PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 10, October 2007, Pages 3249–3254
S 0002-9939(07)08836-3
Article electronically published on June 20, 2007

ON THE GROWTH OF THE NUMBER OF PERIODIC POINTS FOR SMOOTH SELF-MAPS OF A COMPACT MANIFOLD

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(Communicated by Jane M. Hawkins)

ABSTRACT. Let f be a continuous self-map of a smooth compact connected and simply-connected manifold of dimension $m \geq 3$. We show that in the homotopy class of f there is a C^1 map with less then r periodic points, up to any given fixed period r.

1. INTRODUCTION

Shub and Sullivan (cf. [SS]) considered a smooth self-map f of a compact manifold M such that fixed points of f^n are isolated. Under the condition that the sequence of Lefschetz numbers $\{L(f^n)\}_{n=1}^{\infty}$ is unbounded, they proved that f has an infinite number of periodic points with distinct minimal periods. In particular all smooth self-maps of S^m of degree d, with $|d| > 1$, have this property. The natural question arises: what is the growth rate of the number of periodic points (with period not greater than a prescribed r)? There are open sets of C^k diffeomorphisms where generic (residual) maps have arbitrary fast growth of the number of periodic points (cf. [K]). As the growth of unbounded Lefschetz numbers is exponential, Shub and Sullivan conjectured that for any given smooth map f the growth of the number of periodic points is also at least (asymptotically) exponential, namely,

$$\limsup_{r\to\infty} \frac{\log \#\mathrm{Fix}(f^r)}{r} \geq \limsup_{r\to\infty} \frac{\log |L(f^r)|}{r}. \tag{1.1}$$

The main result of this paper is related to the Shub-Sullivan Conjecture and is the following.

Theorem 1.1. *Let $f : M \to M$ be a continuous self-map of a smooth compact connected and simply-connected manifold M, $\dim M \geq 3$. Let $r \in \mathbb{N}$, $r \geq 2$. For M with nonempty boundary we assume additionally that f has no periodic points on the boundary.*

Then f is homotopic to a smooth map g satisfying

$$\#\left(\bigcup_{i\leq r} \mathrm{Fix}(g^i)\right) \leq r-1.$$

Received by the editors March 30, 2006 and, in revised form, June 30, 2006.
2000 *Mathematics Subject Classification.* Primary 37C25, 55M20; Secondary 37C05.
Key words and phrases. Periodic points, C^1 maps, indices of iterations, Nielsen number.
This research was supported by KBN grant No 1 P03A 03929.

Of course the above theorem does not contradict the Shub-Sullivan Conjecture, since our methods do not control what happens in the periods higher than a prescribed r. It only states that for each map f the growth of the number

$$h_r[f] = \min_g \# \left(\bigcup_{i=1}^{r} \mathrm{Fix}(g^i) \right),$$

where g runs over the set of smooth maps homotopic to f, is not faster than linear. By Theorem 1.1 we may generate examples of smooth maps with an unbounded sequence of Lefschetz numbers with a relatively small set of periodic points, up to any given fixed period r. The growth of

$$c_r(f) = \# \left(\bigcup_{i=1}^{r} \mathrm{Fix}(f^i) \right),$$

as well as $\#\mathrm{Fix}(f^r)$, still remains unknown. On the other hand the growth of $c_r(f)$ must be at least linear (with the coefficient depending on f and a manifold), which follows from [BaBo]:

Remark 1.2. Babenko and Bogatyi proved that for f, a self-map of an m-dimensional smooth compact manifold M, such that $\{L(f^n)\}_{n=1}^{\infty}$ is unbounded, the following inequality holds:

$$O(f,r) \geq \frac{r - N_0}{D \cdot 2^{[(m+1)/2]}},$$

where $O(f,r)$ is the number of periodic orbits of period no greater than r, $D = \dim H_*(M,\mathbb{Q})$ and N_0 is a natural number.

As a consequence, the inequality $c_r(f) > O(f,r)$ gives the linear estimate from below for $c_r(f)$.

2. Preliminary results

2.1. **Fixed point indices of iterations.** Let f be a continuous self-map of a compact smooth manifold. We deal with the problem what is the least number of periodic points of a smooth map homotopic to f. It appears that the forms of local fixed point indices of iterations of C^1 maps are crucial for studying the possibility of cancelling the periodic points in the homotopy class.

In 1983 A. Dold (cf. [D1]) gave some necessary conditions (called Dold relations) which must be satisfied by any sequence of indices of iterations. Namely, let $f : U \to X$, where U is an open subset of a finite polyhedron X (or more generally, of a Euclidean Neighborhood Retract).

We denote inductively $U_0 = U$, $U_{n+1} = f^{-1}(U_n)$. We assume that the fixed point set $\mathrm{Fix}(f^n) = \{x \in U_n : f^n(x) = x\}$ is compact for each $n \in \mathbb{N}$. In such a situation the fixed point index $\mathrm{ind}(f^n) = \mathrm{ind}(f^n, U_n)$ is well-defined (cf. [D2]). Dold proved that the sequence of fixed point indices $\{\mathrm{ind}(f^n)\}_{n=1}^{\infty}$ must satisfy the following congruences: for each $n \in \mathbb{N}$,

$$\sum_{k|n} \mu(n/k)\mathrm{ind}(f^k) \equiv 0 \pmod{n}, \tag{2.1}$$

where μ is the classical Möbius function, i.e., $\mu : \mathbb{N} \to \mathbb{Z}$ is defined by three properties: $\mu(1) = 1$, $\mu(k) = (-1)^r$ if k is a product of r different primes, $\mu(k) = 0$ otherwise.

Let f be a C^1 map with an isolated fixed point x_0. Then there are further strong restrictions, except for the Dold relations, on the form of the local indices $\{\mathrm{ind}(f^n, x_0)\}_{n=1}^{\infty}$ (cf. [SS], [CMPY]).

A complete description of sequences which can be realized as indices at an isolated fixed point in $\mathbb{R}^3$ is given in [GNP]. For the sake of this paper we need the existence of smooth maps which have sequences of local indices of iterations in $\mathbb{R}^m$, $m \geq 3$, following two special patterns. The constructions of such maps may be found by the reader in [GNP] (Theorem 3 cases (A) and (D)).

Lemma 2.1. *Let a_1, a_2 and a_d, $d > 1$ be arbitrarily given integers and let $x_0 \in \mathbb{R}^m$, $m \geq 3$. Then there exist: an open neighborhood U of x_0 and C^1 maps $f_{a_1,a_2}; f_{a_d} : U \to \mathbb{R}^m$, such that x_0 is an isolated fixed point for each iteration and*

$$(*) \qquad \mathrm{ind}(f^n_{a_1,a_2}, x_0) = \begin{cases} a_1 & \text{if } 2 \nmid n, \\ a_1 + 2a_2 & \text{if } 2|n, \end{cases}$$

$$(**) \qquad \mathrm{ind}(f^n_{a_d}, x_0) = \begin{cases} 0 & \text{if } d \nmid n, \\ a_d d & \text{if } d|n. \end{cases}$$

Remark 2.2. Let us emphasize that maps with the indices of iterations of the form $(*)$ and $(**)$ do not exist in $\mathbb{R}^m$ for $m < 3$ (cf. [BaBo]); i.e., 3 is the lowest dimension in which such types of sequences may be realized as local indices of iterations of a C^1 map.

2.2. **Adding and cancelling periodic points in a homotopy class.** In the proof of Theorem 1.1 we will also need two statements, proved by the methods of Nielsen Periodic Point Theory in [Je] (cf. also [JM]), which allow one to add or remove some periodic points in a homotopy class. We assume in this section that M is a compact connected and simply-connected manifold of dimension at least 3.

The following theorem (Creating Procedure) enables one to create an additional orbit in the homotopy class of f, by a homotopy f_t which is constant near periodic points of f (up to the given period r) and such that f_1^n near the created n-orbit may be given by an arbitrarily prescribed formula. More precisely:

Theorem 2.3 ([Je], Theorem 3.3)**.** *Given numbers $n, r \in \mathbb{N}$, $n|r$ and a continuous map $f : M \to M$, where $\dim M \geq 3$, such that $\mathrm{Fix}(f^r)$ is finite and a point $x_0 \notin \mathrm{Fix}(f^r)$, then there is a homotopy $\{f_t\}_{0 \leq t \leq 1}$ satisfying:*

(1) $f_0 = f$.
(2) *$\{f_t\}$ is constant in a neighborhood of* $\mathrm{Fix}(f^r)$.
(3) *$f_1^n(x_0) = x_0$ and $f_1^i(x_0) \neq x_0$ for $i = 1, \dots, n-1$.*
(4) *The map f_1^n is given near x_0 by an arbitrarily prescribed formula ϕ with the property $\phi^{\frac{r}{n}}(x) = x \iff x = x_0$.*
(5) *The orbit $x_0, f_1(x_0), \dots, f_1^{n-1}(x_0)$ is isolated in* $\mathrm{Fix}(f_1^r)$.

The next lemma enables one to cancel some subsets of periodic points with indices of iterations equal to zero.

Lemma 2.4 ([Je], Lemma 5.2)**.** *Let f be a continuous self-map of M. Suppose that $S \subset \mathrm{Fix}(f^r)$ satisfies:*

(1) *S is finite and f-invariant, i.e., $f(S) = S$.*
(2) $\mathrm{Fix}(f^r) \setminus S$ *is compact.*
(3) $\mathrm{ind}(f^n, \mathrm{Fix}(f^n) \setminus S) = 0$ *for all $n|r$.*

Then there is a homotopy f_t, starting from $f_0 = f$, constant near S and such that $\mathrm{Fix}(f_1^r) = S$.

We are ready to prove our main theorem. The brief scheme of the proof is the following. We will use an inductive procedure: at the inductive step we add a single fixed point by the Creating Procedure (Theorem 2.3), taking as ϕ an appropriate realization from Lemma 2.1, i.e., inserting locally a C^1 map with a fixed point having indices following the given pattern. This is done in such a way that at the end of the homotopy the other periodic points have indices equal to zero, so may be removed by Lemma 2.4.

3. Proof of Theorem 1.1

We use induction for $r \geq 2$. Let $r = 2$. We assume that $\mathrm{Fix}(f^2)$ is finite, since otherwise we may approximate f by a smooth map which has a finite number of fixed and 2-periodic points by the Kupka-Smale theorem.

Next we create an additional fixed point $z_2 \notin \mathrm{Fix}(f^2)$ by the Creating Procedure ($n = 1$), deforming f to $\bar{f}_2$ in such a way that $\bar{f}_2 = f_{a_1,a_2}$ near z_2 with $a_1 = \mathrm{ind}(f)$, $a_2 = \frac{1}{2}(\mathrm{ind}(f^2) - \mathrm{ind}(f))$ (cf. item 4 of Theorem 2.3 and $(*)$ of Lemma 2.1). Notice that by the Dold relations (cf. Formula 2.1) a_2 is an integer. As a result, by Theorem 2.3, $\bar{f}_2$ is homotopic to f and

$$\mathrm{ind}(\bar{f}_2, z_2) = a_1 = \mathrm{ind}(f),$$
$$\mathrm{ind}(\bar{f}_2^2, z_2) = a_1 + 2a_2 = \mathrm{ind}(f^2).$$

Now $\bar{f}_2$, $S = \{z_2\}$ and $r = 2$ satisfy the assumptions (1)–(3) of Lemma 2.4; hence $\bar{f}_2$ is homotopic, rel. a neighborhood of z_2, to a map f_2 with $\mathrm{Fix}(f_2^2) = \{z_2\}$. The map f_2 is smooth in some neighborhood U_2 of z_2 and f_2^2 has no fixed points outside U_2. Thus, if f_2 were not smooth as the global self-map of M, we could approximate it by a smooth map, and then use a homotopy constant on U_2, without adding new r-fixed points ($r = 1, 2$) in the compact set $M \setminus U_2$.

Now we assume that $\bigcup_{i<r} \mathrm{Fix}(f^i) = \{z_2, ..., z_{r-1}\}$. Again, by the Kupka-Smale approximation argument, we may assume that $\mathrm{Fix}(f^r)$ is finite. Then the set

$$A = \mathrm{Fix}(f^r) \setminus \left(\bigcup_{i<r} \mathrm{Fix}(f^i) \right) = \mathrm{Fix}(f^r) \setminus \{z_2, ..., z_{r-1}\}$$

splits into r-orbits; hence $\mathrm{ind}(f^r, A)$ is divisible by r.

We deform f by the Creating Procedure (again $n = 1$), rel. a neighborhood of $\{z_2, ..., z_{r-1}\}$, to a map $\bar{f}_r$ having a fixed point z_r such that in its neighborhood $\bar{f}_r = f_{a_d}$ with $a_d = \frac{\mathrm{ind}(f^r, A)}{r}$, $d = r$ (cf. item 4 of Theorem 2.3 and $(**)$ of Lemma 2.1).

We will show that $S = \{z_2, ..., z_r\}$ and that the $\bar{f}_r$ satisfy the assumptions of Lemma 2.4. In fact only the condition (3), i.e., $\mathrm{ind}(\bar{f}_r^n, \mathrm{Fix}(\bar{f}_r^n) \setminus S) = 0$ for $n|r$, needs to be checked. For $n < r$, $\mathrm{ind}(\bar{f}^n, z_r) = 0$ (cf. $(**)$ of Lemma 2.1); thus

$$\mathrm{ind}(\bar{f}_r^n, S) = \mathrm{ind}(\bar{f}_r^n, \{z_2, ..., z_{r-1}, z_r\}) = \mathrm{ind}(\bar{f}_r^n, \{z_2, ..., z_{r-1}\}).$$

On the other hand, by item (2) of Theorem 2.3 and homotopy invariance of the fixed point index we get

$$\mathrm{ind}(\bar{f}_r^n, \{z_2, ..., z_{r-1}\}) = \mathrm{ind}(f^n, \{z_2, ..., z_{r-1}\}) = \mathrm{ind}(f^n) = \mathrm{ind}(\bar{f}_r^n).$$

Now $\mathrm{ind}(\bar{f}_r^n) = \mathrm{ind}(\bar{f}_r^n, S)$, which implies, by additivity of the fixed point index, that $\mathrm{ind}(\bar{f}_r^n, \mathrm{Fix}(\bar{f}_r^n) \setminus S) = 0$, as desired.

For $n = r$, $\mathrm{ind}(\bar{f}_r^n, z_r) = \mathrm{ind}(f^r, A)$; thus

$$\begin{aligned}\mathrm{ind}(\bar{f}_r^r, \{z_2, ..., z_{r-1}, z_r\}) &= \mathrm{ind}(\bar{f}_r^r, \{z_2, ..., z_{r-1}\}) + \mathrm{ind}(\bar{f}_r^r, z_r)\\ &= \mathrm{ind}(f^r, \{z_2, ..., z_{r-1}\}) + \mathrm{ind}(f^r, A)\\ &= \mathrm{ind}(f^r, \{z_2, ..., z_{r-1}\}) + \mathrm{ind}(f^r, \mathrm{Fix}(f^r) \setminus \{z_2, ..., z_{r-1}\})\\ &= \mathrm{ind}(f^r, \mathrm{Fix}(f^r)) = \mathrm{ind}(f^r).\end{aligned}$$

Thus,

$$\mathrm{ind}(\bar{f}_r^r, \{z_2, ..., z_{r-1}, z_r\}) = \mathrm{ind}(f^r) = \mathrm{ind}(\bar{f}_r^r).$$

As a consequence, $\mathrm{ind}(\bar{f}_r^r, \mathrm{Fix}(\bar{f}_r^r) \setminus S) = 0$, as desired. Then Lemma 2.4 allows us to deform $\bar{f}_r$ to a map f_r with $\mathrm{Fix}(f_r^r) = S = \{z_2, ..., z_r\}$.

Finally, if f_r is not smooth as the self-map of M, we approximate it by a smooth map in the same way as we did with f_2, which ends the proof.

Remark 3.1. A map g, homotopic to the given f, with the growth of periodic points no faster than linear to a prescribed period r, can be chosen as smooth as desired. Namely, we can assure (cf. Remark 4 in [GNP]) that the maps which realize the indices of iterations from Lemma 2.1 are C^l, where l is any natural number. This implies that we can find g which is a C^l map with l arbitrarily large.

Acknowledgments

The authors thank Professor Wacław Marzantowicz for the inspiration, encouragement and helpful comments.

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