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L-FUNCTIONS OF TWISTED DIAGONAL EXPONENTIAL SUMS OVER FINITE FIELDS

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Abstract. Let $\mathbf{F}_q$ be the finite field of q elements with characteristic p and $\mathbf{F}_{q^m}$ its extension of degree m. Fix a nontrivial additive character Ψ and let $\chi_1, ..., \chi_n$ be multiplicative characters of $\mathbf{F}_p$. For

$$f(x_1, ..., x_n) \in \mathbf{F}_q[x_1, x_1^{-1}, ..., x_n, x_n^{-1}],$$

one can form the twisted exponential sum $S_m^*(\chi_1, ..., \chi_n, f)$. The corresponding L-function is defined by

$$L^*(\chi_1, ..., \chi_n, f; t) = \exp(\sum_{m=0}^{\infty} S_m^*(\chi_1, ..., \chi_n, f)\frac{t^m}{m}).$$

In this paper, by using the p-adic gamma function and the Gross–Koblitz formula on Gauss sums, we give an explicit formula for the L-function $L^*(\chi_1, ..., \chi_n, f; t)$ if f is a Laurent diagonal polynomial. We also determine its p-adic Newton polygon.

1. Introduction

Let $\mathbf{F}_q$ be the finite field of q elements with characteristic p and $\mathbf{F}_{q^m}$ its extension of degree m. Fix a nontrivial additive character Ψ and let $\chi_1, ..., \chi_n$ be multiplicative characters of $\mathbf{F}_q$. If $f(x_1, ..., x_n) \in \mathbf{F}_q[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$, then one can form the twisted exponential sum

$$\begin{aligned} &S_m^*(\chi_1, ..., \chi_n, f) \\ &= \sum_{(x_1,...,x_n)\in(\mathbf{F}_{q^m}^*)^n} \chi_1\left(\mathrm{N}_m(x_1)\right) ... \chi_n\left(\mathrm{N}_m(x_n)\right) \Psi\left(\mathrm{Tr}_m(f(x_1, ..., x_n))\right), \end{aligned} \tag{1.1}$$

where $\mathrm{Tr}_m : \mathbf{F}_{q^m} \to \mathbf{F}_q$ is the trace map and $\mathrm{N}_m : \mathbf{F}_{q^m} \to \mathbf{F}_q$ is the norm map. The corresponding L-function is defined as follows:

$$L^*(\chi_1, ..., \chi_n, f; t) = \exp\left(\sum_{m=0}^{\infty} S_m^*(\chi_1, ..., \chi_n, f)\frac{t^m}{m}\right).$$

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By the renowned Dwork–Grothendieck theorem, one knows that the L-function $L^*(\chi_1, ..., \chi_n, f; t)$ is a rational function.

Finding sharp p-adic estimates of the exponential sums $S_m^*(\chi_1, ..., \chi_n, f)$ is a fundamental question in number theory. This is well known to be reduced to determining the p-adic absolute values of the reciprocal roots and reciprocal poles. The best way to describe this problem is in terms of p-adic Newton polygons (see [11]). Of course one may consider the question of finding the p-adic Newton polygon of Dwork's unit root L-function attached to a family of algebraic varieties over $\mathbf{F}_q$, which is proved by Wan [18, 19, 20] to be p-adic meromorphic as conjectured by Dwork [4]. Such question should be very deep and hard.

On the other hand, Dwork [5] determined the Newton polygon for the L-functions of the one-dimensional Kloosterman sum, and then Sperber [13] generalized Dwork's result to the n-dimensional case while Sperber [14] considered the general hyper-Kloosterman sums. Wan [16, 17, 21] solved a conjecture of Adolphson and Sperber [1]. Note also that Wan [17] presented a p-adic proof of the Dwork–Mazur conjecture [3, 11]. When all the χ_i are trivial, then the p-adic Newton polygon of the L-function of the one-variable polynomial $f(x)$ received attention by Sperber [15], Hong [7, 8, 9] and Yang [22]. A general interesting question is to determine the p-adic Newton polygon of the L-function $L^*(\chi, f; t)$ if f is of one variable, even for small degree, and χ is a nontrivial multiplicative character.

In this paper, we are interested in the case that $f(x_1, ..., x_n)$ is a Laurent diagonal polynomial. We obtain an explicit formula for the L-function $L^*(\chi_1, ..., \chi_n, f; t)$ if f is a Laurent diagonal polynomial by using the p-adic gamma function. We also determine the p-adic Newton polygon for such an L-function $L^*(\chi_1, ..., \chi_n, f; t)$. In the forthcoming works, we will use the results presented here to investigate a conjecture of Adolphson and Sperber [2] on the p-adic Newton polygons of twisted exponential sums.

2. Preliminaries on p-adic gamma functions

In this section, we present some properties of the p-adic gamma function Γ_p. For $s \in \mathbf{Z}^+$ one sets

$$\Gamma(s) = (-1)^s \cdot \prod_{i=1}^{s-1} i. \tag{2.1}$$

It is known (see, for example, [12]) that this extends to a continuous function on $\mathbf{Z}_p$, and we define Γ_p to be this extension. Therefore it follows from continuity that the following functional equation holds:

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -1, & \text{if } x \in p\mathbf{Z}_p, \\ -x, & \text{if } x \notin p\mathbf{Z}_p, \end{cases}$$

and also we have

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\langle x \rangle_p},$$

where $\langle x \rangle_p$ denotes the least nonnegative residue of x modulo p.

Denote by ζ_p a fixed primitive p-th root of unity in $\mathbf{C}_p$, the completion of the algebraic closure of the p-adic field $\mathbf{Q}_p$. Then there is a generator π of the maximal ideal of the ring of integers of $\mathbf{Q}_p(\zeta_p)$ uniquely characterized by $\pi^{p-1} = -p$ and $\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}$. Let ϖ be any fixed Teichmüller character of $\mathbf{F}_q$. Let $\sigma_p(a)$ denote the sum of the digits of the p-adic representation of an integer $a \geq 1$.

Lemma 2.1 (Gross–Koblitz formula, see [6] or [12])**.** *Let $1 \le a < p^e := q$. Then the value of the Gauss sum $g_q(a)$ is explicitly given by*

$$g_q(a) := -\sum_{0\neq x\in\mathbf{F}_q} \omega(x)^{-a}\Psi(\mathrm{Tr}(x)) = \pi^{\sigma_p(a)} \prod_{j=0}^{e-1} \Gamma_p\left(\frac{\langle p^j a\rangle_{q-1}}{q-1}\right).$$

3. Twisted diagonal exponential sums

For brevity, we define $x^\alpha := x_1^{\alpha_1}...x_n^{\alpha_n}$ for any vector $x = (x_1, ..., x_n) \in (\mathbf{F}_{q^m}^*)^n$ and any n-tuple $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{Z}^n$. Since ζ_p is a given primitive pth root of unity in $\mathbf{C}_p$, we know that for any given nontrivial additive character Ψ of $\mathbf{F}_q$, we have $\Psi(c) = \zeta_p{}^{\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(c)}, c \in \mathrm{F}_q$. Let ϖ_l be any fixed Teichmüller character of $\mathbf{F}_{q^l}$ for any given integer $l \ge 1$. Then $\varpi_1 = \varpi$ and for multiplicative characters $\chi_1, ..., \chi_n$, there must be nonnegative integers $b_1, ..., b_n$, such that $\chi_i = \varpi^{b_i}, 1 \le i \le n$. Thus we can rewrite (1.1) as follows:

$$S_m^*(\chi_1, ..., \chi_n, f) = \sum_{x\in(\mathbf{F}_{q^m}^*)^n} \zeta_p{}^{\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_p}(f(x))} \cdot \prod_{i=1}^{n} (\varpi^{b_i} \circ \mathbf{N}_m)(x_i). \tag{3.1}$$

Lemma 3.1. *For any $c \in \mathbf{F}_q$ and any integer $l \ge 1$, we have $\varpi_l(c) = \varpi(c)$.*

Proof. For any positive integer c, let $\mathbf{Z}_c^{\mathrm{unram}}$ denote the ring of integers of the unramified extension of $\mathbf{Q}_p$ of degree c. Since the diagram

$$\begin{array}{ccc} \mathbf{F}_q & \hookrightarrow & \mathbf{F}_{q^l} \\ \downarrow & & \downarrow \\ \mathbf{Z}_{da}^{\mathrm{unram}} & \hookrightarrow & \mathbf{Z}_{dla}^{\mathrm{unram}} \end{array}$$

commutes, the result follows immediately. The proof is complete. □

Now for $x_i \in \mathbf{F}_{q^m}^*$, we have

$$\left(\varpi^{b_i} \circ \mathrm{N}_m\right)(x_i) = \varpi^{b_i}\left(x_i^{\frac{q^m-1}{q-1}}\right) = \varpi\left(x_i^{\frac{q^m-1}{q-1}\cdot b_i}\right).$$

Since $x_i \in \mathbf{F}_{q^m}^*$, we have $x_i^{\frac{q^m-1}{q-1}\cdot b_i} \in \mathbf{F}_q$. By Lemma 3.1 we obtain

$$\left(\varpi^{b_i} \circ \mathrm{N}_m\right)(x_i) = \varpi_m\left(x_i^{\frac{q^m-1}{q-1}\cdot b_i}\right).$$

So by (3.1) our sum (1.1) may be rewritten as

$$S_m^*(\chi_1, ..., \chi_n, f) = \sum_{x\in(\mathbf{F}_{q^m}^*)^n} \zeta_p{}^{\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_p}(f(x))} \cdot \varpi_m\left(x^{\frac{q^m-1}{q-1}\cdot b}\right), \tag{3.2}$$

where $b = (b_1, ..., b_n)$. From the definition of Gauss sums over $\mathbf{F}_{q^m}$, one can deduce that for each $c \in \mathbf{F}_{q^m}$, the Gauss sum satisfies the following interpolation relation:

$$\zeta_p{}^{\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_p}(c)} = \frac{1}{1-q^m} \sum_{k=0}^{q^m-2} g_{q^m}(k) \cdot \varpi_m(c)^k. \tag{3.3}$$

Now let $f(x)=f(x_1,...,x_n)=\sum_{i=1}^n a_ix^{V_i}\in\mathbf{F}_q[x_1,x_1^{-1},...,x_n,x_n^{-1}]$ be a diagonal polynomial. Then by (3.2) and (3.3) we derive that

$$\begin{aligned}
S_m^*(\chi_1,...,\chi_n,f)&=\sum_{x\in(\mathbf{F}_{q^m}^*)^n}\zeta_p^{\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_p}(\sum_{i=1}^n a_ix^{V_i})}\cdot\varpi_m\left(x^{\frac{q^m-1}{q-1}\cdot b}\right)\\
&=\sum_{x\in(\mathbf{F}_{q^m}^*)^n}\varpi_m\left(x^{\frac{q^m-1}{q-1}\cdot b}\right)\cdot\prod_{i=1}^n\zeta_p^{\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_p}(a_ix^{V_i})}\\
&=\sum_{x\in(\mathbf{F}_{q^m}^*)^n}\varpi_m\left(x^{\frac{q^m-1}{q-1}\cdot b}\right)\cdot\prod_{i=1}^n\frac{1}{1-q^m}\sum_{k_i=0}^{q^m-2}g_{q^m}(k_i)\cdot\varpi_m\left(a_ix^{V_i}\right)^{k_i}\\
&=\sum_{x\in(\mathbf{F}_{q^m}^*)^n}\varpi_m\left(x^{\frac{q^m-1}{q-1}\cdot b}\right)\cdot\sum_{k_1=0}^{q^m-2}...\sum_{k_n=0}^{q^m-2}\prod_{i=1}^n\frac{g_{q^m}(k_i)}{1-q^m}\cdot\varpi_m\left(a_ix^{V_i}\right)^{k_i}\\
&=\sum_{k_1=0}^{q^m-2}...\sum_{k_n=0}^{q^m-2}\sum_{x\in(\mathbf{F}_{q^m}^*)^n}\varpi_m\left(x^{\frac{q^m-1}{q-1}\cdot b}\right)\prod_{i=1}^n\frac{g_{q^m}(k_i)}{1-q^m}\cdot\varpi_m\left(a_ix^{V_i}\right)^{k_i}\\
&=\sum_{k_1=0}^{q^m-2}...\sum_{k_n=0}^{q^m-2}\left(\prod_{i=1}^n\frac{g_{q^m}(k_i)}{1-q^m}\cdot\varpi_m(a_i)^{k_i}\right)\cdot\sum_{x\in(\mathbf{F}_{q^m}^*)^n}\varpi_m\left(x^{\sum_{i=1}^n k_iV_i+\frac{q^m-1}{q-1}\cdot b}\right).
\end{aligned}$$

Noting that

$$\sum_{x\in(\mathbf{F}_{q^m}^*)^n}\varpi_m\left(x^{\sum_{i=1}^n k_iV_i+\frac{q^m-1}{q-1}\cdot b}\right)$$

is equal to $(q^m-1)^n$ if $\sum_{i=1}^n k_iV_i+\frac{q^m-1}{q-1}\cdot b\equiv 0 \pmod{q^m-1}$, and 0 otherwise, one then deduces that

$$\begin{aligned}
S_m^*(\chi_1,...,\chi_n,f)&=\sum_{\substack{\sum_{i=1}^n k_iV_i+\frac{q^m-1}{q-1}\cdot b\equiv 0 \pmod{q^m-1}\\ 0\le k_i\le q^m-2 \text{ for } 1\le i\le n}}(q^m-1)^n\prod_{i=1}^n\frac{g_{q^m}(k_i)}{1-q^m}\cdot\varpi_m(a_i)^{k_i}\\
&=(-1)^n\cdot\sum_{\substack{\sum_{i=1}^n k_iV_i+\frac{q^m-1}{q-1}\cdot b\equiv 0 \pmod{q^m-1}\\ 0\le k_i\le q^m-2 \text{ for } 1\le i\le n}}\prod_{i=1}^n g_{q^m}(k_i)\cdot\varpi_m(a_i)^{k_i}. \qquad (3.4)
\end{aligned}$$

Let each V_i be written as a column vector, and let $M=(V_1,...,V_n)$ be the $n\times n$ matrix having V_i as its i-th column for $1\le i\le n$. Write $u_i=\frac{k_i}{q^m-1}$ and $b_i'=\frac{b_i}{q-1}$ for $1\le i\le n$. Then for all $1\le i\le n$, we have $0\le u_i,b_i'<1$. It is clear that

$$\sum_{i=1}^n k_iV_i+\frac{q^m-1}{q-1}\cdot b\equiv 0 \pmod{q^m-1}$$

if and only if

$$Mu+b'\equiv 0 \pmod 1,$$

where $u=(k_1',...,k_n')^T$ and $b'=(b_1',...,b_n')^T$ are written as column vectors. Then from (3.4) we finish the proof of the following result.

Theorem 3.2. *For any integer $m\ge 1$, we have*

$$S_m^*(\chi_1,...,\chi_n,f)=(-1)^n\cdot\sum_{\substack{Mu+b'\equiv 0 \pmod 1\\ (q^m-1)u\in\mathbf{Z}^n}}\prod_{i=1}^n g_{q^m}(u_i(q^m-1))\cdot\varpi_m(a_i)^{u_i(q^m-1)}.$$

4. Explicit formulas of L-functions of twisted diagonal sums and Newton polygons

In this section, we will give an explicit formula for the L-function of the twisted sum of the diagonal Laurent polynomial over $\mathbf{F}_q$. Let $f(x) = \sum_{i=1}^{n} a_i x^{V_i}$ be a diagonal Laurent polynomial and Δ the Newton polyhedron generated by the n indices $V_i, 1 \le i \le n$. Let $S(\Delta, b')$ be the set of solutions u of the following linear system of congruences:

$$Mu + b' \equiv 0 \pmod 1, \ u_i \text{ rational numbers}, \ 0 \le u_i < 1.$$

Now suppose that $(\det M, \ p) = 1$. Then for each $u \in S(\Delta, b')$, there exists a smallest positive integer, denoted by $d(q, b', u)$, such that the multiplication by $q^{d(q,b',u)}$ acts trivially on $u+b'$. We claim that this is equivalent to $(q^{d(q,b',u)}-1)u \in \mathbf{Z}^n$. In fact, since $(q-1)b' = b \in \mathbf{Z}^n$, we have $q^{d(q,b',u)}(u+b') \equiv u+b' \pmod 1 \Leftrightarrow (q^{d(q,b',u)}-1)u \equiv (1-q^{d(q,b',u)})b' \pmod 1 \Leftrightarrow (q^{d(q,b',u)}-1)u \equiv 0 \pmod 1$. So the assertion is proved. Therefore $d(q, b', u)$ is the smallest positive integer d' such that $(q^{d'}-1)u \in \mathbf{Z}^n$.

For any given integer $d \in \mathbf{Z}^+$, let $S(q, b', d) = \{u \in S(\Delta, b') | d(q, b', u) = d\}$. Define a set I of indices by $I := \{d \in \mathbf{Z}^+ | \exists u \in S(\Delta, b'), \text{ such that } d(q, b', u) = d\}$. Since $S(\Delta, b')$ is finite, so is I.

To give the main result, one first needs two lemmas.

Lemma 4.1. *For any integer l and $0 \le k \le q^d - 2$, we have*

$$g_{q^{dl}}\left(k \cdot \frac{q^{dl}-1}{q^d-1}\right) = g_{q^d}(k)^l.$$

Proof. First we have

$$g_{q^{dl}}\left(k \cdot \frac{q^{dl}-1}{q^d-1}\right) = -\sum_{c \in \mathbf{F}_{q^{dl}}^*} \varpi_{dl}\left(\mathrm{N}_{dl}(c)\right)^{-k} \cdot \zeta_p^{\mathbf{Tr}_{\mathbf{F}_{q^{dl}}/\mathbf{F}_p}(c)}.$$

Since $\mathrm{N}_{dl}(c) \in \mathbf{F}_q$, by Lemma 3.1 we have $\varpi_{dl}(\mathrm{N}_{dl}(c)) = \varpi_d(\mathrm{N}_{dl}(c))$. Thus

$$g_{q^{dl}}\left(k \cdot \frac{q^{dl}-1}{q^d-1}\right) = -\sum_{c \in \mathbf{F}_{q^{dl}}^*} \varpi_d^{-k}\left(\mathrm{N}_{dl}(c)\right) \cdot \zeta_p^{\mathbf{Tr}_{\mathbf{F}_{q^{dl}}/\mathbf{F}_p}(c)}.$$

Then it follows from the well-known Hasse–Davenport relation (see, for example, [10]) that

$$g_{q^{dl}}\left(k \cdot \frac{q^{dl}-1}{q^d-1}\right) = \left(-\sum_{c \in \mathbf{F}_{q^d}^*} \varpi_d^{-k}(N_d(c)) \cdot \zeta_p^{\mathbf{Tr}_{\mathbf{F}_{q^d}/\mathbf{F}_p}(c)}\right)^l = \left(g_{q^d}(k)\right)^l = g_{q^d}(k)^l,$$

as desired. The proof is complete. □

Lemma 4.2. *The Gauss sum is invariant under p-action; i.e., we have $g_q(pk) = g_q(k)$.*

Proof. We have

$$g_q(pk) = -\sum_{c \in \mathbf{F}_q^*} \varpi(c)^{-pk} \cdot \zeta_p^{\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(c)} = -\sum_{c \in \mathbf{F}_q^*} \varpi(c^p)^{-k} \cdot \zeta_p^{\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(c)}.$$

Noting that $\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(c) = \mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(c^p)$, we then have

$$g_q(pk) = -\sum_{c\in\mathbf{F}_q^*} \varpi(c^p)^{-k} \cdot \zeta_p^{\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(c^p)}.$$

Let $c' = c^p$. It is easy to see that if c runs over $\mathbf{F}_q^*$, then c' runs also over $\mathbf{F}_q^*$. Therefore

$$g_q(pk) = -\sum_{c\in\mathbf{F}_q^*} \varpi(c')^{-k} \cdot \zeta_p^{\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(c')} = g_q(k).$$

This completes the proof of Lemma 4.2. □

As usual, let $\{r\}$ denote the fractional part of the real number r, namely $\{r\} = r - [r]$. For any $u \in S(q, b', d)$, one can easily check that $\{u\} \in S(q, b', d)$. So we can define the q-action on u by $q \cdot u := \{qu\}$, where $\{qu\} = (\{qu_1\}, ..., \{qu_n\})$. Evidently for any $0 \le i \ne j \le d-1$, we have $\{q^i u\} \ne \{q^j u\}$ and $\{q^d u\} = u$. Namely $\{\{q^l u\} | l \in \mathbf{Z}_{\ge 0}\} = \{u, \{qu\}, ..., \{q^{d-1}u\}\}$. Call it an orbit (closed point) of u under q-action. Hence there are d distinct points in the orbit of $u \in S(q, b', d)$ under q-action. Then $S(q, b', d)$ can be partitioned into distinct orbits. We take an element from each orbit of $S(q, b', d)$ as the representation of this orbit and put these representations together to form a set $O(S(q, b', d))$.

We are now ready to give the main result in this paper.

Theorem 4.3. *Let $f(x) = \sum_{i=1}^n a_i x^{V_i} \in \mathbf{F}_q[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$ be a diagonal Laurent polynomial, and let $M = (V_1, ..., V_n)$ be the $n \times n$ matrix having V_i as its i-th column for $1 \le i \le n$. Let $(p, \det M) = 1$ and $q = p^e$. Then we have*

$$\begin{aligned} &L^*(\chi_1, ..., \chi_n, f; t) \\ &\quad = \prod_{d\in I} \prod_{u\in O(S(q,b',d))} \Bigg(1 - t^d \cdot \pi^{\sum_{i=1}^n \sigma_p(u_i(q^d-1))} \\ &\qquad\qquad \cdot \prod_{i=1}^n \varpi(a_i)^{u_i(q^d-1)} \prod_{j=0}^{de-1} \Gamma_p\Bigg(\frac{\langle p^j u_i(q^d-1)\rangle_{q^d-1}}{q^d-1}\Bigg)\Bigg). \end{aligned} \tag{4.1}$$

Furthermore, if $u \in O(S(q, b', d))$ and ω_u is any one of the d reciprocal roots of the corresponding factor in (4.1)*, then we have*

$$\mathrm{ord}_q \omega_u = \frac{1}{d} \sum_{j=0}^{de-1} \sum_{i=1}^n \{p^j u_i\}.$$

Proof. From Theorem 3.2 we deduce that

$$\begin{aligned} &L^*(\chi_1, ..., \chi_n, f; t) \\ &\quad = \exp\left(\sum_{m=1}^\infty (-1)^n \cdot \sum_{\substack{u\in S(\Delta,b') \\ (q^m-1)u\in\mathbf{Z}^n}} \prod_{i=1}^n \varpi_m(a_i)^{u_i(q^m-1)} \cdot g_{q^m}\left(u_i(q^m-1)\right) \cdot \frac{t^m}{m}\right). \end{aligned} \tag{4.2}$$

Clearly, for any given u satisfying $Mu + b' \equiv 0 \pmod 1$, there exists a unique integer d such that $u \in S(q, b', d)$, i.e., such that $(q^d - 1)u + b' \in \mathbf{Z}^n$, but $(q^{d-1} - 1)u + b' \notin \mathbf{Z}^n$.

For such a u, we have $(q^{dk}-1)u+b' \in \mathbf{Z}^n$ for any integer $k \geq 1$. So by (4.2) we have

$$L^*(\chi_1, ..., \chi_n, f; t) = \exp\left((-1)^n \cdot \sum_{d=1}^{\infty} \sum_{u \in S(q,d,b')} \prod_{k=1}^{\infty} \prod_{i=1}^{n} \varpi(a_i)^{u_i(q^{dk}-1)} \cdot g_{q^{dk}}\left(u_i(q^{dk}-1)\right) \cdot \frac{t^{dk}}{dk}\right). \tag{4.3}$$

By Lemma 4.1, one knows that for any $u \in S(q, b', d)$,

$$g_{q^{dk}}\left(u_i(q^{dk}-1)\right) = g_{q^d}\left(u_i(q^d-1)\right)^k. \tag{4.4}$$

Note also that for any $u \in S(q, b', d)$, we have $\varpi(a_i)^{u_i(q^{dk}-1)} = \varpi(a_i)^{u_i(q^d-1)k}$. Then by (4.3) and (4.4) we have

$$\begin{aligned}
&L^*(\chi_1, ..., \chi_n, f; t) \\
&= \exp\left((-1)^n \cdot \sum_{d=1}^{\infty} \sum_{u \in S(q,b',d)} \sum_{k=1}^{\infty} \frac{t^{dk}}{dk} \cdot \prod_{i=1}^{n} \varpi(a_i)^{u_i(q^d-1)k} \cdot g_{q^d}(u_i(q^d-1))^k\right) \\
&= \exp\left(\sum_{d=1}^{\infty} \frac{(-1)^{n-1}}{d} \sum_{u \in S(q,b',d)} \sum_{k=1}^{\infty} -\frac{\prod_{i=1}^{n} \varpi(a_i)^{u_i(q^d-1)k} \cdot g_{q^d}(u_i(q^d-1))^k \cdot t^{dk}}{k}\right) \\
&= \exp\left(\sum_{d=1}^{\infty} \frac{(-1)^{n-1}}{d} \sum_{u \in S(q,b',d)} \log\left(1 - t^d \prod_{i=1}^{n} \varpi(a_i)^{u_i(q^d-1)} \cdot g_{q^d}\left(u_i(q^d-1)\right)\right)\right) \\
&= \prod_{d=1}^{\infty} \prod_{u \in S(q,b',d)} \exp\left(\frac{(-1)^{n-1}}{d} \log\left(1 - t^d \prod_{i=1}^{n} \varpi(a_i)^{u_i(q^d-1)} \cdot g_{q^d}(u_i(q^d-1))\right)\right) \\
&= \prod_{d=1}^{\infty} \prod_{u \in S(q,b',d)} \left(1 - t^d \prod_{i=1}^{n} \varpi(a_i)^{u_i(q^d-1)} \cdot g_{q^d}\left(u_i(q^d-1)\right)\right)^{\frac{(-1)^{n-1}}{d}}.
\end{aligned} \tag{4.5}$$

It follows immediately from (4.5) that

$$\begin{aligned}
&L^*(\chi_1, ..., \chi_n, f; t)^{(-1)^{n-1}} \\
&= \prod_{d=1}^{\infty} \prod_{u \in S(q,b',d)} \left(1 - t^d \prod_{i=1}^{n} \varpi(a_i)^{u_i(q^d-1)} \cdot g_{q^d}\left(u_i(q^d-1)\right)\right)^{\frac{1}{d}} \\
&= \prod_{d \in I} \prod_{u \in S(q,b',d)} \left(1 - t^d \prod_{i=1}^{n} \varpi(a_i)^{u_i(q^d-1)} \cdot g_{q^d}\left(u_i(q^d-1)\right)\right)^{\frac{1}{d}}.
\end{aligned} \tag{4.6}$$

Now let $u \in O(S(q, b', d))$. Then note that $\omega(a_i)^q = \omega(a_i)$ and by Lemma 4.2 we know that for each of the d points in the orbit of $u \in S(q, b', d)$ under q-action, the corresponding factor in (4.6) is the same. So we can remove the power $\frac{1}{d}$ in (4.6) by restricting u to run over the set $O(S(q, b', d))$. Therefore we get

$$\begin{aligned}
&L^*(\chi_1, ..., \chi_n, f; t)^{(-1)^{n-1}} \\
&= \prod_{d \in I} \prod_{u \in O(S(q,b',d))} \left(1 - t^d \prod_{i=1}^{n} \varpi(a_i)^{u_i(q^d-1)} \cdot g_{q^d}\left(u_i(q^d-1)\right)\right).
\end{aligned} \tag{4.7}$$

But Lemma 2.1 applied to the finite field $\mathbf{F}_{q^d}$ gives

$$g_{q^d}\left(u_i(q^d-1)\right) = \pi^{\sum_{i=1}^n \sigma_p(u_i(q^d-1))} \prod_{j=0}^{de-1} \Gamma_p\left(\frac{\langle p^j u_i(q^d-1)\rangle_{q^d-1}}{q^d-1}\right). \tag{4.8}$$

So by (4.7) and (4.8) we then get the desired formula (4.1).

Now from (4.1) we deduce that

(4.9)

$$\begin{aligned}
&\mathrm{ord}_q\omega_u \\
&= \frac{1}{d}\cdot \mathrm{ord}_q\left(\pi^{\sum_{i=1}^n \sigma_p(u_i(q^d-1))}\cdot \prod_{i=1}^n \varpi(a_i)^{u_i(q^d-1)} \prod_{j=0}^{de-1}\Gamma_p\left(\frac{\langle p^j u_i(q^d-1)\rangle_{q^d-1}}{q^d-1}\right)\right) \\
&= \frac{1}{d}\cdot \mathrm{ord}_q\left(\pi^{\sum_{i=1}^n \sigma_p(u_i(q^d-1))}\right) \\
&= \frac{1}{d(p-1)}\sum_{i=1}^n \sigma_p(u_i(q^d-1)) \\
&= \frac{1}{d(p-1)}\sum_{i=1}^n \frac{p-1}{q^d-1}\sum_{j=0}^{de-1}\langle p^j u_i(q^d-1)\rangle_{q^d-1} \\
&= \frac{1}{d}\sum_{i=1}^n \frac{1}{q^d-1}\sum_{j=0}^{de-1}\langle p^j u_i(q^d-1)\rangle_{q^d-1}.
\end{aligned}$$

Since

$$\langle p^j u_i(q^d-1)\rangle_{q^d-1} - \{p^j u_i\}(q^d-1) \equiv [p^j u_i](q^d-1) \equiv 0 \pmod{q^d-1}$$

and note that $0 \le \{p^j u_i\}(q^d-1) \le q^d-1$, we have

$$\langle p^j u_i(q^d-1)\rangle_{q^d-1} = \{p^j u_i\}(q^d-1). \tag{4.10}$$

So we get by (4.9) as well (4.10),

$$\mathrm{ord}_q\omega_u = \frac{1}{d}\sum_{i=1}^n \frac{1}{q^d-1}\sum_{j=0}^{de-1}\{p^j u_i\}(q^d-1) = \frac{1}{d}\sum_{j=0}^{de-1}\sum_{i=1}^n \{p^j u_i\},$$

as required. The proof of Theorem 4.3 is complete. □

For any rational number $0 \le r \le n$, we define a number h_r by

$$h_r := \sharp\left\{u \in S(\Delta, b') \mid \sum_{i=1}^n \sum_{j=0}^{de-1}\{p^j u_i\} = dr\right\}.$$

Therefore by Theorem 4.3 we derive the following result.

Theorem 4.4. *Let $f(x) = \sum_{i=1}^n a_i x^{V_i} \in \mathbf{F}_q[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$ be a diagonal Laurent polynomial, and let $M = (V_1, ..., V_n)$ be the $n \times n$ matrix having V_i as its i-th column for $1 \le i \le n$ satisfying $(p, \det M) = 1$. Then for each rational number $0 \le r \le n$, the p-adic Newton polygon of the L-function $L^*(\chi_1, ..., \chi_n, f; t)$ has a side of slope r with horizontal length h_r.*

Corollary 4.5. *Let $f(x) = \sum_{i=1}^n a_i x^{V_i} \in \mathbf{F}_q[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$ be a diagonal Laurent polynomial, and let $M = (V_1, ..., V_n)$ be the $n \times n$ matrix having V_i as its i-th column for $1 \le i \le n$. Let $(p, \det M) = 1$. Then each of the following is true:*

(i) *We have*

$$L^*(f;t) = \prod_{d\in I_0}\prod_{u\in O(S(q,d))}\Bigg(1-t^d\cdot\pi^{\sum_{i=1}^n\sigma_p(u_i(q^d-1))} \cdot\prod_{i=1}^n\varpi(a_i)^{u_i(q^d-1)}\prod_{j=0}^{de-1}\Gamma_p\Bigg(\frac{\langle p^j u_i(q^d-1)\rangle_{q^d-1}}{q^d-1}\Bigg)\Bigg), \tag{4.11}$$

where I_0 is defined by $I_0 := \{d\in\mathbf{Z}^+|\exists u\in S(\Delta,0),$ such that $d(q,0,u)=d\}$ and $S(q,d) := S(q,0,d)$.

(ii) *If $u\in O(S(q,d))$ and ω_u is any one of the d reciprocal roots of the corresponding factor in* (4.11)*, then we have*

$$\mathrm{ord}_q\omega_u = \frac{1}{d}\sum_{j=0}^{de-1}\sum_{i=1}^n\{p^j u_i\}.$$

(iii) *For each rational number $0\le r\le n$, the p-adic Newton polygon of the L-function $L^*(f;t)$ has a side of slope r with horizontal length h_r.*

Finally we point out that if f is degenerate, which means that $p|\det M$, then we can change it to the nondegenerate case by using some suitable transformations. For instance, let $f(x_1,x_2) = x_1x_2^{-1}+x_1^{-1}x_2^{1-p}$. Then $|\det(M)| = p$ and so f is degenerate. First let $y_1 = x_1x_2^{-1}, y_2 = x_2$. Then it is an invertible transformation. Second let $y_1' = y_1, y_2' = y_2^p$. Finally let $z_1 = y_1', z_2 = y_2'y_1'^{-1}$. This is also invertible. Hence we get $f(x_1,x_2) = z_1+z_2 := h(z_1,z_2)$ and

$$L^*(\chi_1,\chi_2,f;t) = L^*(\chi_1',\chi_2',h;t) \tag{4.12}$$

for some multiplicative characters χ_1',χ_2' of $\mathbf{F}_q$. But $h = z_1+z_2$ is nondegenerate. Therefore we can use Theorems 4.3 and 4.4 to compute the L-function $L^*(\chi_1',\chi_2',h;t)$ (hence $L^*(\chi_1,\chi_2,f;t)$ by (4.12)) and determine its p-adic Newton polygon.

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