CROSS $i$-SECTIONS OF STAR BODIES AND DUAL MIXED VOLUMES

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Abstract. In this paper, we establish an extension of Funk’s section theorem. Our result has the following corollary: If $K$ is a star body in $\mathbb{R}^n$ whose central $i$-slices have the same volume (with appropriate dimension) as the central $i$-slices of a centered body $M$, then the dual quermassintegrals satisfy $\tilde{W}_j(M) \leq \tilde{W}_j(K)$, for any $0 \leq j < n - i$, with equality if and only if $K = M$. The case that $K$ is a centered body implies Funk’s section theorem.

1. Introduction

Let $G(n, i)$ denote the Grassmann manifold of $i$-dimensional subspaces of $\mathbb{R}^n$, and let $\text{vol}_i(\cdot)$ be the $i$-dimensional Lebesgue measure. Let $B$ denote the Euclidean unit ball, and let $S^{n-1}$ be the Euclidean sphere in $\mathbb{R}^n$. A compact, convex set in $\mathbb{R}^n$ is said to be a convex body if it has a non-empty interior. Let $K^n$, $K^n_e$ be the classes of convex bodies and of origin-symmetric convex bodies, respectively. For $K \in K^n$ and $\xi \in G(n, i)$, let $K|\xi$ denote the image of the orthogonal projection of $K$ onto $\xi$. We will also work with general star bodies $L$, which are star-shaped bodies, meaning that $tL \subset L$ for all $t \in [0, 1]$, with the additional requirement that their radial functions $\rho_L(u) = \max \{\lambda \geq 0 : \lambda u \in L\}$ for $u \in S^{n-1}$ are continuous functions on $S^{n-1}$. Denote by $S^n$, $S^n_e$ the classes of star bodies and of origin-symmetric star bodies, respectively. For $M \in S^n$ and $\xi \in G(n, i)$, let $M \cap \xi$ denote the intersection of $M$ and $\xi$.

The well-known Aleksandrov’s projection theorem (see [1] and [4]) states that if $K, L \in K^n_e$ and if for some $i$ such that $0 < i < n$,

$$\text{vol}_i(K|\xi) = \text{vol}_i(L|\xi) \forall \xi \in G(n, i),$$

then $K = L$.

In [2], Chakerian and Lutwak gave the following extension of Aleksandrov’s projection theorem (in fact, they established a more general result):

**Theorem 1.1** (Chakerian and Lutwak [2]). If $K \in K^n_e$, $L \in K^n$ and if for some $i$ such that $0 < i < n$,

$$\text{vol}_i(K|\xi) = \text{vol}_i(L|\xi) \forall \xi \in G(n, i),$$

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then
\[ \text{vol}_n(K) \geq \text{vol}_n(L), \]
with equality if and only if \( L \) is a translate of \( K \).

A dual form of Aleksandrov’s projection theorem is the well-known Funk’s section theorem (see [3] and [4]), which states that if \( K, L \in S^n \) and if for some \( i \) such that \( 0 < i < n \),
\[ \text{vol}_i(K \cap \xi) = \text{vol}_i(L \cap \xi) \quad \forall \xi \in G(n, i), \]
then \( K = L \).

In this paper, we will give an extension of Funk’s section theorem, which is a dual form of Chakerian and Lutwak’s result (Theorem 1.1). Our result is the following theorem, which is a special case of Corollary 3.5 of this paper:

**Theorem 1.2.** If \( K \in S^n \), \( L \in S^n \) and if for some \( i \) such that \( 0 < i < n \),
\[ \text{vol}_i(K \cap \xi) = \text{vol}_i(L \cap \xi) \quad \forall \xi \in G(n, i), \]
then
\[ \text{vol}_n(K) \leq \text{vol}_n(L), \]
with equality if and only if \( K = L \).

Obviously, Funk’s section theorem is an immediate consequence: If \( K, L \) are both in \( S^n \), and
\[ \text{vol}_i(K \cap \xi) = \text{vol}_i(L \cap \xi) \quad \forall \xi \in G(n, i), \]
then Theorem 1.2 gives both
\[ \text{vol}_n(K) \leq \text{vol}_n(L) \quad \text{and} \quad \text{vol}_n(L) \leq \text{vol}_n(K). \]
But the equality conditions of the inequality of Theorem 1.2 now show that \( K = L \).

The key idea used by Chakerian and Lutwak in [2] to prove Theorem 1.1 is from Weil [14] [13] [10] and Schneider and Weil [13]. However, it seems that there is no “dual” method to arrive at Theorem 1.2, and we use an entirely “non-dual” approach to prove it. Our main tools are Radon transforms and the dual mixed volume theory which was developed by Lutwak in [9] and [10].

2. Notation and background material

Let \( C(S^{n-1}) \) be the space of continuous functions on the unit sphere \( S^{n-1} \), and let \( C_e(S^{n-1}) \) be the subspace of \( C(S^{n-1}) \) that contains the even continuous functions on \( S^{n-1} \). The subset of \( C_e(S^{n-1}) \) that contains the infinitely differentiable functions will be denoted by \( C^\infty(S^{n-1}) \). Denote by \( C(G(n, i)) \) the space of continuous functions on \( G(n, i) \). For \( f \in C(S^{n-1}) \), \( g \in C(G(n, i)) \), \( 1 \leq i \leq n - 1 \), the \( i \)-dimensional spherical Radon transform \( R_i f \) and its dual transform \( R_i^d g \) are defined by
\[
(2.1) \quad (R_i f)(\xi) = \int_{S^{n-1}} f(u) d\sigma_i(u), \quad (R_i^d g)(u) = \int_{\xi \in G(n, i)} g(\xi) d\nu_i(\xi),
\]
where \( \sigma_i \) is the Haar probability measure on \( S^{n-1} \) (and we have identified \( S^{n-1} \) with \( S^{i-1} \)) \( \cap \xi \), and \( \nu_i \) is the Haar probability measure on the homogeneous space \( \{ \xi \in G(n, i) : u \in \xi \} \).

The corresponding duality relation reads (see [7] [8] or [12])
\[
(2.2) \quad \int_{G(n, i)} (R_i f)(\xi) g(\xi) d\xi = \int_{S^{n-1}} f(u)(R_i^d g)(u) du.
\]
This allows us to define $R_i \mu$ and $R_i^\nu$ for arbitrary finite Borel measures $\mu$ on $S^{n-1}$ and $\nu$ on $G(n, i)$ as follows:

\begin{equation}
\int_{G(n, i)} (R_i \mu)(\xi)g(\xi)d\xi = \int_{S^{n-1}} (R_i^g \nu)(u)d\mu(u), \; g \in C(G(n, i));
\end{equation}

\begin{equation}
\int_{S^{n-1}} (R_i^\nu)(u)f(u)du = \int_{G(n, i)} (R_i f)(\xi)d\nu(\xi), \; f \in C(S^{n-1}).
\end{equation}

We will also write (2.2), (2.3), and (2.4) briefly as

$(R_i, f, g) = (f, R_i^g), \; (R_i \mu, g) = (\mu, R_i^g), \; (R_i^\nu, f) = (\nu, R_i f)$.

We shall say that an origin-symmetric star body $K$ in $\mathbb{R}^n$ is an $i$-intersection body (see [5] and [18]) if there is a non-negative Borel measure $\mu$ on $G(n, i)$ such that $\rho_K^{-1} = R_i^\mu$. Denote by $\mathcal{I}_n^i$ the class of $i$-intersection bodies. The case $i = n - 1$ is associated with the notion of intersection body due to Lutwak [9] (we refer to a slightly more general notion than the original notion by Lutwak). We shall simply denote the class of intersection bodies by $\mathcal{I}_n^i$ (rather than $\mathcal{I}_n^{n-1}$).

The $k$-radial sum of two star bodies $L_1, L_2$ is defined as the star body $L$ satisfying $\rho_L^k = \rho_{L_1}^k + \rho_{L_2}^k$. When $k = 1$ this operation will simply be referred to as the radial sum and denoted by $+$. It is well known (e.g. [5], [11]) that the class of $i$-intersection bodies is closed under taking $k$-radial sums.

We shall assume that the space of star bodies in $\mathbb{R}^n$ is always endowed with the natural radial metric $\delta$, defined by $\delta(L_1, L_2) = \max_{u \in S^{n-1}} |\rho_{L_1}(u) - \rho_{L_2}(u)|$. For $i$ star bodies $K_1, \ldots, K_i$ and $\xi \in G(n, i)$, the dual mixed volume, $\tilde{V}_\xi(K_1 \cap \xi, \ldots, K_i \cap \xi)$, of $K_1 \cap \xi, \ldots, K_i \cap \xi$ is defined by

\begin{equation}
\tilde{V}_\xi(K_1 \cap \xi, \ldots, K_i \cap \xi) = \frac{1}{i} \int_{S^{n-1} \cap \xi} \rho_{K_1}(u) \cdots \rho_{K_i}(u)du.
\end{equation}

If $K_1 = \cdots = K_i$, then we get the $i$th section volume function of $K$:

\begin{equation}
\text{vol}_i(K \cap \xi) = \frac{1}{i} \int_{S^{n-1} \cap \xi} \rho_K(u)du.
\end{equation}

Thus, by (2.1), the Radon transform $R_i$ has the following close connection with the central sections of star bodies:

\begin{equation}
(R_i \rho_K)(\xi) = \frac{1}{\omega_i} \text{vol}_i(K \cap \xi), \; \xi \in G(n, i).
\end{equation}

When $i = n$ in (2.5), the dual mixed volume of the star bodies $K_1, \ldots, K_n$ is denoted by $\tilde{V}(K_1, \ldots, K_n)$, and the dual mixed volume $\tilde{V}(K, n - i; B, i)$ is called the dual quermassintegral of $K$, denoted by $\tilde{W}_i(K)$, where there are $n - i$ copies of $K$ and $i$ copies of the unit ball $B$ in $\mathbb{R}^n$.

3. Main result and its proof

In this section, we shall establish the following theorem, which has Theorem 1.2 as a direct consequence.

\textbf{Theorem 3.1.} If $K_1, \ldots, K_i \in S^a, M \in S^a_i$, and

$\tilde{V}_\xi(K_1 \cap \xi, \ldots, K_i \cap \xi) = \text{vol}_i(M \cap \xi) \forall \xi \in G(n, i)$,
then
$$\tilde{W}_{n-i}(M) = \tilde{V}(K_1, \ldots, K_i, B_{n-i}),$$
and for all $j$ such that $0 \leq j < n-i$,
$$\tilde{W}_j(M)^i \leq \tilde{W}_j(K_1) \cdots \tilde{W}_j(K_i),$$
with equality, for any $j$, implying that $K_1, \ldots, K_i$ all are dilations of $M$.

To prove Theorem 3.1, the class of generalized intersection bodies, introduced by Zhang in [17], is needed. Therefore, we give the notion and some properties of generalized intersection bodies in advance.

For the case $i = n-1$, the $i$-dimensional spherical Radon transform, $R_{n-1}$, will be simply denoted by $R$. When restricted to $C^\infty(S^{n-1})$, the spherical Radon transform $R : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$ is a continuous bijection (see [8] and [17], Lemma 5.7). For $K \in S^n$, the distribution $R^{-1}\rho_K$ is called the dual generating distribution of $K$, denoted by $\tilde{\mu}_K$. If $\tilde{\mu}_K$ is a measure, then $\rho_K = R\tilde{\mu}_K$ determines an intersection body $K$. A body $K \in S^n$ is called a generalized intersection body if the dual generating distribution $\tilde{\mu}_K$ is a signed measure. Denote by $\mathcal{T}_d^n$ the set of generalized intersection bodies. If $Q \in \mathcal{T}_d^n$, then $\tilde{\mu}_Q = R^{-1}\rho_Q$ is a signed measure on $S^{n-1}$. Thus by the Jordon decomposition of $\tilde{\mu}_Q$, there exist measures $\mu_1, \mu_2$ on $S^{n-1}$ such that $\tilde{\mu}_Q = \mu_1 - \mu_2$. Now from the fact that $\tilde{\mu}_Q$ is finite, it is easily shown that $\mu_1$ and $\mu_2$ are both finite (see [6]). For $i = 1, 2$, define
$$\tilde{\mu}_i(E) = \frac{1}{2}[\mu_i(E) + \mu_i(-E)],$$
where $E$ is a Borel set on $S^{n-1}$. Then $\tilde{\mu}_1, \tilde{\mu}_2$ are even finite measures on $S^{n-1}$. Since $\tilde{\mu}_Q$ also is even, we have $\tilde{\mu}_Q = \tilde{\mu}_1 - \tilde{\mu}_2$. Let $Q_1, Q_2$ be intersection bodies determined by $\tilde{\mu}_1, \tilde{\mu}_2$, respectively. Then $\rho_Q = \rho_{Q_1} - \rho_{Q_2}$. But this implies that $Q_1 = Q + Q_2$. That is to say, we can define a body in $\mathcal{T}_d^n$ in the following different way: A body $K \in S^n$ is said to be a generalized intersection body if there exists an $M \in \mathcal{T}^n$ such that $K + M \in \mathcal{T}^n$.

As main ingredients in the proof of Theorem 3.1, the following lemmas are also required.

**Lemma 3.2** (Milman [11]). Let $K_1 \in \mathcal{T}_{n-i_1}$, and $K_2 \in \mathcal{T}_{n-i_2}$ for $i_1, i_2 < n$ such that $i_3 = i_1 + i_2 < n$. Then the origin-symmetric star body $K_3$ defined by $\rho_{K_3} = \rho_{K_1}^{i_1} \rho_{K_2}^{i_2}$ satisfies $K_3 \in \mathcal{T}_{n-i_3}$.

**Lemma 3.3** (Zhang [17]). The generalized intersection bodies are dense in the class of origin-symmetric star bodies.

**Lemma 3.4.** If $K_1, \ldots, K_i \in S^n$ $(1 \leq i \leq n-1)$, $L \in S^n$, and
\begin{equation}
\tilde{v}_\xi(K_1 \cap \xi, \ldots, K_i \cap \xi) = \text{vol}_i(L \cap \xi) \forall \xi \in G(n, i),
\end{equation}
then
\begin{equation}
\tilde{V}(K_1, \ldots, K_i, Q_1, \ldots, Q_{n-i}) = \tilde{V}(L, \ldots, L, Q_1, \ldots, Q_{n-i}),
\end{equation}
for all $Q_j \in S^n$. 

Define a star body $K$ by $\rho^i_K \equiv \rho_K, \ldots, \rho_i$. Then
\begin{equation}
\bar{V}_i(K_1 \cap \xi, \cdots, K_n \cap \xi) = \text{vol}_i(K \cap \xi \ \forall \xi \in G(n, i)).
\end{equation}

Let $Q_j \in T^n (j = 1, \cdots, n - i)$. Applying Lemma 3.2 to $\{Q_j\}$ successively shows that the origin-symmetric star body $Q$ defined by $\rho^i_Q = \rho_{Q_1} \cdots \rho_{Q_{n-i}}$ is an $i$-intersection body. Thus there exists a non-negative measure $\mu$ on $G(n, i)$ so that $\rho^i_Q = R^i_i \mu$.

From (3.1), (3.3), (2.4) and (2.7), we have
\begin{equation}
\bar{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i}) - \bar{V}(L_1, \cdots, L_i, Q_1, \cdots, Q_{n-i})
\end{equation}
\begin{equation}
= \frac{1}{n} \int_{S^{n-1}} \rho^i_K \rho^i_Q^{-i} du - \frac{1}{n} \int_{S^{n-1}} \rho^i_L \rho^i_Q^{-i} du
\end{equation}
\begin{equation}
= \frac{1}{n} \left[ (\rho^i_K, \rho^i_Q) - (\rho^i_L, \rho^i_Q) \right]
\end{equation}
\begin{equation}
= \frac{1}{n} \left[ (\rho^i_K, R^i_i \mu) - (\rho^i_L, R^i_i \mu) \right]
\end{equation}
\begin{equation}
= \frac{1}{n} \left[ (R^i_i \rho^i_K, \mu) - (R^i_i \rho^i_L, \mu) \right]
\end{equation}
\begin{equation}
= \frac{1}{n \omega_i} \int_{G(n, i)} \left[ \text{vol}_i(K \cap \xi) - \text{vol}_i(L \cap \xi) \right] d\mu(\xi)
\end{equation}
\begin{equation}
= \frac{1}{n \omega_i} \int_{G(n, i)} \left[ \bar{V}_i(K_1 \cap \xi, \cdots, K_i \cap \xi) - \text{vol}_i(L \cap \xi) \right] d\mu(\xi).
\end{equation}

This proves that (3.2) must hold for all $Q_j \in T^n$.

Next we show that (3.2) holds for all $Q_j \in T^n$. Since the dual mixed volume $\bar{V}$ is symmetric in its arguments, we need only to show that (3.2) holds for one of $\{Q_j\} \subset T^n$. Without loss of generality, suppose $Q \in T^n$ and that $Q$ is the last argument of $\bar{V}$. By the alternative definition of generalized intersection body, there exists an $M \in T^n$ such that $Q \cap M \in T^n$. Since (3.2) holds for any body in $\{Q_j\} \subset T^n$, we have
\begin{equation}
\bar{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i-1}, Q \cap M) = \bar{V}(L_1, \cdots, L_i, Q_1, \cdots, Q_{n-i-1}, Q \cap M).
\end{equation}

But from the definition of the radial sum and (2.5),
\begin{equation}
\bar{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i-1}, Q \cap M) = \bar{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i-1}, Q)
\end{equation}
\begin{equation}
\quad + \bar{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i-1}, M),
\end{equation}
\begin{equation}
\bar{V}(L_1, \cdots, L_i, Q_1, \cdots, Q_{n-i-1}, Q \cap M) = \bar{V}(L_1, \cdots, L_i, Q_1, \cdots, Q_{n-i-1}, Q)
\end{equation}
\begin{equation}
\quad + \bar{V}(L_1, \cdots, L_i, Q_1, \cdots, Q_{n-i-1}, M).
\end{equation}

Again by the fact that (3.2) holds for any body in $\{Q_j\} \subset T^n$, from $M \in T^n$ it follows that
\begin{equation}
\bar{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i-1}, M) = \bar{V}(L_1, \cdots, L_i, Q_1, \cdots, Q_{n-i-1}, M).
\end{equation}
Then from (3.4), (3.5), and (3.6) we get
\[ \widetilde{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i-1}, Q) = \widetilde{V}(L, \cdots, L, Q_1, \cdots, Q_{n-i-1}, Q), \]
where \( Q \in \mathcal{T}_n^g \). This proves that (3.2) holds for all \( Q_j \in \mathcal{T}_n^g \).

By Lemma 3.3, every member of \( \mathcal{S}_n^e \) is the limit of generalized intersection bodies in the radial metric \( \widetilde{\delta} \), and it follows from the continuity of dual mixed volumes that (3.2) must hold for all \( Q_j \in \mathcal{S}_n^e \) \((j = 1, \cdots, n - i)\). \( \square \)

**Proof of Theorem 3.1.** Taking \( Q_1 = \cdots = Q_{n-i} = B \) in Lemma 3.4 immediately gives
\[ \widetilde{W}_{n-i}(L) = \widetilde{V}(K_1, \cdots, K_i, B, \cdots, B). \]
Suppose \( 0 \leq j < n - i \). Take \( Q_1 = \cdots = Q_j = B \) and \( Q_{j+1} = \cdots = Q_{n-i} = L \) in Lemma 3.4, and get
\[ (3.7) \quad \widetilde{W}_j(L) = \widetilde{V}(K_1, \cdots, K_i, L, \cdots, L, B, \cdots, B). \]

Applying Hölder’s inequality to the right-hand side of (3.7), we get
\[ (3.8) \quad \widetilde{W}_j(L)^{n-j-1} \leq \widetilde{W}_j(L)^{n-j-1} \prod_{l=1}^{i} \widetilde{V}(K_l, \cdots, K_i, L, B, \cdots, B). \]

Again applying the Hölder inequality to the dual mixed volumes in the product on the right-hand side of (3.8), we have
\[ \widetilde{W}_j(L)^i \leq \widetilde{W}_j(K_1) \cdots \widetilde{W}_j(K_i); \]
the equality condition now follows from that of the Hölder inequality, that is, \( K_1, \cdots, K_i \) are dilations of \( L \). This proves the theorem. \( \square \)

Of interest is the special case \( K_1 = \cdots = K_i = K \) of Theorem 3.1:

**Corollary 3.5.** Let \( L \in \mathcal{S}_n^e, K \in \mathcal{S}_n^e \), and
\[ \text{vol}_i(K \cap \xi) = \text{vol}_i(L \cap \xi) \ \forall \xi \in G(n, i). \]
Then \( \widetilde{W}_{n-i}(L) = \widetilde{W}_{n-i}(K) \), and for all \( j \) such that \( 0 \leq j < n - i \),
\[ \widetilde{W}_j(L) \leq \widetilde{W}_j(K), \]
with equality, for any \( j \), if and only if \( K = L \).

Obviously, taking \( j = 0 \) in Corollary 3.5, we immediately get Theorem 1.2 presented in Section 1.

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