ON THE SUM FORMULA FOR THE $q$-ANALOGUE
OF NON-STRIC T MULTIPLE ZETA VALUES

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Abstract. In this article, the $q$-analogues of the linear relations of non-strict
multiple zeta values called “the sum formula” and “the cyclic sum formula”
are established.

1. Introduction

For any multi-index $k = (k_1, k_2, \ldots, k_r)$ ($k_i \in \mathbb{Z}$, $k_i > 0$), the weight $\text{wt}(k)$
and depth $\text{dep}(k)$ of $k$ are by definition the integers $k = k_1 + k_2 + \cdots + k_r$ and
$r$ respectively. We denote by $I(k, r)$ the set of multi-indices $k$ of weight $k$, and
depth $r$, and by $I_0(k, r)$ the subset of admissible indices, i.e., indices with the extra
requirement that $k_1 \geq 2$.

For an admissible index $(k_1, \ldots, k_r)$, the multiple zeta value $\zeta(k_1, \ldots, k_r)$ ($\text{MZV}$, for
short) and the non-strict multiple zeta value are defined as follows:

$$\zeta(k_1, \ldots, k_r) := \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},$$

$$\zeta^*(k_1, \ldots, k_r) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

The latter is also called multiple zeta-star value (MZSV, for short) in $[1][19]$. Both
values can be written as a $\mathbb{Z}$-linear combination of each other.

These values are known to be related to many objects of mathematics and
quantum physics, for example, connection formulæ for hypergeometric functions $[20]$,
knot invariants $[16]$, Feynman diagrams $[15]$ and so on. They also appear in the
coefficients of the Drinfel’d’s KZ-associator $[7]$. The properties of the KZ-associator
are related to the representations of the fundamental group of a configuration space.

Study of MZSVs has been initiated by Leonhard Euler $[8]$, and he got many
results including the well-known formula:

$$\zeta^*(k - 1, 1) = \frac{k + 1}{2} \zeta(k) - \frac{1}{2} \sum_{r=2}^{k-2} \zeta(r) \zeta(k - r).$$

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It seems that Euler wanted to give an answer to the question, “When are MZSVs in the algebra generated by Riemann zeta values $\zeta(k)$?” It is a basic and an important question even now. The next two equivalent formulae were conjectured in [11], and proved by Andrew Granville [10] and Don Zagier independently.

**Sum Formula.** For positive integers $0 < r < k$, there holds

$$
(1) \quad \sum_{k \in I_0(k, r)} \zeta(k) = \zeta(k), \quad \sum_{k \in I_0(k, r)} \zeta^*(k) = \left(\frac{k-1}{r-1}\right) \zeta(k).
$$

These formulae are so fundamental that they are re-proved again and again [2, 12, 13, 17, 18, 20, 22]. For the MZSV case, there are two more proofs. One is (1)

$$
(2) \quad \sum_{i=1}^{k-r-2} \sum_{j=0}^{k-r-1} \zeta^*(k_1 - j, k_i + 1, \ldots, k_r, k_1, \ldots, k_{i-1}, j + 1) = k \zeta(k + 1),
$$

where the empty sum means zero.

**Generating Function.** For the multiple polylogarithms with equality defined by

$$
\text{Li}_{k_1, \ldots, k_r}^*(t) := \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{t^{n_1}}{n_1^{k_1} \cdots n_r^{k_r}},
$$

the generating function and its special value at $t = 1$ are expressed as follows:

$$
(3) \quad \sum_{k>r>0} \sum_{k \in I_0(k, r)} \text{Li}^*_k(t) x^{k-r-1} y^{r-1} = \frac{1}{1-x-y} \int_0^t (1-s)^{-y} \text{$_2F_1$}(1-x-y, 1-y, 2-x-y; s) ds,
$$

where $\text{$_2F_1$}$ is Gauß’s hypergeometric series.

In this article, we define the $q$-analogues of MZSVs and construct the $q$-analogues of the above formulae.

For $0 < q < 1$ and $\alpha \in \mathbb{C}$, $[\alpha]$ is defined by $[\alpha] := (1-q^\alpha)/(1-q)$ $q^{-1}$ $\alpha$. The $q$-Pochhammer symbol is defined by $(\alpha; q)_{\infty} := \prod_{n=0}^{\infty} (1 - \alpha q^n)$ and $(\alpha; q)_n := (\alpha; q)_{\infty}/(\alpha q^n; q)_{\infty}$ for any integer $n$. Then the $q$-analogues of MZV are defined by

$$
\zeta_q[k_1, \ldots, k_r] := \sum_{n_1 > \cdots > n_r > 0} \frac{q^{n_1(k_1-1)+\cdots+n_r(k_r-1)}}{|n_1|^{k_1} \cdots |n_r|^{k_r}}.
$$

A $q$-analogue of the relations for MZVs are considered in [3, 5, 6, 21]. A finite $q$-analogue of $\zeta^*(k)$ is studied in [4].
**Definition 1.** For any admissible index \((k_1, \ldots, k_r)\), the \(q\)-analogue of MZSV and the multiple polylogarithm with equality are as follows:

\[
\zeta_q^* [k_1, \ldots, k_r] := \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{q^{n_1 (k_1 - 1) + \cdots + n_r (k_r - 1)}}{[n_1]_{k_1} \cdots [n_r]_{k_r}} ,
\]

\[
\text{Li}_{k_1, \ldots, k_r}^* [t] := \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{t^{n_1}}{[n_1]_{k_1} \cdots [n_r]_{k_r}} .
\]

As the \(q\)-analogue of (2), we have the next formula:

**Theorem 1** (Cyclic Sum Formula). For \((k_1, \ldots, k_r) \in I_0(k, r)\),

\[
\sum_{i=1}^{r} \sum_{j=0}^{k_i-2} \zeta_q^* [k_i - j, k_{i+1}, \ldots, k_r, k_1, \ldots, k_{i-1}, j + 1] = \sum_{l=0}^{r} (k - l) \binom{r}{l} (1 - q)^l \zeta_q [k - l + 1] ,
\]

where the empty sum means zero.

Moreover, there also holds the \(q\)-analogue of (3):

**Theorem 2** (Generating Function of Multiple Polylogarithms).

\[
\sum_{k \geq r > 0} \left\{ \sum_{k \in I_0(k, r)} \text{Li}_k^* [t] \right\} u^{k-r-1} v^{r-1} = \frac{1}{1 - u - v} \int_0^\infty \frac{t \phi_1 (a, b, aq; s, q) \, dq}{(bs; q)_\infty} d_q s ,
\]

where \(2\phi_1\) is Heine’s \(q\)-hypergeometric series [9], \(q^{-a-1} = \frac{1}{1 - q(a+v)}\) and \(b = \frac{1 - (1-q)u}{1 - (1-q)(a+v)}\), and the integral is the Jackson \(q\)-integral [9]:

\[
\int_0^\infty f(s) \, dq s := (1-q)t \sum_{n=0}^\infty f(q^n t) q^n .
\]

As the corollary of these theorems, we obtain the \(q\)-analogue of (4):

**Corollary 3** (Sum Formula). For integers \(0 < r < k\),

\[
\sum_{k \in I_0(k, r)} \zeta_q^* [k] = \frac{1}{k-1} \binom{k-1}{r-1} \sum_{l=0}^{r-1} \binom{r-1}{l} (k - 1 - l) (1 - q)^l \zeta_q [k - l] .
\]
Summing up these equations by rotating the indices, and we have the theorem.

To prove (3), by using the equation

$$\frac{1}{n_1} q^{n_1 - n_{r+1}} = \left( \frac{1}{n_1 - n_{r+1}} - \frac{1}{n_1} \right) \frac{1}{n_{r+1}}.$$ 

we have

$$T(k_1, k_2, \ldots, k_r)$$

$$= \sum_{n_1 \geq \cdots \geq n_{r+1} \geq 1, \ n_1 \neq n_{r+1}} q^{n_1(k_1-1)+n_2(k_2-1)+\cdots+n_{r+1}(k_{r+1}-1)} \left( \frac{1}{n_1 - n_{r+1}} - \frac{1}{n_1} \right) \frac{1}{n_{r+1}}$$

$$= \sum_{n_1 \geq \cdots \geq n_{r+1} \geq 1, \ n_1 \neq n_{r+1}} q^{n_1(k_1-2)+n_2(k_2-1)+\cdots+n_{r+1}(k_{r+1}-1)} q^{n_1}$$

$$= \sum_{n_1 \geq \cdots \geq n_{r+1} \geq 1, \ n_1 \neq n_{r+1}} q^{n_1(k_1-3)+n_2(k_2-1)+\cdots+n_{r+1}(k_{r+1}-1)+n_{r+1}(2-1)} q^{n_1}$$

$$= \cdots \cdots \cdots$$

$$= \sum_{n_1 \geq \cdots \geq n_{r+1} \geq 1, \ n_1 \neq n_{r+1}} q^{n_2(k_2-1)+\cdots+n_{r}(k_{r}-1)+n_{r+1}(k_{r+1}-2)} q^{n_1}$$

$$= \sum_{j=0}^{k_1-2} \zeta_q[k_1-j, k_2, \ldots, k_r, j+1] + (k_1-1) \sum_{n_1=1}^{\infty} \frac{q^{n(k-r)}[n]}{[n+1]}.$$

By using the equation

$$\frac{1}{n_1} q^{n_1} = \left( \frac{q^{n_1 - n_{r+1}}}{n_1 - n_{r+1}} - \frac{q^{n_1 - n_{r+1}}}{n_1} \right) \frac{q^{n_{r+1}}}{n_{r+1}}.$$
the first term is
\[
\sum_{n_1 \geq \cdots \geq n_{r+1} \geq 1 \atop n_1 \neq n_{r+1}} q^{n_2(k_2-1)+\cdots+n_r(k_r-1)+n_{r+1}(k_{r+1}-1)} \frac{q^{n_1}}{[n_1]^1 [n_2]^2 \cdots [n_r]^r [n_{r+1}]^{r+1}} \frac{q^{n_1}}{n_1 - n_{r+1}}
\]
\[
= \sum_{n_2 \geq \cdots \geq n_{r+2} \geq 1 \atop n_2 \neq n_{r+2}} q^{n_2(k_2-1)+\cdots+n_r(k_r-1)+n_{r+2}(k_{r+2}-1)} \times \left( \sum_{n_1=n_2}^{n_1=n_2} q^{n_1-n_{r+1}} \frac{q^{n_1}}{n_1} - q^{n_1} \frac{q^{n_1}}{n_{r+1}} \right)
+ \sum_{n_2=1}^{\infty} q^{n_2(k_2+\cdots+k_r+k_{r+1}-r)} \left( \sum_{n_1=n_2+1}^{\infty} q^{n_1-n_2} \frac{q^{n_1}}{n_1} - \frac{q^{n_1}}{n_2} \right)
\]
\[
= \sum_{n_2 \geq \cdots \geq n_{r+2} \geq 1 \atop n_2 \neq n_{r+2}} q^{n_2(k_2-1)+\cdots+n_r(k_r-1)+n_{r+2}(k_{r+2}-1)} \frac{q^{n_2-n_{r+2}}}{[n_2]^2 \cdots [n_r]^r [n_{r+2}]^{r+2}} \frac{q^{n_2-n_{r+2}}}{n_2 - n_{r+2}} + \sum_{n=1}^{\infty} q^{n(k-r+1)} \frac{1}{[n]^{k+1}}.
\]

Moreover, by substituting the equation
\[
\sum_{n=1}^{\infty} q^{n(k-m)} \frac{1}{[n]^{k+1}} = \sum_{l=0}^{m} \binom{m}{l} (1-q)^l \zeta_q[k-l+1],
\]
we obtain \(6\).

Furthermore, for \(k = (k_1, \ldots, k_r) \in I_0(k, r)\), we set
\[
J_0(k) := \bigcup_{i=1}^{r} \bigcup_{j=0}^{k_{i-2}} \{(k_i - j, k_{i+1}, \ldots, k_r, k_1, \ldots, k_{i-1}, j + 1) \in I_0(k + 1, r + 1)\}
\]
Then
\[
I_0(k + 1, r + 1) = \bigcup_{k \in I_0(k, r)} J_0(k), \quad \text{and} \quad J_0(k) \cap J_0(k') = \emptyset \quad \text{if} \quad k \neq k'.
\]

From \(4\) we have
\[
\sum_{k' \in J_0(k)} \zeta_q^* [k'] = \frac{\#J_0(k)}{k-r} \sum_{l=0}^{r} (k-l) \binom{r}{l} (1-q)^l \zeta_q[k-l+1],
\]
and summing up about \(k\) we obtain
\[
\sum_{k' \in I_0(k+1, r+1)} \zeta_q^* [k'] = \sum_{k \in I_0(k, r)} \sum_{k' \in J_0(k)} \zeta_q^* [k']
= \frac{\#I_0(k + 1, r + 1)}{k-r} \sum_{l=0}^{r} (k-l) \binom{r}{l} (1-q)^l \zeta_q[k-l+1]
= \frac{1}{k-r} \sum_{l=0}^{r} (k-l) \binom{r}{l} (1-q)^l \zeta_q[k-l+1].
\]
Thus we have Corollary \(3\)
We denote the generating functions of $\text{Li}_k^*$ as follows:

$$
\Psi^*(u,v,t,q) := \sum_{k \geq r > 0} \left( \sum_{k \in l(k,r)} \text{Li}_k^*[t] \right) u^{k-r} v^{r-1},
$$

$$
\Psi_0^*(u,v,t,q) := \sum_{k > r > 0} \left( \sum_{k \in l_0(k,r)} \text{Li}_k^*[t] \right) u^{k-r-1} v^{r-1}.
$$

To investigate the above generating functions we use the $q$-differential equation, where the $q$-differential operator $D_q$ is defined by

$$(D_q f)(t) := \frac{f(t) - f(qt)}{t - qt}.$$ 

From the $q$-differential equation for $\text{Li}_k^*$,

$$D_q \text{Li}_{k_1,k_2,...,k_r}^*[t] = \begin{cases} 
\frac{1}{t} \text{Li}_{k_1-1,k_2,...,k_r}^*[t] & (k_1 \geq 2), \\
\frac{1}{t-1} \text{Li}_{k_2,...,k_r}^*[t] & (k_1 = 1 \text{ and } r \geq 2), \\
\frac{1}{1-t} & (k_1 = 1 \text{ and } r = 1),
\end{cases}$$

$\Psi^*$ and $\Psi_0^*$ satisfy the following $q$-differential equations:

$$D_q \Psi_0^*(u,v,t,q) = \frac{1}{t} \Psi^*(u,v,t,q),$$

$$D_q (\Psi^* - u\Psi_0^*)(u,v,t,q) = \frac{1}{1-t} + \frac{1}{t(1-t)} v\Psi^*(u,v,t,q).$$

By eliminating $\Psi^*$ from the above equations, we have that $\Psi_0^*$ satisfies the inhomogeneous linear $q$-differential equation of second order:

$$(7) \quad q t(1-t) D_q^2 f + \{(1-t)(1-u) - v\} D_q f = 1.$$

$\Psi_0^*$ is characterized as the regular solution of (7) around the origin and the value at the origin is 0.

We must find such a solution of (7) in another way. At first we put $g := D_q f$ and solve the equation

$$(8) \quad q(t)(1-t)D_q g + \{(1-t)(1-u) - v\} g = 1,$$

by variation of parameter. We choose $C_0 t^a (t; q)_\infty / (bt; q)_\infty$ for the solution of the homogeneous equation

$$q(t)(1-t)D_q h + \{(1-t)(1-u) - v\} h = 0,$$

where $q^{-a-1} = \frac{1}{1 - (1-q)(u+v)}$, $b = \frac{1 - (1-q)u}{1 - (1-q)(u+v)}$ and $C_0 \in \mathbb{C}$. We assume that

$$g(t) = C(t) t^a (t; q)_\infty / (bt; q)_\infty,$$

and substitute this into (8); then we have

$$C'(t) = q^{-a-1} t^{-a-1} (bt; q)_\infty / (t; q)_\infty.$$

The Jackson integral of $C'(t)$ is as follows:

$$
\int_0^t q^{-a-1-s-a-1} \frac{(bq; q)_\infty}{(s; q)_\infty} d_q s = q^{-a-1} \int_0^t s^{-a-1} \sum_{n=0}^{\infty} \frac{(bq; q)_n}{(q; q)_n} s^n \, d_q s
$$

$$
= q^{-a-1} \sum_{n=0}^{\infty} \frac{(bq; q)_n}{(q; q)_n} \frac{t^{n-a}}{[n-a]}
$$

$$
= \frac{t^{-a}}{1 - u - v} \phi_1(q^{-a}, bq, q^{-a+1}; t, q),
$$

where the first equality is by virtue of the $q$-binomial theorem $[9]$. So we obtain the solution of (8) which is regular at the origin:

$$
g(t) = \frac{1}{1 - u - v} \frac{(t; q)_\infty}{(bt; q)_\infty} \phi_1(q^{-a}, bq, q^{-a+1}; t, q).
$$

We consider the Jackson integral again and get the solution of (7):

$$
f(t) = \frac{1}{1 - u - v} \int_0^t \frac{(s; q)_\infty}{(bs; q)_\infty} \phi_1(q^{-a}, bq, q^{-a+1}; s, q) d_q s.
$$

By executing the Jackson integral, we have

$$
f(t) = \frac{1}{1 - u - v} \sum_{n=0}^{\infty} \frac{(1 - q^{-a})(bq; q)_n}{(1 - q^{n-a})(q; q)_n} \int_0^t \frac{(s; q)_\infty}{(bs; q)_\infty} \phi_1(q^{-a}, bq, q^{-a+1}; q^{n+1}; t, q) d_q s
$$

$$
= \frac{1}{1 - u - v} \sum_{n=0}^{\infty} \frac{(1 - q^{-a})(bq; q)_n}{(1 - q^{n-a})(q; q)_n} (1 - q) t^{n+1} \sum_{j=0}^{\infty} q^{j(n+1)} \phi_1(bt, q^{-a}, bq, q^{-a+1}; q^{n+1}, q^{-a}, q^{-a+1}, t, q)
$$

(9)

which is zero at $t = 0$. Thus we obtain the theorem.

In the same way as $[21]$, the special value of $\text{Li}^q$ and the generating function are expressed by the combination of the $q$-analogue of MZSVs: Substitute $t = q$ and the value is

$$
\text{Li}^q_{k_1, k_2, \ldots, k_r} [q] = \sum_{a_1=0}^{k_1-2} \sum_{a_2=0}^{k_2-1} \cdots \sum_{a_r=0}^{k_r-1} \binom{k_1-2}{a_1} \binom{k_2-1}{a_2} \cdots \binom{k_r-1}{a_r} 
$$

$$
\times (1 - q)^{k_1 + \cdots + k_r - a_1 - \cdots - a_r} \zeta_q [a_1, a_2, \ldots, a_r],
$$

and the generating function is

$$
\Phi^q(u, v; q, q) = \frac{1}{1 + (1 - q)x} \sum_{k > r > 0} \left\{ \sum_{k \in I_0(k, r)} \zeta_q^{r}[k] \right\} x^{k-r-1} y^{r-1},
$$

where $x = \frac{1}{1 - (1 - q)u}$ and $y = \frac{1}{1 - (1 - q)u}$. 

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On the other hand, by substituting $t = q$ to \([1]\),

\[
\Psi^r(u, v; q, q) = \frac{1}{1 - u - v} \sum_{n=0}^{\infty} \frac{(1 - q^{-a})(bq; q)_n}{(1 - q^{n-a})(q; q)_n} (1 - q)q^{n+1} \frac{(q; q)_\infty}{(bq; q)_\infty} \\
\times \sum_{j=0}^{\infty} q^{j(n+1)} \frac{(bq; q)_j}{(q; q)_j} \\
= \frac{1 - q}{1 - u - v} \sum_{n=0}^{\infty} \frac{(1 - q^{-a})(bq; q)_n}{(1 - q^{n-a})(q; q)_n} q^{n+1} \frac{(q; q)_\infty}{(bq; q)_\infty} (bq^{n+2}; q) \\
= \frac{1 - q}{1 - u - v} \sum_{n=0}^{\infty} \frac{(1 - q^{-a})(1 - bq^{n+1})}{(1 - q^{n-a})} q^{n+1} \\
= \sum_{n=1}^{\infty} \frac{q^n}{[n] - (u + v)} \frac{1 - (1 - q)(u + v)}{[n] - (1 - q)u - v} \\
= \frac{1}{1 + (1 - q)x} \sum_{n=1}^{\infty} q^n \frac{1 - (1 - q)y}{([n] - y)([n] - xq^n - y)}.
\]

Hence we have

\[
\sum_{k>r>0} \left\{ \sum_{k \in I_0(k, r)} \zeta^r_s[k] \right\} x^{k-r-1} y^{r-1} = \sum_{n=1}^{\infty} q^n \frac{1 - (1 - q)y}{([n] - y)([n] - xq^n - y)},
\]

and expanding the right hand by geometric series, we obtain Corollary \(\text{3}\).

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