L-FUNCTIONS OF TWISTED DIAGONAL EXPONENTIAL SUMS OVER FINITE FIELDS

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Abstract. Let \( \mathbb{F}_q \) be the finite field of \( q \) elements with characteristic \( p \) and \( \mathbb{F}_q^m \) its extension of degree \( m \). Fix a nontrivial additive character \( \Psi \) and let \( \chi_1, \ldots, \chi_n \) be multiplicative characters of \( \mathbb{F}_p \). For \( f(x_1, \ldots, x_n) \in \mathbb{F}_q[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \), one can form the twisted exponential sum \( S^*_m(\chi_1, \ldots, \chi_n, f) \). The corresponding \( L \)-function is defined by

\[
L^*(\chi_1, \ldots, \chi_n, f; t) = \exp \left( \sum_{m=0}^{\infty} S^*_m(\chi_1, \ldots, \chi_n, f) \frac{t^m}{m} \right).
\]

In this paper, by using the \( p \)-adic gamma function and the Gross–Koblitz formula on Gauss sums, we give an explicit formula for the \( L \)-function \( L^*(\chi_1, \ldots, \chi_n, f; t) \) if \( f \) is a Laurent diagonal polynomial. We also determine its \( p \)-adic Newton polygon.

1. Introduction

Let \( \mathbb{F}_q \) be the finite field of \( q \) elements with characteristic \( p \) and \( \mathbb{F}_q^m \) its extension of degree \( m \). Fix a nontrivial additive character \( \Psi \) and let \( \chi_1, \ldots, \chi_n \) be multiplicative characters of \( \mathbb{F}_q \). If \( f(x_1, \ldots, x_n) \in \mathbb{F}_q[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \), then one can form the twisted exponential sum

\[
S^*_m(\chi_1, \ldots, \chi_n, f) = \sum_{(x_1, \ldots, x_n) \in (\mathbb{F}_q^m)^n} \chi_1(N_m(x_1)) \cdots \chi_n(N_m(x_n)) \Psi(\text{Tr}_m(f(x_1, \ldots, x_n))),
\]

where \( \text{Tr}_m : \mathbb{F}_q^m \to \mathbb{F}_q \) is the trace map and \( N_m : \mathbb{F}_q^m \to \mathbb{F}_q \) is the norm map. The corresponding \( L \)-function is defined as follows:

\[
L^*(\chi_1, \ldots, \chi_n, f; t) = \exp \left( \sum_{m=0}^{\infty} S^*_m(\chi_1, \ldots, \chi_n, f) \frac{t^m}{m} \right).
\]

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By the renowned Dwork–Grothendieck theorem, one knows that the $L$-function $L^*(\chi_1, ..., \chi_n, f; t)$ is a rational function.

Finding sharp $p$-adic estimates of the exponential sums $S^*_{m} (\chi_1, ..., \chi_n, f)$ is a fundamental question in number theory. This is well known to be reduced to determining the $p$-adic absolute values of the reciprocal roots and reciprocal poles. The best way to describe this problem is in terms of $p$-adic Newton polygons (see [11]). Of course one may consider the question of finding the $p$-adic Newton polygon of Dwork’s unit root $L$-function attached to a family of algebraic varieties over $\mathbb{F}_q$, which is proved by Wan [18, 19, 20] to be $p$-adic meromorphic as conjectured by Dwork [4]. Such question should be very deep and hard.

On the other hand, Dwork [5] determined the Newton polygon for the $L$-functions of the one-dimensional Kloosterman sum, and then Sperber [13] generalized Dwork’s result to the $n$-dimensional case while Sperber [14] considered the general hyper-Kloosterman sums. Wan [16, 17, 21] solved a conjecture of Adolphson and Sperber [1]. Note also that Wan [17] presented a $p$-adic proof of the Dwork–Mazur conjecture [3, 11]. When all the $\chi_i$ are trivial, then the $p$-adic Newton polygon of the $L$-function of the one-variable polynomial $f(x)$ received attention by Sperber [15], Hong [7, 8, 9] and Yang [22]. A general interesting question is to determine the $p$-adic Newton polygon of the $L$-function $L^*(\chi_1, ..., \chi_n, f; t)$ if $f$ is of one variable, even for small degree, and $\chi$ is a nontrivial multiplicative character.

In this paper, we are interested in the case that $f(x_1, ..., x_n)$ is a Laurent diagonal polynomial. We obtain an explicit formula for the $L$-function $L^*(\chi_1, ..., \chi_n, f; t)$ if $f$ is a Laurent diagonal polynomial by using the $p$-adic gamma function. We also determine the $p$-adic Newton polygon for such an $L$-function $L^*(\chi_1, ..., \chi_n, f; t)$. In the forthcoming works, we will use the results presented here to investigate a conjecture of Adolphson and Sperber [2] on the $p$-adic Newton polygons of twisted exponential sums.

2. Preliminaries on $p$-adic gamma functions

In this section, we present some properties of the $p$-adic gamma function $\Gamma_p$. For $s \in \mathbb{Z}^+$ one sets

$$\Gamma_p(s) = (-1)^s \prod_{i=1}^{s-1} \frac{1}{i}.$$  \hspace{1cm} (2.1)

It is known (see, for example, [12]) that this extends to a continuous function on $\mathbb{Z}_p$, and we define $\Gamma_p$ to be this extension. Therefore it follows from continuity that the following functional equation holds:

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -1, & \text{if } x \in p\mathbb{Z}_p, \\ -x, & \text{if } x \not\in p\mathbb{Z}_p, \end{cases}$$

and also we have

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\langle x \rangle_p},$$

where $\langle x \rangle_p$ denotes the least nonnegative residue of $x$ modulo $p$.

Denote by $\zeta_p$ a fixed primitive $p$-th root of unity in $\mathbb{C}_p$, the completion of the algebraic closure of the $p$-adic field $\mathbb{Q}_p$. Then there is a generator $\pi$ of the maximal ideal of the ring of integers of $\mathbb{Q}_p(\zeta_p)$ uniquely characterized by $\pi^{p-1} = -p$ and $\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}$. Let $\varpi$ be any fixed Teichmüller character of $\mathbb{F}_q$. Let $\sigma_p(a)$ denote the sum of the digits of the $p$-adic representation of an integer $a \geq 1$. 


Lemma 2.1 (Gross–Koblitz formula, see [6] or [12]). Let $1 \leq a < p^e := q$. Then the value of the Gauss sum $g_q(a)$ is explicitly given by

$$g_q(a) := - \sum_{0 \neq x \in \mathbb{F}_q} \omega(x)^{-a} \Psi(\text{Tr}(x)) = \pi_p^e(a) \prod_{j=0}^{e-1} \Gamma_p \left( \frac{(pq^j)a - 1}{q - 1} \right).$$

3. Twisted diagonal exponential sums

For brevity, we define $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for any vector $x = (x_1, \ldots, x_n) \in (\mathbb{F}_{q^m})^n$ and any $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. Since $\zeta_p$ is a given primitive $p^{th}$ root of unity in $\mathbb{C}_p$, we know that for any given nontrivial additive character $\Psi$ of $\mathbb{F}_q$, we have $\Psi(c) = \zeta_p^{\text{Tr}_{\mathbb{F}_q/p}(\sigma)}$, for $c \in \mathbb{F}_q$. Let $\varpi$ be any fixed Teichmüller character of $\mathbb{F}_{q^l}$ for any given integer $l \geq 1$. Then $(\varpi_1) = \varpi$ and for multiplicative characters $\chi, \ldots, \chi_n$, there must be nonnegative integers $b_1, \ldots, b_n$, such that $\chi_i = \varpi^{b_i}$, $1 \leq i \leq n$. Thus we can rewrite (1.1) as follows:

$$(3.1) \quad S_m^*(\chi_1, \ldots, \chi_n, f) = \sum_{x \in (\mathbb{F}_{q^m})^n} \zeta_p^{\text{Tr}_{\mathbb{F}_q/p}(f(x))} \prod_{i=1}^n (\varpi^{b_i} \circ \mathcal{N}_m)(x_i).$$

Lemma 3.1. For any $c \in \mathbb{F}_q$ and any integer $l \geq 1$, we have $(\varpi_l(c) = \varpi(c))$.

Proof. For any positive integer $c$, let $\mathbb{Z}^{\text{unram}}$ denote the ring of integers of the unramified extension of $\mathbb{Q}_p$ of degree $c$. Since the diagram

$$\begin{array}{ccc}
\mathbb{F}_q & \hookrightarrow & \mathbb{F}_{q^l} \\
\mathbb{Z}^{\text{unram}}_{\mathbb{F}_q} & \hookrightarrow & \mathbb{Z}^{\text{unram}}_{\mathbb{F}_{q^l}}
\end{array}$$

commutes, the result follows immediately. The proof is complete. \hfill \Box

Now for $x_i \in \mathbb{F}_{q^m}$, we have

$$(\varpi^{b_i} \circ \mathcal{N}_m)(x_i) = \varpi^{b_i} \left( \frac{x_i^{\frac{q^m-1}{q-1} - b_i}}{x_i^{\frac{q^m-1}{q-1}}} \right) = \varpi \left( x_i^{\frac{q^m-1}{q-1} - b_i} \right).$$

Since $x_i \in \mathbb{F}_{q^m}$, we have $x_i^{\frac{q^m-1}{q-1} - b_i} \in \mathbb{F}_q$. By Lemma 3.1 we obtain

$$(\varpi^{b_i} \circ \mathcal{N}_m)(x_i) = \varpi_m \left( x_i^{\frac{q^m-1}{q-1} - b_i} \right).$$

So by (3.1) our sum (1.1) may be rewritten as

$$(3.2) \quad S_m^*(\chi_1, \ldots, \chi_n, f) = \sum_{x \in (\mathbb{F}_{q^m})^n} \zeta_p^{\text{Tr}_{\mathbb{F}_q/p}(f(x))} \varpi_m \left( x_i^{\frac{q^m-1}{q-1} - b_i} \right),$$

where $b = (b_1, \ldots, b_n)$. From the definition of Gauss sums over $\mathbb{F}_{q^m}$, one can deduce that for each $c \in \mathbb{F}_{q^m}$, the Gauss sum satisfies the following interpolation relation:

$$(3.3) \quad \zeta_p^{\text{Tr}_{\mathbb{F}_q/p}(c)} = \frac{1}{1 - q^m} \sum_{k=0}^{q^m - 2} g_q(k) \cdot \varpi_m(c)^k.$$
Now let \( f(x) = f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x^{V_i} \in F_q[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \) be a diagonal polynomial. Then by (3.2) and (3.3) we derive that

\[
S^*_m(\chi_1, \ldots, \chi_n, f) = \sum_{x \in (F_q^m)^n} \epsilon_p^{\sum_{i=1}^{n} a_i x^{V_i}} \cdot \varpi_m \left( x \frac{q^m - 1}{q - 1} b \right)
\]

\[
= \sum_{x \in (F_q^m)^n} \varpi_m \left( x \frac{q^m - 1}{q - 1} b \right) \cdot \prod_{i=1}^{n} \epsilon_p^{\sum_{k_i=0}^{q^m-2} g_q m(k_i) \cdot \varpi_m \left( a_i x^{V_i} \right) k_i}
\]

\[
= \sum_{x \in (F_q^m)^n} \varpi_m \left( x \frac{q^m - 1}{q - 1} b \right) \cdot \prod_{k_i=0}^{q^m-2} \sum_{k_n=0}^{q^m-2} \cdot \sum_{i=1}^{n} \prod_{k_i = 0}^{q^m-2} g_q m(k_i) \cdot \varpi_m \left( a_i x^{V_i} \right) k_i
\]

\[
= \sum_{x \in (F_q^m)^n} \varpi_m \left( x \frac{q^m - 1}{q - 1} b \right) \cdot \prod_{k_i=0}^{q^m-2} \sum_{k_n=0}^{q^m-2} \left( \prod_{i=1}^{n} g_q m(k_i) \cdot \varpi_m(a_i) k_i \right) \cdot \sum_{x \in (F_q^m)^n} \varpi_m \left( x \frac{\sum_{i=1}^{n} k_i V_i + \frac{q^m - 1}{q - 1} b}{q - 1} \right)
\]

Noting that

\[
\sum_{x \in (F_q^m)^n} \varpi_m \left( x \frac{\sum_{i=1}^{n} k_i V_i + \frac{q^m - 1}{q - 1} b}{q - 1} \right)
\]

is equal to \((q^m - 1)^n\) if \(\sum_{i=1}^{n} k_i V_i + \frac{q^m - 1}{q - 1} \cdot b \equiv 0 \pmod{q^m - 1}\), and 0 otherwise, one then deduces that

\[
S^*_m(\chi_1, \ldots, \chi_n, f) = \sum_{\sum_{i=1}^{n} k_i V_i + \frac{q^m - 1}{q - 1} \cdot b \equiv 0 \pmod{q^m - 1}, 0 \leq k_i \leq q^m-2 \text{ for } 1 \leq i \leq n} (q^m - 1)^n \prod_{i=1}^{n} g_q m(k_i) \cdot \varpi_m(a_i) k_i
\]

\[
= (-1)^n \cdot \sum_{\sum_{i=1}^{n} k_i V_i + \frac{q^m - 1}{q - 1} \cdot b \equiv 0 \pmod{q^m - 1}, 0 \leq k_i \leq q^m-2 \text{ for } 1 \leq i \leq n} \prod_{i=1}^{n} g_q m(k_i) \cdot \varpi_m(a_i) k_i.
\]

(3.4)

Let each \( V_i \) be written as a column vector, and let \( M = (V_1, \ldots, V_n) \) be the \( n \times n \) matrix having \( V_i \) as its \( i \)-th column for \( 1 \leq i \leq n \). Write \( u_i = \frac{k_i}{q^m-1} \) and \( b_i' = \frac{b_i}{q^m-1} \) for \( 1 \leq i \leq n \). Then for all \( 1 \leq i \leq n \), we have \( 0 \leq u_i, b_i' < 1 \). It is clear that

\[
\sum_{i=1}^{n} k_i V_i + \frac{q^m - 1}{q - 1} \cdot b \equiv 0 \pmod{q^m - 1}
\]

if and only if

\[
M u + b' \equiv 0 \pmod{1},
\]

where \( u = (k_1', \ldots, k_n')^T \) and \( b' = (b_1', \ldots, b_n')^T \) are written as column vectors. Then from (3.4) we finish the proof of the following result.

**Theorem 3.2.** For any integer \( m \geq 1 \), we have

\[
S^*_m(\chi_1, \ldots, \chi_n, f) = (-1)^n \cdot \sum_{(M u + b' \equiv 0 \pmod{1})} \prod_{i=1}^{n} g_q m(u_i(q^m - 1)) \cdot \varpi_m(a_i) u_i(q^m - 1).
\]
4. Explicit formulas of $L$-functions of twisted diagonal sums and Newton polygons

In this section, we will give an explicit formula for the $L$-function of the twisted sum of the diagonal Laurent polynomial over $\mathbb{F}_q$. Let $f(x) = \sum_{i=1}^{n} a_i x^{V_i}$ be a diagonal Laurent polynomial and $\Delta$ the Newton polyhedron generated by the $n$ indices $V_i, 1 \leq i \leq n$. Let $S(\Delta, b')$ be the set of solutions $u$ of the following linear system of congruences:

$$Mu + b' \equiv 0 \pmod{1}, \ u_i \text{ rational numbers, } 0 \leq u_i < 1.$$  

Now suppose that $(\det M, p) = 1$. Then for each $u \in S(\Delta, b')$, there exists a smallest positive integer, denoted by $d(q, b', u)$, such that the multiplication by $q^{d(q, b', u)}$ acts trivially on $u + b'$. We claim that this is equivalent to $(q^{d(q, b', u)} - 1)u \in \mathbb{Z}^n$. In fact, since $(q - 1)b' = b \in \mathbb{Z}^n$, we have $q^{d(q, b', u)}(u + b') \equiv u + b' \pmod{1} \Leftrightarrow (q^{d(q, b', u)} - 1)u \equiv (1 - q^{d(q, b', u)})b' \pmod{1} \Leftrightarrow (q^{d(q, b', u)} - 1)u \equiv 0 \pmod{1}$. So the assertion is proved. Therefore $d(q, b', u)$ is the smallest positive integer $d'$ such that

$$(q^{d'} - 1)u \in \mathbb{Z}^n.$$  

For any given integer $d \in \mathbb{Z}^+$, let $S(q, b', d) = \{u \in S(\Delta, b')|d(q, b', u) = d\}$. Define a set $I$ of indices by $I := \{d \in \mathbb{Z}^+| \exists u \in S(\Delta, b') \text{ such that } d(q, b', u) = d\}$. Since $S(\Delta, b')$ is finite, so is $I$.

To give the main result, one first needs two lemmas.

Lemma 4.1. For any integer $l$ and $0 \leq k \leq q^d - 2$, we have

$$g_{q^d} \left( k \cdot \frac{q^d - 1}{q^d - 1} \right) = g_{q^d}(k)^l.$$  

Proof. First we have

$$g_{q^d} \left( k \cdot \frac{q^d - 1}{q^d - 1} \right) = - \sum_{c \in \mathbb{F}_{q^d}^*} \varpi_d \left( N_d(c) \right)^{-k} \cdot \zeta_p^{\text{Tf}_{q^{d^l}/F_p}(c)}.$$  

Since $N_d(c) \in \mathbb{F}_q$, by Lemma 3.1 we have $\varpi_d(N_d(c)) = \varpi_d(N_d(c))$. Thus

$$g_{q^d} \left( k \cdot \frac{q^d - 1}{q^d - 1} \right) = - \sum_{c \in \mathbb{F}_{q^d}^*} \varpi_d^{-k} \left( N_d(c) \right) \cdot \zeta_p^{\text{Tf}_{q^{d^l}/F_p}(c)}.$$  

Then it follows from the well-known Hasse–Davenport relation (see, for example, \cite{10}) that

$$g_{q^d} \left( k \cdot \frac{q^d - 1}{q^d - 1} \right) = \left( - \sum_{c \in \mathbb{F}_{q^d}^*} \varpi_d^{-k} \left( N_d(c) \right) \cdot \zeta_p^{\text{Tf}_{q^{d^l}/F_p}(c)} \right)^l = (g_{q^d}(k))^l = g_{q^d}(k)^l,$$  

as desired. The proof is complete. \qed

Lemma 4.2. The Gauss sum is invariant under $p$-action; i.e., we have $g_q(pk) = g_q(k)$.

Proof. We have

$$g_q(pk) = - \sum_{c \in \mathbb{F}_q^*} \varpi(c)^{-pk} \cdot \zeta_p^{\text{Tf}_{q^{d^l}/F_p}(c)} = - \sum_{c \in \mathbb{F}_q^*} \varpi(c)^{-k} \cdot \zeta_p^{\text{Tf}_{q^{d^l}/F_p}(c)}.$$  

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Namely Noting that \( Tr_{\mathbb{F}_q}/\mathbb{F}_p (c) = Tr_{\mathbb{F}_q}/\mathbb{F}_p (c^p) \), we then have
\[
g_q(pk) = - \sum_{c \in \mathbb{F}_q^*} \varphi(c^p - k) \cdot \zeta_p \cdot Tr_{\mathbb{F}_q}/\mathbb{F}_p (c^p).
\]
Let \( c' = c^p \). It is easy to see that if \( c \) runs over \( \mathbb{F}_q^* \), then \( c' \) runs also over \( \mathbb{F}_q^* \).
Therefore
\[
g_q(pk) = - \sum_{c \in \mathbb{F}_q^*} \varphi(c') - k \cdot \zeta_p \cdot Tr_{\mathbb{F}_q}/\mathbb{F}_p (c') = g_q(k).
\]
This completes the proof of Lemma 4.2.

As usual, let \( \{ r \} \) denote the fractional part of the real number \( r \), namely \( r - \lfloor r \rfloor \). For any \( u \in S(q, b, d) \), one can easily check that \( \{ u \} \in S(q, b, d) \). So we can define the \( q \)-action on \( u \) by \( q \cdot u := \{ qu \} \), where \( \{ qu \} = \{ \{ qu_1 \}, \ldots, \{ qu_n \} \rangle \). Evidently for any \( 0 \leq i \neq j \leq d - 1 \), we have \( \{ qu \} \neq \{ qu' \} \) and \( \{ qu^d \} = u \). Namely \( \{ qu \} | b \in \mathbb{Z}_{\geq 0} \} = \{ u, \{ qu \}, \ldots, \{ qu^{d-1} \} \} \). Call it an orbit (closed point) of \( u \) under \( q \)-action. Hence there are \( d \) distinct points in the orbit \( u \in S(q, b, d) \) under \( q \)-action. Then \( S(q, b, d) \) can be partitioned into distinct orbits. We take an element from each orbit of \( S(q, b, d) \) as the representation of this orbit and put these representations together to form a set \( O(S(q, b, d)) \).

We are now ready to give the main result in this paper.

**Theorem 4.3.** Let \( f(x) = \sum_{i=1}^{n} a_i x^i \in \mathbb{F}_q[x, x_1^{-1}, \ldots, x_n, x_n^{-1}] \) be a diagonal Laurent polynomial, and let \( M = (V_1, \ldots, V_n) \) be the \( n \times n \) matrix having \( V_i \) as its \( i \)-th column for \( 1 \leq i \leq n \). Let \( (p, \det M) = 1 \) and \( q = p^e \). Then we have

\[
L^*(\chi_1, \ldots, \chi_n; f; t) = \prod_{d \in I} \prod_{u \in O(S(q, b, d))} \left( 1 - t^d \cdot \pi \sum_{i=1}^{n} \sigma_p (u_i (q^d - 1)) \right) \cdot \prod_{i=1}^{n} \varphi (a_i) u_i (q^d - 1) \prod_{j=0}^{de-1} \Gamma_p \left( \frac{(p^j u_i (q^d - 1) q^d)}{q^d - 1} \right).
\]

Furthermore, if \( u \in O(S(q, b, d)) \) and \( \omega_u \) is any one of the \( d \) reciprocal roots of the corresponding factor in \( \mathbb{F}_q \), then we have

\[
\text{ord}_{\omega_u} \frac{1}{d} \sum_{j=0}^{de-1} \sum_{i=1}^{n} (p^j u_i).
\]

**Proof:** From Theorem 3.2 we deduce that

\[
L^*(\chi_1, \ldots, \chi_n; f; t) = \exp \left( \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{u \in S(\Delta, b')} \prod_{i=1}^{n} \varphi_m (a_i) u_i (q^m - 1) \cdot g_{q^m} (u_i (q^m - 1)) \cdot \frac{t^m}{m} \right).
\]

Clearly, for any given \( u \) satisfying \( Mu + b' \equiv 0 \pmod{1} \), there exists a unique integer \( d \) such that \( u \in S(q, b, d) \), i.e., such that \( (q^d - 1)u + b' \in \mathbb{Z}^n \), but \( (q^d - 1)u + b' \not\in \mathbb{Z}^n \).
For such a $u$, we have $(q^{dk} - 1)u + b' \in \mathbb{Z}^n$ for any integer $k \geq 1$. So by (4.2) we have
\[ L^*(\chi_1, \ldots, \chi_n, f; t) \]
\[ = \exp \left( (-1)^n \sum_{d=1}^{\infty} \sum_{u \in S(q, b', d)} \prod_{i=1}^{n} \prod_{k=1}^{n} \varpi(\alpha_i)^u(q^{dk} - 1)^i \cdot g_{q^{dk}}(u_i(q^{dk} - 1)) \frac{t^{dk}}{dk} \right). \]

By Lemma 4.1, one knows that for any $u \in S(q, b', d)$,
\[ g_{q^{dk}}(u_i(q^{dk} - 1)) = g_{q^d}(u_i(q^d - 1))^k. \]

Note also that for any $u \in S(q, b', d)$, we have $\varpi(\alpha_i)^u(q^{dk} - 1) = \varpi(\alpha_i)^u(q^d - 1)^k$. Then by (4.3) and (4.4) we have
\[ L^*(\chi_1, \ldots, \chi_n, f; t) \]
\[ = \exp \left( (-1)^n \sum_{d=1}^{\infty} \sum_{u \in S(q, b', d)} \prod_{i=1}^{n} \prod_{k=1}^{n} \varpi(\alpha_i)^u(q^d - 1)^k \cdot g_{q^d}(u_i(q^d - 1))^k \right) \cdot t^{dk} \right). \]

It follows immediately from (4.5) that
\[ L^*(\chi_1, \ldots, \chi_n, f; t)(-1)^{n-1} \]
\[ = \prod_{d=1}^{\infty} \prod_{u \in S(q, b', d)} \left( 1 - t^d \prod_{i=1}^{n} \varpi(\alpha_i)^u(q^d - 1) \cdot g_{q^d}(u_i(q^d - 1)) \right)^{\frac{1}{d}} \]
\[ = \prod_{d \in D} \prod_{u \in O(S(q, b', d))} \left( 1 - t^d \prod_{i=1}^{n} \varpi(\alpha_i)^u(q^d - 1) \cdot g_{q^d}(u_i(q^d - 1)) \right)^{\frac{1}{d}}. \]

Now let $u \in O(S(q, b', d))$. Then note that $\omega(\alpha_i)^q = \omega(\alpha_i)$ and by Lemma 4.2 we know that for each of the $d$ points in the orbit of $u \in S(q, b', d)$ under $q$-action, the corresponding factor in (4.6) is the same. So we can remove the power $\frac{1}{d}$ in (4.6) by restricting $u$ to run over the set $O(S(q, b', d))$. Therefore we get
\[ L^*(\chi_1, \ldots, \chi_n, f; t)(-1)^{n-1} \]
\[ = \prod_{d \in I} \prod_{u \in O(S(q, b', d))} \left( 1 - t^d \prod_{i=1}^{n} \varpi(\alpha_i)^u(q^d - 1) \cdot g_{q^d}(u_i(q^d - 1)) \right)^{\frac{1}{d}}. \]
But Lemma 2.1 applied to the finite field \( F_{q^d} \) gives

\[
g_{q^d}(u(q^d - 1)) = \prod_{i=1}^{n} \sigma_p(u_i(q^d - 1)) \prod_{j=0}^{d-1} \Gamma_p \left( \frac{(p^j u_i(q^d - 1))_{q^d-1}}{q^d - 1} \right).
\]

So by (4.7) and (4.8) we then get the desired formula (4.1).

Now from (4.1) we deduce that

\[
\text{ord}_q \omega_u = \frac{1}{d} \cdot \sum_{i=1}^{n} \sigma_p(u_i(q^d - 1)) \prod_{j=0}^{d-1} \Gamma_p \left( \frac{(p^j u_i(q^d - 1))_{q^d-1}}{q^d - 1} \right)
\]

and note that \( 0 \leq \{p^j u_i\}(q^d - 1) \leq q^d - 1 \), we have

\[
\langle p^j u_i(q^d - 1) \rangle_{q^d-1} = \{p^j u_i\}(q^d - 1).
\]

So we get by (4.9) as well (4.10),

\[
\text{ord}_q \omega_u = \frac{1}{d} \sum_{i=1}^{n} \frac{1}{q^d - 1} \sum_{j=0}^{d-1} \{p^j u_i\}(q^d - 1) = \frac{d}{d} \sum_{i=1}^{n} \sum_{j=0}^{d-1} \{p^j u_i\},
\]

as required. The proof of Theorem 4.3 is complete.

For any rational number \( 0 \leq r \leq n \), we define a number \( h_r \) by

\[
h_r := \# \left\{ u \in S(\Delta, b) \mid \sum_{i=1}^{n} \sum_{j=0}^{d-1} \{p^j u_i\} = dr \right\}.
\]

Therefore by Theorem 4.3 we derive the following result.

**Theorem 4.4.** Let \( f(x) = \sum_{i=1}^{n} a_i x^V_i \in F_q[x_1, x_1^{-1}, ..., x_n, x_n^{-1}] \) be a diagonal Laurent polynomial, and let \( M = (V_1, ..., V_n) \) be the \( n \times n \) matrix having \( V_i \) as its \( i \)-th column for \( 1 \leq i \leq n \) satisfying \( (p, \det M) = 1 \). Then for each rational number \( 0 \leq r \leq n \), the \( p \)-adic Newton polygon of the \( L \)-function \( L^*(\chi_1, ..., \chi_n; f; t) \) has a side of slope \( r \) with horizontal length \( h_r \).

**Corollary 4.5.** Let \( f(x) = \sum_{i=1}^{n} a_i x^V_i \in F_q[x_1, x_1^{-1}, ..., x_n, x_n^{-1}] \) be a diagonal Laurent polynomial, and let \( M = (V_1, ..., V_n) \) be the \( n \times n \) matrix having \( V_i \) as its \( i \)-th column for \( 1 \leq i \leq n \). Let \( (p, \det M) = 1 \). Then each of the following is true:
where $I_0$ is defined by $I_0 := \{d \in \mathbb{Z}^+ | \exists u \in S(\Delta, 0), \text{ such that } d(q, 0, u) = d\}$ and $S(q, d) := S(q, 0, d)$.

(ii) If $u \in O(S(q, d))$ and $\omega_u$ is any one of the $d$ reciprocal roots of the corresponding factor in (4.11), then we have

$$\text{ord}_q \omega_u = \frac{1}{d} \sum_{j=0}^{d-1} \sum_{i=1}^{n} \{p^j u_i\}.$$ 

(iii) For each rational number $0 \leq r \leq n$, the $p$-adic Newton polygon of the $L$-function $L^*(f; t)$ has a side of slope $r$ with horizontal length $h_r$.

Finally we point out that if $f$ is degenerate, which means that $p|\det M$, then we can change it to the nondegenerate case by using some suitable transformations. For instance, let $f(x_1, x_2) = x_1 x_2^{-1} + x_1^{-1} x_2^{-p}$. Then $|\det(M)| = p$ and so $f$ is degenerate. First let $y_1 = x_1 x_2^{-1}, y_2 = x_2$. Then it is an invertible transformation. Second let $y_1' = y_1, y_2' = y_2^p$. Finally let $z_1 = y_1', z_2 = y_2' y_1'^{-1}$. This is also invertible. Hence we get $f(x_1, x_2) = z_1 + z_2 := h(z_1, z_2)$ and

$$L^*(\chi_1, \chi_2, f; t) = L^*(\chi_1', \chi_2', h; t)$$

for some multiplicative characters $\chi_1', \chi_2'$ of $\mathbb{F}_q$. But $h = z_1 + z_2$ is nondegenerate. Therefore we can use Theorems 4.3 and 4.4 to compute the $L$-function $L^*(\chi_1, \chi_2, h; t)$ (hence $L^*(\chi_1, \chi_2; f; t)$ by (4.12)) and determine its $p$-adic Newton polygon.

REFERENCES


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