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A NEW PROOF OF THE TRANSFORMATION LAW OF JACOBI'S THETA FUNCTION $\theta_3(w,\tau)$

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Abstract. We present a new proof, using Residue Calculus, of the transformation law of the Jacobi theta function $\theta_3(w,\tau)$ defined in the upper half plane. Our proof is inspired by Siegel's proof of the transformation law of the Dedekind eta function.

1. Introduction

Let $w = \sigma + it$ be a complex number. The Jacobi theta function $\theta_3(w,\tau)$ is defined by

$$\theta_3(w,\tau) = \prod_{n=1}^{\infty}(1-q^{2n})(1+2q^{2n-1}\cos 2w + q^{4n-2}) \ , \tag{1}$$

where $q = e^{\pi i \tau}$ and τ is in the upper half plane [6]. The transformation law (see [6, XXI], [2, XVI]) is given by

$$\theta_3\left(\frac{w}{\tau}, \frac{-1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{-\frac{w^2}{\pi i \tau}} \theta_3(w,\tau) \ . \tag{2}$$

The transformation law (2) was first obtained by Jacobi who obtained it from the theory of elliptic functions. Notice that at $w = 0$, we have

$$\theta_3\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} \theta_3(\tau) \ . \tag{3}$$

In [5], Siegel gives a nice proof, using residue calculus, of the transformation law of the Dedekind eta function under inversion. Radamacher [4] generalized Siegel's method to determine the transformation formula of the Dedekind eta function under any modular transformation. The referee also pointed out that Siegel's method has been enormously generalized by Sarachi Kongsiriwong [3]. In this paper, we give a new detailed proof of (2) inspired by Siegel's proof. First, we prove (2) for $\tau = iy$, where $y > 2|t|/\pi$, and then we extend the result to all τ in the upper half plane by analytic continuation.

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2. The proof

In proving the transformation law of the logarithmic derivative, we will encounter some problems with the zeroes of the theta function. The zeroes of $\theta_3(w,\tau)$ are the points

$$w = \frac{\pi}{2} + \frac{\pi\tau}{2} + m\pi + n\pi\tau,$$

for $m, n \in Z$.

To deal with this problem, we fix w such that $\mathrm{Re}w \neq \pi/2 + n\pi$ and prove (2) for $\tau = iy$ using the logarithmic derivative. We then extend the result by analytic continuation to the whole τ plane. Once we have it for all w such that $\mathrm{Re}w \neq \pi/2 + n\pi$, we use analytic continuation again in the w plane to extend the result to all w.

Fix w such that $\mathrm{Re}w \neq \pi/2 + n\pi$.

Theorem 1. *If $\tau = iy$ and $y > 2|t|/\pi$, where $w = \sigma + it$, then the transformation formula is*

$$\theta_3\left(\frac{w}{iy}, \frac{i}{y}\right) = (y)^{\frac{1}{2}} e^{\frac{w^2}{\pi y}} \theta_3(w, iy) \quad . \tag{4}$$

Proof. If $\mathrm{Re}w \neq \pi/2 + n\pi$, then it is sufficient to prove that

$$\log\theta_3(w, iy) - \log\theta_3\left(\frac{w}{iy}, \frac{i}{y}\right) + \frac{w^2}{\pi y} = -\frac{1}{2}\log y \quad . \tag{5}$$

If we simplify (1), we obtain

$$\begin{aligned} \theta_3(w,\tau) &= \prod_{n=1}^{\infty}(1-q^{2n})(e^{2iw}+q^{2n-1})(e^{-2iw}+q^{2n-1}) \\ &= \prod_{n=1}^{\infty}(1-q^{2n})(1+\frac{q^{2n-1}}{e^{2iw}})(1+\frac{q^{2n-1}}{e^{-2iw}}). \end{aligned} \tag{6}$$

Since $y > 2|t|/\pi$, then $\left|q^{2n-1}/e^{2iw}\right| < 1$. Thus $\log\theta_3(w, iy)$ given by (6) can be expanded in the form

(7)

$$\begin{aligned}
\log\theta_3(w,iy) &= \sum_{n=1}^{\infty}\log(1-q^{2n}) + \sum_{n=1}^{\infty}\log(1+\frac{q^{2n-1}}{e^{2iw}}) + \sum_{n=1}^{\infty}\log(1+\frac{q^{2n-1}}{e^{-2iw}}) \\
&= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(q^{2n})^m}{m} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\frac{(q^{2n-1})^m}{e^{2imw}} \\
&\quad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\frac{(q^{2n-1})^m}{e^{-2imw}} \\
&= -\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{q^{2m}}{1-q^{2m}}\right) + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{me^{2imw}}q^{-m}\left(\frac{q^{2m}}{1-q^{2m}}\right) \\
&\quad + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{me^{-2imw}}q^{-m}\left(\frac{q^{2m}}{1-q^{2m}}\right) \\
&= -\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{e^{-2\pi ym}}{1-e^{-2\pi ym}}\right) + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\frac{e^{\pi my}}{e^{2imw}}\left(\frac{e^{-2\pi ym}}{1-e^{-2\pi ym}}\right) \\
&\quad + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\frac{e^{\pi my}}{e^{-2imw}}\left(\frac{e^{-2\pi ym}}{1-e^{-2\pi ym}}\right) \\
&= \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{1}{1-e^{2\pi ym}}\right) + \sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{-2iwm}\left(\frac{e^{\pi ym}}{1-e^{2\pi ym}}\right) \\
&\quad + \sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{2iwm}\left(\frac{e^{\pi ym}}{1-e^{2\pi ym}}\right).
\end{aligned}$$

To obtain a similar expansion for $\log\theta_3(w/iy, i/y)$, we use (7) to obtain

(8)

$$\begin{aligned}
\log\theta_3(\frac{w}{iy},\frac{i}{y}) &= \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{1}{1-e^{2\pi m/y}}\right) + \sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{-2mw/y}\left(\frac{e^{\pi m/y}}{1-e^{2\pi m/y}}\right) \\
&\quad + \sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{2wm/y}\left(\frac{e^{\pi m/y}}{1-e^{2\pi m/y}}\right).
\end{aligned}$$

We substitute in (5) the expressions (7) and (8), so we have to prove that

(9)

$$\begin{aligned}
&\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{1}{1-e^{2\pi ym}}\right) + \sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{-2iwm}\left(\frac{e^{\pi ym}}{1-e^{2\pi ym}}\right) \\
&\quad + \sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{2iwm}\left(\frac{e^{\pi ym}}{1-e^{2\pi ym}}\right) - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{1}{1-e^{2\pi m/y}}\right) \\
&\quad - \sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{-2mw/y}\left(\frac{e^{\pi m/y}}{1-e^{2\pi m/y}}\right) \\
&\quad - \sum_{m=1}^{\infty}\frac{(-1)^m}{m}e^{2wm/y}\left(\frac{e^{\pi m/y}}{1-e^{2\pi m/y}}\right) + \frac{w^2}{\pi y} = -\frac{1}{2}\log y.
\end{aligned}$$

To prove (9), consider
(10)
$$F_n(z) = -\frac{1}{8z}\cot \pi i N z \cot \pi N z/y + \frac{1}{z}\left(\frac{e^{-iNz(\pi/y+2w/y+2w)}}{1-e^{-2\pi i N z/y}}\right)\left(\frac{e^{Nz(\pi+2iw)}}{1-e^{2\pi z N}}\right),$$
where $N = n + 1/2$.

We will calculate the residues of $F_n(z)$ at the poles $z = 0$, $z = ik/N$ and $z = ky/N$ for $k = \pm 1, \pm 2, \dots$.

First, we calculate the residue of $F_n(z)$ at $z = 0$. We use Bernoulli numbers to calculate the residue of the second summand of $F_n(z)$. The residue at $z = 0$ of the first summand of (10) is

$$\frac{i}{24}\left(y - \frac{1}{y}\right). \tag{11}$$

Now for the second summand of (10) we will use the fact that

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \tag{12}$$

where $B_0 = 1$, $B_1 = -1/2$ and $B_2 = 1/6$.

Notice that
$$\frac{1}{z}\left(\frac{e^{-iNz(\pi/y+2w/y+2w)}}{1-e^{-2\pi i N z/y}}\right)\left(\frac{e^{Nz(\pi+2iw)}}{1-e^{2\pi z N}}\right) = \frac{-y}{4\pi^2 i N^2 z^3}\left(\frac{-2\pi i N z/y}{e^{-2\pi i N z/y}-1}\right)\left(\frac{2\pi N z}{e^{2\pi z N}-1}\right)e^{Nz(-i\pi/y-2iw/y+\pi)}.$$

Using (12) and the Taylor expansion of e^z, we find that the residue at $z = 0$ of the second summand of (10) is

$$\begin{aligned} \frac{-y}{4\pi^2 i N^2}\{&\frac{N^2}{2}\left(-\frac{\pi^2}{y^2} - \frac{4w^2}{y^2} + \pi^2 - \frac{4\pi w}{y^2} - \frac{2\pi^2 i}{y} - \frac{4\pi i w}{y}\right) \\ &+ \left(-\frac{\pi^2 N^2 i}{y} - \frac{\pi^2 N^2}{3y^2} + \frac{\pi^2 N^2}{3}\right) \\ &+ \left(\frac{\pi i N^2}{y} - \pi N^2\right)\left(-\frac{i\pi}{y} - \frac{2iw}{y} + \pi\right)\}. \end{aligned} \tag{13}$$

We simplify (13) to obtain

$$\frac{w^2}{2\pi^2 i y} - \frac{i}{24}\left(y - \frac{1}{y}\right). \tag{14}$$

As a result, we add (11) and (14) to get that the residue at $z = 0$ of $F_n(z)$ is

$$Res[F_n(z), 0] = \frac{w^2}{2\pi^2 i y}. \tag{15}$$

We note that

$$\mathrm{Res}[F_n(z), \frac{ik}{N}] = \frac{1}{8\pi k}\cot\frac{\pi i k}{y} - \frac{(-1)^k}{2\pi i k}e^{2kw/y}\frac{e^{\pi k/y}}{1-e^{2\pi k/y}}. \tag{16}$$

Using (16), we obtain

$$
\begin{aligned}
\sum_{\substack{k=-n;\ k\neq 0\\ z=\frac{ik}{N}}}^{n} \operatorname{Res}[F_n(z), \frac{ik}{N}] &= 2\sum_{k=1}^{n} \frac{1}{8\pi k}\cot\frac{\pi ik}{y} - \sum_{k=1}^{n}\frac{(-1)^k}{2\pi ik}e^{2kw/y}\frac{e^{\pi k/y}}{1-e^{2\pi k/y}} \\
&\quad - \sum_{k=1}^{n}\frac{(-1)^k}{2\pi ik}e^{-2kw/y}\frac{e^{-\pi k/y}}{1-e^{2\pi k/y}} \\
&= \frac{1}{4\pi i}\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{2\pi i}\sum_{k=1}^{n}\frac{1}{k(1-e^{2\pi k/y})} \\
&\quad - \frac{1}{2\pi i}\sum_{k=1}^{n}\frac{(-1)^k}{k}e^{2kw/y}\frac{e^{\pi k/y}}{1-e^{2\pi k/y}} \\
&\quad - \frac{1}{2\pi i}\sum_{k=1}^{n}\frac{(-1)^k}{k}e^{-2kw/y}\frac{e^{\pi k/y}}{1-e^{2\pi k/y}}.
\end{aligned}
\tag{17}
$$

The residue of $F_n(z)$ at $z = ky/N$ is

$$
\operatorname{Res}[F_n(z), \frac{ky}{N}] = -\frac{1}{8\pi k}\cot\pi iky + \frac{(-1)^k}{2\pi ik}e^{-2ikw}\frac{e^{\pi ky}}{1-e^{2\pi ky}}. \tag{18}
$$

Using (18), we obtain

(19)

$$
\begin{aligned}
&\sum_{\substack{k=-n;\ k\neq 0\\ z=\frac{ky}{N}}}^{n} \operatorname{Res}\ [F_n(z), \frac{ky}{N}] \\
&= 2\sum_{k=1}^{n} -\frac{1}{8\pi k}\cot\pi iky + \sum_{k=1}^{n}\frac{(-1)^k}{2\pi ik}e^{-2ikw}\frac{e^{\pi ky}}{1-e^{2\pi ky}} + \sum_{k=1}^{n}\frac{(-1)^k}{2\pi ik}e^{2ikw}\frac{e^{\pi ky}}{1-e^{2\pi ky}} \\
&= -\frac{1}{4\pi i}\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{2\pi i}\sum_{k=1}^{n}\frac{1}{k(1-e^{2\pi ky})} + \frac{1}{2\pi i}\sum_{k=1}^{n}\frac{(-1)^k}{k}e^{-2ikw}\frac{e^{\pi ky}}{1-e^{2\pi ky}} \\
&\quad + \frac{1}{2\pi i}\sum_{k=1}^{n}\frac{(-1)^k}{k}e^{2ikw}\frac{e^{\pi ky}}{1-e^{2\pi ky}}.
\end{aligned}
$$

It follows from (15), (17) and (19) that

$$
\begin{aligned}
2\pi i \sum_{\substack{k=-n\\ z=\frac{ky}{N};z=\frac{ik}{N}}}^{n} \operatorname{Res}F_n(z) &= \sum_{k=1}^{n}\frac{1}{k}\left(\frac{1}{1-e^{2\pi yk}}\right) + \sum_{k=1}^{n}\frac{(-1)^k}{k}e^{-2iwk}\left(\frac{e^{\pi yk}}{1-e^{2\pi yk}}\right) \\
&+ \sum_{k=1}^{n}\frac{(-1)^k}{k}e^{2iwk}\left(\frac{e^{\pi yk}}{1-e^{2\pi yk}}\right) - \sum_{k=1}^{n}\frac{1}{k}\left(\frac{1}{1-e^{2\pi k/y}}\right) \\
&- \sum_{k=1}^{n}\frac{(-1)^k}{k}e^{-2kw/y}\left(\frac{e^{\pi k/y}}{1-e^{2\pi k/y}}\right) \\
&- \sum_{k=1}^{n}\frac{(-1)^k}{k}e^{2wk/y}\left(\frac{e^{\pi k/y}}{1-e^{2\pi k/y}}\right) + \frac{w^2}{\pi y}.
\end{aligned}
$$

What remains to prove is that

$$\lim_{n\to\infty}\oint_c F_n(z)dz = -\frac{1}{2}\log y,$$

where C is the parallelogram of vertices $y, i, -y$ and $-i$ taken counterclockwise.

Now it is easy to see that $\lim_{n\to\infty} zF_n(z)$ is $1/8$ on the edges connecting y to i and $-y$ to $-i$ and the limit $-1/8$ on the other two edges. Moreover, $F_n(z)$ is uniformly bounded on C for all n. Hence by the bounded convergence theorem we have

$$\begin{aligned}
\lim_{n\to\infty}\oint_c F_n(z)dz &= \oint_c zF_n(z)\frac{dz}{z} \\
&= \frac{1}{8}\left[-\int_{-i}^{y}\frac{dz}{z}+\int_{y}^{i}\frac{dz}{z}-\int_{i}^{-y}\frac{dz}{z}+\int_{-y}^{-i}\frac{dz}{z}\right] \\
&= \frac{1}{4}\left[-\int_{-i}^{y}\frac{dz}{z}+\int_{y}^{i}\frac{dz}{z}\right] \\
&= \frac{1}{4}\left[-\left(\log y+\frac{\pi i}{2}\right)+\left(\frac{\pi i}{2}-\log y\right)\right] \\
&= -\frac{1}{2}\log y.
\end{aligned}$$

This completes the proof. □

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