1)^k \sum_{I\in S'\left[{n-1 \atop k-1}\right]} a_I + (-1)^m \\ &= \sum_{k=0}^{m}(-1)^k \sum_{I\in S\left[{n \atop k}\right]} a_I - \sum_{k=0}^{m-1}(-1)^k \sum_{I\in S'\left[{n-1 \atop k}\right]} a_I + (-1)^m.\end{aligned}$$

The case of $n$ odd is similar. □

**Corollary 2.3.** *Let $a_1, a_2, \ldots, a_n$ be elements of a commutative ring.*

(i) $(a_1, \ldots, a_n) = \sum_{k=0}^{[n/2]}(-1)^k \sum_{I\in S\left[{n \atop k}\right]} a_I.$

(ii)

$$[\![a_1, \ldots, a_n]\!] = \begin{cases} \sum_{k=0}^{[n/2]}(-1)^k \sum_{I\in\widetilde{S}\left[{n \atop k}\right]} a_I + 2, & n \text{ odd}, \\ \sum_{k=0}^{[n/2]}(-1)^k \sum_{I\in\widetilde{S}\left[{n \atop k}\right]} a_I - 2 + (-1)^{n/2}, & n \text{ even}. \end{cases}$$

*Proof.* (i) For $n = 1$ the left hand side and the right hand side equal $a_1$. Both sides satisfy the same recurrence relation.

(ii) By equation (6)

$$[\![a_1, \ldots, a_n]\!] = (a_1, \ldots, a_n) - (a_2, \ldots, a_{n-1}) - (-1)^n 2$$

so the result now follows from (i) and Proposition 2.2 (iii). □

**Proposition 2.4.** *Let $1 \le p < q$ be relatively prime, $\theta = p/q$, and $a_k = 2\cos(2\pi k\theta)$. Then*

$$a_1 a_2 \cdots a_q = \begin{cases} 0 & q \equiv 0 \pmod 4, \\ 2 & q \equiv 1, 3 \pmod 4, \\ -4 & q \equiv 2 \pmod 4. \end{cases}$$

*Proof.* Let $T_q$ be the $q$th Chebyshev polynomial of the first kind. The constant term of $T_q$ is 0 for $q$ odd and $(-1)^{q/2}$ for $q$ even. The result now follows from the identity (see e.g. [R, §1.2])

$$\prod_{i=1}^{q}(x - a_i) = 2(T_q(x/2) - 1). \qquad \square$$

The statement of the main theorem follows. Its proof will be given at the end of the next section.

**Theorem 2.5.** *Let $1 \le p < q$ be relatively prime, $a_k = 2\cos(2\pi k\theta)$, and $\theta = p/q$. For $1 \le k < q/2$,*

$$\sum_{I \in \widetilde{S}\left[{q \atop k}\right]} a_I = 0.$$

**Corollary 2.6.** *Let $1 \le p < q$ be relatively prime, $\theta = p/q$, $\lambda > 0$, and $a_k = \lambda\cos(2\pi k\theta)$. Then*

$$[\![a_1, a_2, \dots, a_q]\!] = \begin{cases} 0 & q \equiv 0 \pmod 4, \\ 2(1 + (\lambda/2)^q) & q \equiv 1, 3 \pmod 4, \\ -4(1 + (\lambda/2)^q) & q \equiv 2 \pmod 4) \end{cases}$$

*and $\Delta_{\theta,\lambda}(0) = (-1)^{q/2} 2(1 + (\lambda/2)^q)$ for $q$ even.*

*Proof.* Suppose $q$ is even. By Theorem 2.5 all the terms of

$$\sum_{k=0}^{[q/2]} (-1)^k \sum_{I \in \widetilde{S}\left[{q \atop k}\right]} a_I$$

are zero except the terms for $k = 0$ and $k = q/2$. The term for $k = 0$ is $a_1 a_2 \cdots a_q$. The term for $k = q/2$ is $(-1)^{q/2}$. Thus when $q = 4m$ we have by Proposition 2.4

$$[\![a_1, a_2, \dots, a_q]\!] = a_1 a_2 \cdots a_q - (-1)^q 2 + (-1)^{q/2} 2 = 0,$$

and when $q = 4m + 2$,

$$[\![a_1, a_2, \dots, a_q]\!] = a_1 a_2 \cdots a_q - (-1)^q 2 + (-1)^{q/2} 2 = -4(1 + (\lambda/2)^q).$$

To obtain the final claim we apply equation (6). □

From the corollary and equation (1) we have the theorem which corrects an error in [CEY, p. 232].

**Theorem 2.7.** *The coefficient of $x^0$ in $\det(xI_q - h_{\theta,\lambda}(z_1, z_2))$ is*

$$-\left(z_1^q + z_1^{-q} + (\lambda/2)^q (z_2^q + z_2^{-q})\right) + (-1)^{q/2} 2(1 + (\lambda/2)^q)$$

*when $q$ is even and $-(z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q + z_2^{-q}))$ when $q$ is odd.*

## 3. Proof of the main theorem

**Theorem 3.1.** *Suppose $a_1, a_2, \dots, a_q$ are elements in a commutative ring and let $a_{q+1} = a_1$. For $I \subset \{1, 2, \dots, q\}$, let $I^c = \{1, 2, \dots, q\} \setminus I$ be the complement of $I$ in $\{1, 2, \dots, q\}$. Then*

$$\sum_{I \in \widetilde{S}\left[{q \atop k}\right]} a_{I^c} = \sum_{i_1=1}^{q-2(k-1)} \sum_{i_2=i_1+2}^{q-2(k-2)} \cdots \sum_{i_k=i_{k-1}+2}^{q} \prod_{j=1}^{k} a_{i_j} a_{i_j+1}$$
$$- a_1 a_2 \left[ \sum_{i_1=3}^{q-2(k-2)} \sum_{i_2=i_1+2}^{q-2(k-3)} \cdots \sum_{i_{k-2}=i_{k-3}+2}^{q-2} \prod_{j=1}^{k-2} a_{i_j} a_{i_j+1} \right] a_q a_{q+1}.$$

*Proof.* Recall that elements of $\widetilde{S}\left[\begin{smallmatrix}q\\k\end{smallmatrix}\right]$ are obtained by deleting $k$ adjacent pairs $\{i, i+1\}$ from $\{1, 2, \ldots, q\}$, counting $q$ and 1 as adjacent. So if $I^c \in \widetilde{S}\left[\begin{smallmatrix}q\\k\end{smallmatrix}\right]$, then $I = \{i_1, j_1, i_2, j_2, \ldots, i_k, j_k\}$ with $1 \le i_1, j_1 = i_1 + 1 < i_2, \ldots, j_{k-1} = i_{k-1} + 1 < i_k \le q$ and either $j_k = i_k + 1$ if $i_k < q$ or $j_k = 1$ if $i_k = q$.

Now let $T\left[\begin{smallmatrix}q\\k\end{smallmatrix}\right] = \{\, \{i_1, j_1, i_2, j_2, \ldots, i_k, j_k\} \mid 1 \le i_1, j_1 = i_1 + 1 < i_2, \ldots, j_{k-1} = i_{k-1} + 1 < i_k \le q, j_k = i_k + 1 \,\}$. Define $\phi : \{1, 2, \ldots, q, q+1\} \to \{1, 2, \ldots, q\}$ by $\phi(q+1) = 1$ and $\phi(i) = i$ for $i \le q$. Then $a_{\phi(I)} = a_I$ for $I \in T\left[\begin{smallmatrix}q\\k\end{smallmatrix}\right]$.

If $I = \{i_1, j_1, i_2, j_2, \ldots, i_k, j_k\}$ and $i_1 = 1$ and $i_k = q$, then $\phi(I)^c \notin \widetilde{S}\left[\begin{smallmatrix}q\\k\end{smallmatrix}\right]$ because $\phi(j_k) = \phi(i_1) = 1$ and the pairs must be disjoint. So let $T'\left[\begin{smallmatrix}q\\k\end{smallmatrix}\right] = \{\, \{1, 2, i_1, j_1, \ldots, i_{k-1}, j_{k-1}, q, q+1\} \mid 3 \le i_1, j_1 = i_1 + 1 < i_2, \ldots, i_{k-1} \le q-2, j_{k-1} = i_{k-1} + 1 \,\}$.

For $I \in T\left[\begin{smallmatrix}q\\k\end{smallmatrix}\right] \setminus T'\left[\begin{smallmatrix}q\\k\end{smallmatrix}\right]$, $\phi(I)^c \in \widetilde{S}\left[\begin{smallmatrix}q\\k\end{smallmatrix}\right]$ and $\phi : T\left[\begin{smallmatrix}q\\k\end{smallmatrix}\right] \setminus T'\left[\begin{smallmatrix}q\\k\end{smallmatrix}\right] \to \widetilde{S}\left[\begin{smallmatrix}q\\k\end{smallmatrix}\right]$ is a bijection. This with the identity $a_{\phi(I)} = a_I$ proves the theorem. □

**Lemma 3.2.** (i) *For* $q \ge 1$,

$$\sum_{i=1}^{q} (x^{-i} + x^i)^{-1} (x^{-i-1} + x^{i+1})^{-1} = \frac{x^{-q} - x^q}{(x^{-2} - x^2)(x^{-q-1} + x^{q+1})}.$$

(ii) *For* $k \ge 1$

$$\prod_{i=1}^{2k} (x^{-i} + x^i)^{-1} = \prod_{i=1}^{k} \frac{(x^{-i} - x^i)}{(x^{-2i} - x^{2i})(x^{-(k+i)} + x^{k+i})}.$$

*Proof.* (i) One checks directly that the formula holds when $q = 1$; then (i) follows by induction on $q$.

(ii) follows from the identity

$$\frac{x^{-i} - x^i}{(x^{-2i} - x^{2i})(x^{-k-i} + x^{k+i})} = \frac{1}{(x^{-i} + x^i)(x^{-k-i} + x^{k+i})}.$$ □

**Corollary 3.3.** *For* $q \ge 5$

$$\frac{(x^{-1} + x)^{-1}(x^{-2} + x^2)^{-1}}{(x^{-q} + x^q)(x^{-q-1} + x^{q+1})} \sum_{i=3}^{q-2} (x^{-i} + x^i)^{-1} (x^{-i-1} + x^{i+1})^{-1}$$
$$= \frac{(x^{-3} - x^3)(x^{-q+4} - x^{q-4})}{(x^{-4} - x^4)(x^{-6} - x^6)(x^{-q+1} + x^{q-1})(x^{-q} + x^q)(x^{-q-1} + x^{q+1})}.$$

*Proof.* By Lemma 3.2 (i)

$$\begin{aligned}
&\sum_{i=3}^{q-2} (x + -i + x^i)^{-1} (x^{-i-1} + x^{i+1})^{-1} \\
&\quad = \frac{x^{-q+2} - x^{q-2}}{(x^{-2} - x^2)(x^{-q+1} + x^{q-1})} - \frac{x^{-2} - x^2}{(x^{-2} - x^2)(x^{-3} + x^3)} \\
&\quad = \frac{(x^{-q+4} - x^{q-4})(x^{-1} + x)}{(x^{-2} - x^2)(x^{-3} + x^3)(x^{-q+1} + x^{q-1})}.
\end{aligned}$$

The result then follows by multiplying both sides by

$$(x^{-1} + x)(x^{-2} + x^2)(x^{-3} + x^3)(x^{-q+1} + x^{q-1}).$$ □

**Theorem 3.4.** *For $k \geq 1$ and $q \geq 2k-1$,*

$$\sum_{i_1=1}^{q-2(k-1)} \sum_{i_2=i_1+2}^{q-2(k-2)} \cdots \sum_{i_k=i_{k-1}+2}^{q} \prod_{j=1}^{k} (x^{-i_j}+x^{i_j})^{-1}(x^{-i_j-1}+x^{i_j+1})^{-1}$$

$$= \frac{\prod_{i=k-1}^{2k-2}(x^{-(q-i)}-x^{q-i})}{\prod_{i=1}^{k}(x^{-2i}-x^{2i})\prod_{i=-1}^{k-2}(x^{-(q-i)}+x^{q-i})}. \tag{7}$$

*Proof.* We prove the equation by induction on $k$. When $k=1$ the equation holds by Lemma 3.2 (i). Lemma 3.2 (ii) shows that for arbitrary $k$ the formula holds for $q=2k-1$; so we fix $k$ and proceed by induction on $q$. Let $S_{k,q}$ and $T_{k,q}$ denote respectively the left hand and right hand sides of equation (7).

If we write $S_{k,q}$ as a sum of two terms, the first in which $i_k < q$ and the second when $i_k = q$, we see that $S_{k,q}$ satisfies the recurrence relation

$$S_{k,q} = S_{k,q-1} + (x^{-q}+x^{q})^{-1}(x^{-q-1}+x^{q+1})^{-1} S_{k-1,q-2}.$$

Thus we have only to show that $T_{k,q}$ satisfies the same relation. Now

$$T_{k,q-1} = \frac{\prod_{i=k}^{2k-1}(x^{-(q-i)}-x^{q-i})}{\prod_{i=1}^{k}(x^{-2i}-x^{2i})\prod_{i=0}^{k-1}(x^{-(q-i)}+x^{q-i})}$$

and

$$T_{k-1,q-2} = \frac{\prod_{i=k}^{2k-2}(x^{-(q-i)}-x^{q-i})}{\prod_{i=1}^{k-1}(x^{-2i}-x^{2i})\prod_{i=1}^{k-1}(x^{-(q-i)}+x^{q-i})}.$$

The proof of the recurrence relation for $T_{k,q}$ is thus reduced to verifying that

$$\frac{(x^{-(q-(k-1))}-x^{q-(k-1)})(x^{-(q-(k-1))}+x^{q-(k-1)})}{(x^{-q-1}+x^{q+1})(x^{-q}+x^{q})}$$
$$= \frac{x^{-(q-(2k-1))}-x^{q-(2k-1)}}{x^{-q}+x^{q}} + \frac{x^{-2k}-x^{2k}}{(x^{-q}+x^{q})(x^{-q-1}+x^{q+1})}. \qquad \square$$

**Theorem 3.5.** *For $k \geq 3$ and $q \geq 2k-1$,*

$$(x^{-1}+x^{1})^{-1}(x^{-2}+x^{2})^{-1}(x^{-q}+x^{q})^{-1}(x^{-q-1}+x^{q+1})^{-1}$$
$$\times \sum_{i_1=3}^{q-2(k-2)} \sum_{i_2=i_1+2}^{q-2(k-3)} \cdots \sum_{i_{k-2}=i_{k-3}+2}^{q-2} \prod_{j=1}^{k-2} (x^{-i_j}+x^{i_j})^{-1}(x^{-i_j-1}+x^{i_j+1})^{-1}$$

$$= \frac{(x^{-k+1}-x^{k-1})(x^{-k}-x^{k})\prod_{i=k+1}^{2k-2}(x^{-q+i}-x^{q-i})}{\prod_{i=1}^{k}(x^{-2i}-x^{2i})\prod_{i=-1}^{k-2}(x^{-q+i}+x^{q-i})}. \tag{8}$$

*Proof.* Let us denote the left and right hand sides of the identity by $S_{k,q}$ and $T_{k,q}$ respectively. By Corollary 3.3 $S_{3,q}=T_{3,q}$. We write $S_{k,q}$ as the sum of two terms: in the first $i_{k-2} < q-2$ and in the second $i_{k-2} - q - 2$. As in the proof of the previous theorem we obtain a recurrence relation, in this case:

$$S_{k,q} = S_{k,q-1}(x^{-q-1}+x^{q+1})^{-1}(x^{-q+1}+x^{q-1})$$
$$+ S_{k-1,q-2}(x^{-q}+x^{q})^{-1}(x^{-q-1}+x^{q+1})^{-1}.$$

It is routine to verify that $T_{k,q}$ satisfies the same recurrence relation. $\square$

Subtracting equation (8) from equation (7) yields.

**Corollary 3.6.**

$$\sum_{i_1=1}^{q-2(k-1)} \sum_{i_2=i_1+2}^{q-2(k-2)} \cdots \sum_{i_k=i_{k-1}+2}^{q} \prod_{j=1}^{k} (x^{-i_j}+x^{i_j})^{-1}(x^{-i_j-1}+x^{i_j+1})^{-1}$$
$$-(x^{-1}+x^{1})^{-1}(x^{-2}+x^{2})^{-1}(x^{-q}+x^{q})^{-1}(x^{-q-1}+x^{q+1})^{-1}$$
$$\times \sum_{i_1=3}^{q-2(k-2)} \sum_{i_2=i_1+2}^{q-2(k-3)} \cdots \sum_{i_{k-2}=i_{k-3}+2}^{q-2} \prod_{j=1}^{k-2} (x^{-i_j}+x^{i_j})^{-1}(x^{-i_j-1}+x^{i_j+1})^{-1}$$
$$(9) \quad = \frac{(x^q-x^{-q}) \prod_{i=k+1}^{2k-2} (x^{-q+i}-x^{q-i})}{\prod_{k=1}^{q} (x^{2i}-x^{-2i}) \prod_{i=-1}^{k-2} (x^{-q+i}+x^{q-i})}.$$

*Proof of Theorem* 2.5. We recall that $1 \leq p < q$ and $p$ and $q$ are relatively prime. We set $\theta = p/q$ and $a_j = 2\cos(2\pi j\theta)$. We shall split the proof into two cases.

*Case 1:* $q \not\equiv 0 \pmod 4$. When $q \not\equiv 0 \pmod 4$ $a_j \neq 0$ for all $j$; moreover when $x = e^{2\pi i\theta}$, $x^{4i} \neq 1$ and $x^{2(q-i)} \neq -1$ for all $i$. Thus the denominator on the right hand side of (9) does not vanish but the numerator does. Hence by Theorem 3.1

$$\sum_{I \in \widetilde{S}\left[{q \atop k}\right]} (a_{I^c})^{-1} = 0.$$

Upon multiplying by $a_1 a_2 \cdots a_q$ we obtain that

$$\sum_{I \in \widetilde{S}\left[{q \atop k}\right]} a_I = a_1 a_2 \cdots a_q \sum_{I \in \widetilde{S}\left[{q \atop k}\right]} (a_{I^c})^{-1} = 0.$$

*Case 2:* $q \equiv 0 \pmod 4$. Again we wish to show that $\sum_{I \in \widetilde{S}\left[{q \atop k}\right]} a_I = 0$ and so we must multiply both sides of equation (9) by $\prod_{i=1}^{q}(x^{-i}+x^{i})$ and evaluate at $x = e^{2\pi i\theta}$.

The denominator of the right hand side of (9) is zero when $x^{4q} = 1$ or $x^{2(q-j)} = -1$, i.e. when $i = j = q/4$; the corresponding factors are $x^{-q/2} - x^{q/2}$ and $x^{-3q/4} + x^{3q/4}$ respectively.

Apart from the factor $x^{-q} - x^{q}$, the numerator of the right hand side of equation (9) is zero only when $x^{2(q-i)} = 1$, i.e. when $i = q/2$. This produces the factor $x^{-q/2} - x^{q/2}$ which cancels one of the zeros in the denominator. The other zero is cancelled when we multiply by $\prod_{i=1}^{q}(x^{-i}+x^{i})$. Hence the product of $\prod_{i=1}^{q}(x^{-i}+x^{i})$ and the right side of (9) is zero when $x = e^{2\pi i\theta}$. □

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