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CODIMENSION GROWTH OF TWO-DIMENSIONAL NON-ASSOCIATIVE ALGEBRAS

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Abstract. Let F be a field of characteristic zero and let A be a two-dimensional non-associative algebra over F. We prove that the sequence $c_n(A), n=1,2,\dots$, of codimensions of A is either bounded by $n+1$ or grows exponentially as 2^n. We also construct a family of two-dimensional algebras indexed by rational numbers with distinct T-ideals of polynomial identities and whose codimension sequence is $n+1$, $n \geq 2$.

1. Introduction

Let A be a non-necessarily associative algebra over a field of characteristic zero. A natural and well-established way of measuring the polynomial identities satisfied by A is through the study of the asymptotic behavior of its sequence of codimensions $c_n(A)$, $n = 1, 2, \dots$. More precisely, if $F\{X\}$ is the free (non-associative) algebra on a countable set $X = \{x_1, x_2, \dots\}$ and P_n is the space of multilinear polynomials in the variables $x_1, \dots, x_n$, $c_n(A)$ is the dimension of P_n modulo the polynomial identities satisfied by A.

It is well known that in case A is an associative algebra satisfying a non-trivial polynomial identity (PI-algebra), the sequence of codimensions of A is exponentially bounded ([12]). Also, in case A is a Lie algebra, such a sequence can have overexponential growth ([11]). Nevertheless for both associative and Lie algebras, no intermediate growth (between polynomial and exponential) and no exponential growth between 1 and 2 is allowed ([8], [9], [10]).

The class of algebras whose sequence of codimensions has exponential growth is quite wide. It includes all associative PI-algebras, a wide class of Lie algebras, all finite-dimensional algebras, etc. For any such algebra A one defines the PI-exponent of A as $\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$, if it exists. It was proved in [4] and [5] that for any associative PI-algebra, the PI-exponent exists and is a non-negative integer (see also [6]). The same conclusion holds for any finite-dimensional Lie algebra ([14]). Nevertheless an example was constructed in [13] showing that Lie algebras

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exist whose sequence of codimensions grows exponentially but the rate of growth is not integer.

In the general case of non-associative algebras in [2] we constructed a family of algebras A_α whose sequence of codimensions has exponential growth, the exponent of A_α exists and equals α, where α can be any real number greater than 1.

Also, examples of non-associative algebras with intermediate growth of the codimensions can be constructed. For instance in [3], for any real number $0 < \beta < 1$, we constructed an algebra whose sequence of codimensions grows as n^{n^β}.

In [3] it was also shown that for any finite-dimensional algebra A, $\dim A = d$, $c_n(A)$ is either polynomially bounded or $c_n(A) \geq \frac{1}{n^2} 2^{\frac{n}{3d^3}}$. On the other hand, for any $\beta > 1, \varepsilon > 0$, there exists a finite-dimensional algebra B with $\beta - \varepsilon \leq \exp(B) \leq \beta + \varepsilon$ ([2]).

In this paper we want to analyze in detail the codimension sequence of an arbitrary two-dimensional algebra. As in the case of associative or Lie algebras, it turns out that there is no intermediate growth and no exponential growth between 1 and 2. Hence the PI-exponent of any two-dimensional algebra is an integer. In fact we prove that the codimension growth of any two-dimensional algebra is either bounded by $n+1$ or is exponential with exponent equal to 2. We also construct a family of two-dimensional algebras indexed by rational numbers with distinct T-ideals of polynomial identities and whose codimension sequence is $n+1$, $n \geq 2$.

This last result shows that the classification, up to PI-equivalence, of the two-dimensional algebras is a difficult problem even in the case of linear codimension growth.

2. Preliminaries

Throughout this paper F will be a field of characteristic zero and A a non-necessarily associative algebra. We let $X = \{x_1, x_2, \dots\}$ be a countable set and $F\{X\}$ the free non-associative algebra on X over F. For every $n \geq 1$, we consider P_n, the space of multilinear polynomials of $F\{X\}$ in the first n variables $x_1, \dots, x_n$. Note that $\dim P_n = C_n n!$, where C_n is the number of all possible arrangements of parentheses in a given monomial of length n. It is well known that $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Given an algebra A, let $\mathrm{Id}(A) = \{f \in F\{X\} | f \equiv 0 \text{ on } A\}$ be the T-ideal of $F\{X\}$ of polynomial identities of A. Since $\operatorname{char} F = 0$, it is well known that the sequence of spaces $P_n \cap \mathrm{Id}(A)$, $n = 1, 2, \dots$, carries all information about $\mathrm{Id}(A)$.

The symmetric group S_n acts on the space P_n by permuting variables: if $\sigma \in S_n$, $f(x_1, \dots, x_n) \in P_n$,

$$\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

The space $P_n \cap \mathrm{Id}(A)$ is invariant under this action, and one studies the structure of $P_n(A) = P_n / P_n \cap \mathrm{Id}(A)$ as an S_n-module. The S_n-character of $P_n(A)$, denoted $\chi_n(A)$, is called the n-th cocharacter of A. Its degree $c_n(A) = \dim P_n(A)$ is the n-th codimension of A. By complete reducibility one writes

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the irreducible S_n-character associated to the partition λ of n and $m_\lambda \geq 0$ is the corresponding multiplicity.

Recall that if $\lambda \vdash n$, $d_\lambda = \chi_\lambda(1)$ denotes the degree of the irreducible character χ_λ.

The next lemma provides a lower bound for the degree of a character corresponding to a rectangle of height two.

Lemma 2.1. *Let $\lambda \vdash n$ be of the form $\lambda = (m, m)$ or $(m+1, m)$ or $(m+2, m)$ or $(m+3, m)$. Then if $n \geq 5$,*

$$d_\lambda \geq \frac{1}{n^2} 2^n.$$

Proof. By the hook formula ([7]) we have that for $k = 0, 1, 2, 3$,

$$d_{(m+k,m)} = \frac{k+1}{m+k+1}\binom{2m+k}{m}.$$

Consider the well-known equality

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n. \tag{1}$$

It is clear that in the sum on the left-hand side of (1) the greatest contribution is given by $\binom{n}{m}$ if $n = 2m$ or $n = 2m+1$ and by $\binom{n}{m+1}$ in the case $n = 2m+2$ or $n = 2m+3$. In any case such a term is greater than $\frac{2^n}{n+1}$.

Therefore, if $k = 0$ or $k = 1$ and $m \geq 2$,

$$d_{(m+k,m)} > \frac{k+1}{(m+k+1)(n+1)} 2^n > \frac{1}{n^2} \cdot 2^n.$$

Whereas if $k = 2$ or $k = 3$ and $m \geq 1$,

$$\begin{aligned} d_{(m+k,m)} &= \frac{(k+1)(m+1)}{(m+k)(m+k+1)}\binom{n}{m+1} \\ &> \frac{(k+1)(m+1)}{(m+k)(m+k+1)(n+1)} 2^n > \frac{1}{n^2} \cdot 2^n. \end{aligned}$$

□

The asymptotics of d_λ for $\lambda = (m+k, m)$, $k = 0, 1, 2, 3$, can be computed, and it turns out that $d_{(m+k,m)} \simeq \frac{4(k+1)}{n\sqrt{n}} 2^n$, where $n = 2m+k$. Hence the above lemma can be actually improved, and it can be shown that

$$d_\lambda \geq \frac{C}{n\sqrt{n}} 2^n$$

for some explicit constant $C > 0$.

We next state the following result of Bahturin and Drensky that is the starting point for our investigation.

Lemma 2.2 ([1]). *Let A be a finite-dimensional algebra,* $\dim A = d$. *Then $c_n(A) \leq d^n$.*

3. The main result

Throughout this section A will be a two-dimensional algebra over a field F of characteristic zero. We start by studying the case when A contains a proper ideal with trivial multiplication.

Given a partition λ of n and a Young tableau T_λ of shape λ, let e_{T_λ} denote the essential idempotent corresponding to T_λ. Recall that $e_{T_\lambda} = \bar{R}_{T_\lambda}\bar{C}_{T_\lambda}$, where

$$\bar{R}_{T_\lambda} = \sum_{\sigma \in R_{T_\lambda}} \sigma, \quad \bar{C}_{T_\lambda} = \sum_{\tau \in C_{T_\lambda}} (\operatorname{sgn}\tau)\tau,$$

and R_{T_λ} and C_{T_λ} are the stabilizers of the rows and the columns of T_λ, respectively.

Lemma 3.1. *Let A be a two-dimensional algebra with a proper ideal I such that $I^2 = 0$. Then*

$$\chi_n(A) = p \cdot \chi_{(n)} + q \cdot \chi_{(n-1,1)},$$

where $p \le 2$, $q \le 1$. Hence $c_n(A) \le n + 1$.

Proof. Since I is a proper ideal, $\dim_F I = 1$, and let $\{a, b\}$ be a basis of the algebra A, with $a \in I$. Since $I^2 = 0$, then $a^2 = (ab)a = a(ba) = 0$.

Let

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{2}$$

be the decomposition of the n-th cocharacter of A into irreducibles.

We first claim that $m_\lambda = 0$ whenever $\lambda = (\lambda_1, \dots, \lambda_t) \vdash n$ is such that $n - \lambda_1 > 1$.

In fact, let T_λ be a Young tableau of shape $\lambda \vdash n$ with $\lambda_3 \neq 0$. For any $f(x_1, \dots, x_n) \in P_n$ such that $e_{T_\lambda} f(x_1, \dots, x_n) \neq 0$, we have that $e_{T_\lambda} f = f_1 + \dots + f_m$ with each polynomial f_i alternating on at least three variables. Since $\dim A < 3$ it follows that $e_{T_\lambda} f$ is an identity of A and, so, $m_\lambda = 0$ in this case. Similarly if $\lambda \vdash n$ is such that $\lambda_2 \ge 2$, then for any tableau T_λ and for any $f \in P_n$ with $e_{T_\lambda} f \neq 0$, the polynomial $e_{T_\lambda} f$ will be a sum of polynomials each alternating on two distinct sets of variables of order two. Also in this case $e_{T_\lambda} f \equiv 0$ on A since $\{a, b\}$ is a basis of A and $a^2 = (ab)a = a(ba) = 0$.

We have proved that if $m_\lambda \neq 0$ in (2), then $\lambda = (n)$ or $\lambda = (n-1, 1)$.

Consider the partition $\lambda = (n-1, 1)$ and let M_1 and M_2 be two irreducible S_n-submodules of P_n corresponding to λ. It is well known that M_1 and M_2 are cyclic modules, and let $f_1(x_1, \dots, x_n)$ and $f_2(x_1, \dots, x_n)$ be generators of M_1 and M_2, respectively. Let e_{T_i} be a Young tableau of shape λ such that $e_{T_i} f_i \neq 0$, $i = 1, 2$. Since $e_{T_i} = \bar{R}_{T_i}\bar{C}_{T_i}$, it follows that $\bar{C}_{T_i} f_i \neq 0$. After multiplying on the left by suitable $\sigma, \tau \in S_n$, we may assume that $\sigma\bar{C}_{T_1} f_1$ and $\tau\bar{C}_{T_2} f_2$ are alternating on x_1 and x_2. Therefore, by replacing $\sigma\bar{C}_{T_1} f_1$ and $\tau\bar{C}_{T_2} f_2$ with f_1 and f_2, respectively, we may assume that f_1 and f_2 are alternating in the variables x_1 and x_2.

Since I is an ideal of A and $a \in I$, we have that $f_i(a, b, \dots, b) = \alpha_i a$, for some $\alpha_i \in F$, $i = 1, 2$. We claim that A satisfies the identity

$$\alpha_2 f_1 - \alpha_1 f_2 \equiv 0.$$

In fact, if two or more variables are evaluated in a, we will get 0 since $I^2 = 0$. If we substitute $x_1 = x_2 = b$ we will also get 0 since f_1 and f_2 are alternating on x_1 and x_2. If we substitute $x_1 = a, x_2 = b$, we will get 0 by the definition of the scalars α_1, α_2. Finally, if $x_1 = b$ and $x_2 = a$, we will get 0 since f_i is alternating on x_1 and x_2 and, so, $f_i(a, b, \dots) = -f_i(b, a, \dots)$. This establishes the claim. Thus $M_1 = M_2$

modulo $P_n \cap \mathrm{Id}(A)$, and since both modules correspond to the partition $(n-1,1)$, we obtain that $m_{(n-1,1)} \leq 1$.

We next consider the partition $\lambda = (n)$.

Let $f_i, i = 1,2,3$, be multilinear polynomials generating three submodules of P_n corresponding to $\lambda = (n)$. In this case all variables of $f_i(x_1, \dots, x_n)$ are symmetric for $i = 1,2,3$.

We split the proof into two cases according to the multiplication table of A.

Case 1. Suppose that $ab = \alpha a$, $ba = \beta a$ and $b^2 = \gamma b$, for some $\alpha, \beta, \gamma \in F$. Let $f_i(a, b, \dots, b) = \alpha_i a$ and $f_i(b, \dots, b) = \beta_i b$, $i = 1,2,3$.

We claim that the algebra A satisfies the identity

$$\delta_1 f_1 + \delta_2 f_2 + \delta_3 f_3 \equiv 0, \tag{3}$$

where $(\delta_1, \delta_2, \delta_3)$ is a solution of the following system of linear equations in the variables z_1, z_2, z_3:

$$\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 = 0,$$
$$\beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3 = 0.$$

In fact, since the f_is are symmetric on all variables and $I^2 = 0$, only two essentially different substitutions are possible: $x_1 = \cdots = x_n = b$ and $x_1 = a, x_2 = \cdots = x_n = b$. It is clear that the polynomial in (3) will vanish on these substitutions by the definition of the scalars $\delta_1, \delta_2, \delta_3$. Hence $m_{(n)} \leq 2$, and we are done in this case.

Case 2. Suppose that $ab = \alpha a$, $ba = \beta a$ and $b^2 = \delta a + \gamma b$, for some $\alpha, \beta, \gamma, \delta \in F$, with $\delta \neq 0$. We note that by choosing the basis $\{\delta a, b\}$ we may assume that $a^2 = 0$, $ab = \alpha a$, $ba = \beta a$ and $b^2 = a + \gamma b$.

Set $b' = \varepsilon a + b$ for some $\varepsilon \in F$ and compute $(b')^2 = (\varepsilon a + b)(\varepsilon a + b) = (\varepsilon\alpha + \varepsilon\beta + 1 - \varepsilon\gamma)a + \gamma(\varepsilon a + b)$. If $\gamma \neq \alpha + \beta$, then we can set $\varepsilon = \frac{1}{\gamma - (\alpha+\beta)}$ and $\{a, b'\}$ is a basis of A satisfying the hypothesis of Case 1.

Therefore we may assume that $\gamma = \alpha + \beta$, and we have that

$$a^2 = 0, \ ab = \alpha a, \ ba = \beta a \ \text{ and } \ b^2 = a + (\alpha + \beta)b.$$

Note that in this case the basis $\{a, a+b\}$ has the same multiplication table as $\{a, b\}$.

Let $f_i(b, \dots, b) = \alpha_i a + \beta_i b$, $i = 1,2,3$, and let $(\delta_1, \delta_2, \delta_3)$ be a solution of the system

$$\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 = 0,$$
$$\beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3 = 0.$$

We claim that the algebra A satisfies the identity

$$\delta_1 f_1 + \delta_2 f_2 + \delta_3 f_3 \equiv 0.$$

In fact, if we substitute $x_1 = \cdots = x_n = b$ or $x_1 = \cdots = x_n = a + b$, we will get 0. On the other hand, if we set $x_1 = a$ and $x_i = b, i = 2, \dots, n$, we will also get 0 since by the symmetry of f_i on all variables we have

$$f_i(a, b, \dots, b) = \frac{f_i(a+b, \dots, a+b)}{n} - f_i(b, \dots, b).$$

This proves the claim and $m_{(n)} \leq 2$ also in this case. Therefore $m_{(n)} \leq 2$ and the lemma is proved. Since by the hook formula, $\chi_{(n)}(1) = d_{(n)} = 1$ and $\chi_{(n-1,1)}(1) = d_{(n-1,1)} = n - 1$, $c_n(A) \leq n + 1$. □

In order to simplify the notation, in what follows we shall substitute alternation on a given set of variables in a polynomial with the marking of the variables with the same symbol ($^-, ^\sim$, etc.). For instance, the polynomial

$$\sum_{\sigma,\tau\in S_2}(\mathrm{sgn}\sigma\tau)(((x_{\sigma(1)}x_{\tau(1)})x_{\sigma(2)})x_{\tau(2)})$$

will be simply written as $(((\bar{x}_1\tilde{x}_1)\bar{x}_2)\tilde{x}_2)$.

We now study the case when A contains a proper ideal with non-trivial multiplication.

Lemma 3.2. *Let A be a two-dimensional algebra with a proper ideal I such that $I^2 = I$. Then either $c_n(A) = 1$ or $\frac{1}{n^2}2^n \le c_n(A) \le 2^n$.*

Proof. Since $\dim A = 2$, by Lemma 2.2, $c_n(A) \le 2^n$. Hence we need only to show that for all $n \ge 1$, either $c_n(A) = 1$ or $c_n(A) \ge \frac{1}{n^2}2^n$.

Since I is proper, $\dim_F I = 1$ and let $\{a\}$ be a basis of I. Since $a^2 \ne 0$, we may assume that $a^2 = a$ and let $\{a, b\}$ be a basis of A. Then $ba = \alpha a$ for some $\alpha \in F$ and, if we set $b' = -\alpha a + b$, it follows that $b'a = 0$. As a consequence, we may assume that the basis $\{a, b\}$ of A is such that $a^2 = a$ and $ba = 0$.

We next split the proof in two distinct cases according to whether $ab = 0$ or $ab \ne 0$.

Suppose first that $ab = 0$. In case $b^2 = \gamma b$ for some $\gamma \in F$, b generates a 1-dimensional ideal J. Hence $A = I \oplus J$ is a commutative associative algebra and $c_n(A) = 1$. The lemma is proved in this case.

We may therefore assume that $b^2 = \alpha a + \gamma b$, for some non-zero $\alpha \in F$.

First consider the polynomial

$$g(x_1, x_2) = \sum_{\sigma,\tau\in S_2}(\mathrm{sgn}\sigma\tau)(((x_{\sigma(2)}x_{\tau(2)})x_{\sigma(1)})x_{\tau(1)})$$

or in our other notation,

$$g(x_1, x_2) = (((\bar{x}_2\tilde{x}_2)\bar{x}_1)\tilde{x}_1).$$

Then it is easily computed that $g(b, a) = \alpha a$.

We shall prove that $c_n(A) \ge \frac{1}{n^2}2^n$. Since this inequality holds trivially for $n \le 4$, we may assume that $n \ge 5$. Write $n = 4m + k$, with $0 \le k \le 3$ and consider the polynomial $g_n(x_1, x_2)$ which is the left-normed product of m copies of the polynomial g and k copies of the variable x_1. In other words

$$g_n(x_1, x_2) = (\cdots((\cdots(\underbrace{g(x_1, x_2)g(x_1, x_2))\cdots g(x_1, x_2))}_{m\text{-times}}\underbrace{x_1)\cdots x_1)}_{k\text{-times}}.$$

It is readily checked that $g_n(b, a) = \alpha^{m+k}a \ne 0$. Moreover the complete linearization of $g_n(x_1, x_2)$ generates an irreducible S_n-submodule M of P_n corresponding to the partition $\lambda = (2m + k, 2m) \vdash n$ and $M \notin \mathrm{Id}(A)$. By Lemma 2.1 we obtain that $c_n(A) \ge \dim M = \chi_\lambda(1) = d_\lambda \ge \frac{1}{n^2}2^n$.

Suppose now that $ab \ne 0$. Since a forms a basis of I, then $ab = \beta a$ for some $\beta \in F$, $\beta \ne 0$ and, by replacing b with $\beta^{-1}b$, we may assume that $ab = a$. Hence in this case

$$a^2 = a,\ ba = 0, \text{ and } ab = a.$$

If $n = 2m$ is even, we consider the commutator $[x_1, x_2] = x_1x_2 - x_2x_1$ and the polynomial $h_m(x_1, x_2)$ obtained by taking the left-normed product of m copies of $[x_1, x_2]$. Hence

$$h_m(x_1, x_2) = (\cdots \underbrace{([x_1, x_2][x_1, x_2]) \cdots [x_1, x_2]}_{m\text{-times}}),$$

and it is immediate to check that $h_m(a, b) = a \neq 0$. Let M be the S_n-submodule of P_n generated by the complete linearization of $h_m(x_1, x_2)$. Then M is an irreducible module corresponding to the partition $\lambda = (m, m) \vdash n$ and by the above $M \notin \mathrm{Id}(A)$. But then by Lemma 2.1, $c_n(A) \geq \dim M = d_\lambda \geq \frac{1}{n^2}2^n$, and we are done.

If $n = 2m + 1$, then it is enough to consider the polynomial $h_m(x_1, x_2)x_1$ which still satisfies $h_m(a, b)a = a \neq 0$, and the conclusion follows as above. $\square$

The following is a well-known result of linear algebra. We prove it here for completeness.

Lemma 3.3. *Let φ and ψ be linear transformations of a two-dimensional vector space V over an algebraically closed field F. If $[\varphi, \psi]$ is a nilpotent transformation, then φ and ψ have a common eigenvector.*

Proof. Fix a basis $\{a_1, a_2\}$ of V on which the matrix of φ has Jordan canonical form A and let $B = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$ be the matrix of ψ in this basis.

Suppose first that $A = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$. If $\alpha = \beta$, then A is a scalar matrix and any non-zero vector of V is an eigenvector of φ. Hence we may assume that $\alpha \neq \beta$. By direct calculation

$$[A, B] = \begin{pmatrix} 0 & (\alpha - \beta)y \\ (\beta - \alpha)z & 0 \end{pmatrix},$$

and, since $[A, B]$ is nilpotent, we obtain that either $y = 0$ or $z = 0$. In the first case a_2 is a common eigenvector of φ and ψ, and in the second case a_1 is such a common eigenvector.

Suppose now that A is of the form $A = \left(\begin{smallmatrix} \alpha & 1 \\ 0 & \alpha \end{smallmatrix}\right)$. Then

$$[A, B] = \begin{pmatrix} z & (t - x) \\ 0 & z \end{pmatrix},$$

and from nilpotency we obtain $z = 0$. Hence a_1 is a common eigenvector of φ and ψ, and we are done. $\square$

We are now in a position to prove the main result of this note.

Theorem 3.1. *Let A be a two-dimensional algebra over a field of characteristic zero. Then either $c_n(A) \leq n + 1$ or $\frac{1}{n^2}2^n \leq c_n(A) \leq 2^n$.*

Proof. Since the codimensions of an algebra are not changed upon extension of the base field (see for instance [4]), we may assume that A is an algebra over the algebraically closed field F.

For any $a \in A$ let L_a and R_a be the linear transformations of A of left and right multiplication by a, respectively. Hence, for any $b \in A$, $L_a(b) = ab$ and $R_a(b) = ba$.

Suppose first that for any $a, b \in A$, the commutator $[L_a, L_b]$ is not nilpotent.

If $n = 2m + 1$ is odd, we consider the element $v \in A$ such that $[L_a, L_b]^m(v) \neq 0$. We then construct the right-normed polynomial

$$g_m(x_1, x_2, x_3) = (\tilde{x}_1(\tilde{x}_2 \cdots (\bar{x}_1(\bar{x}_2x_3)) \dots))$$

that we can also write in the form

$$g_m(x_1, x_2, x_3) = [L_{x_1}, L_{x_2}]^m(x_3).$$

It is clear that $g_m(a, b, v) = [L_a, L_b]^m(v) \neq 0$. If we let M to be the S_n-module generated by the complete linearization of $g_m(x_1, x_2, x_3)$, then M is irreducible corresponding to the partition $(m + 1, m)$ and, by the above, $M \notin \mathrm{Id}(A)$. By Lemma 2.2 and Lemma 2.1 we then obtain that $2^n \geq c_n(A) \geq d_{(m+1,m)} \geq \frac{1}{n^2}2^n$.

Now let $n = 2m + 2$ be even. Note that if there exists a non-zero element $c \in A$ such that $cd = dc = 0$ for all $d \in A$, then c generates a proper ideal of A and we will be done by Lemma 3.1 and Lemma 3.2. Therefore, since $g_m(a, b, v) \neq 0$, we may assume that either $x_4 g_m(x_1, x_2, x_3)$ or $g_m(x_1, x_2, x_3)x_4$ is not an identity of A. Let N be the S_n-submodule of P_n generated by the complete linearization of $x_4 g_m(x_1, x_2, x_3)$ or $g_m(x_1, x_2, x_3)x_4$. It can be easily proved that for any k-dimensional algebra A its cocharacter $\chi_n(A)$ lies in the strip of width k for all $n = 1, 2, \ldots$. Hence the cocharacter of A lies in a strip of height two, N contains an irreducible submodule M corresponding to the partition $(m+2, m)$ or $(m+1, m+1)$ and $M \notin \mathrm{Id}(A)$. As above we then obtain that $2^n \geq c_n(A) \geq \frac{1}{n^2}2^n$.

In case the commutator $[R_a, R_b]$ is not nilpotent, for any $a, b \in A$, then the above argument can be applied in order to construct a left-normed polynomial that will lead to the desired conclusion.

Therefore we may assume that both commutators $[L_a, L_b]$ and $[R_a, R_b]$ are nilpotent, for all $a, b \in A$. We then apply Lemma 3.3, and we conclude that A has a one-dimensional left ideal Fa and a one-dimensional right ideal Fb.

If $Fa = Fb$, then Fa is a proper two-sided ideal of A, and by Lemma 3.1 and Lemma 3.2 the conclusion of the theorem follows.

Therefore we may assume that $Fa \neq Fb$. Clearly $\{a, b\}$ is a basis of A. Moreover $ba = 0$ since $ba \in Fa \cap Fb = 0$, and by eventually multiplying a or b by a suitable scalar we may assume that only the following possibilities occur: either $a^2 = 0$ or $a^2 = a$ and either $b^2 = 0$ or $b^2 = b$. We next examine the possible four cases.

Suppose first that $a^2 = 0$ and $b^2 = 0$. If $ab = 0$, then A has trivial multiplication and $c_n(A) = 0$, for all $n > 1$. If instead $ab \neq 0$, then $A^2 = Fab$ is a one-dimensional ideal of A and we are done by Lemma 3.1 and Lemma 3.2.

Suppose now that $a^2 = a$ and $b^2 = 0$ and let $ab = \alpha a + \beta b$. If $\alpha = 0$, then Fb is a one-dimensional ideal of A. If $\beta = 0$, then Fa is a one-dimensional ideal of A. In any case we are done by Lemma 3.1 and Lemma 3.2. Therefore we may assume that $\alpha \neq 0$ and $\beta \neq 0$.

If $\alpha \neq 0$ and $n = 2m$, define the polynomial

$$g_m(x_1, x_2) = (\bar{\bar{x}}_1(\cdots(\tilde{x}_1(\cdots((\bar{x}_1\bar{x}_2)\tilde{x}_2)\cdots\bar{\bar{x}}_2)\cdots))) \tag{4}$$

obtained by first taking the left-normed product $y = (((x_1 \underbrace{x_2)x_2)\cdots x_2}_{m\text{-times}})$, then the right-normed product $(\underbrace{x_1 \cdots (x_1(x_1}_{(m-1)\text{-times}} y)))$ and then by alternating the variables equidistant from the center. Note that

$$g_m(a, b) = (a \cdots (a(((ab)b)\cdots b))).$$

Now, $(((ab)b)\cdots b) = \alpha^m a + \alpha^{m-1}\beta b$ and $(a\cdots(a(a(\alpha^m a + \alpha^{m-1}\beta b)))) = \gamma a + \alpha^{m-1}\beta^m b \neq 0$, for some $\gamma \in F$. Hence $g_m(a, b) \neq 0$, and the complete linearization of $g_m(x_1, x_2)$ generates an irreducible module not contained in $\mathrm{Id}(A)$. Clearly this module corresponds to the partition $\lambda = (m, m)$.

In case $n = 2m+1$ is odd, we only need to consider the polynomial $x_1 g_m(x_1, x_2)$ obtained from (4) by adding the variable x_1. Also in this case we can construct an irreducible module corresponding to the partition $\lambda = (m+1, m)$ which is not contained in $\mathrm{Id}(A)$. But then, by Lemma 2.2 and Lemma 2.1 we get the desired conclusion of the theorem.

The case when $a^2 = 0$ and $b^2 = b$ is clearly obtained from the above by exchanging the roles of a and b.

We finally consider the case when $a^2 = a$ and $b^2 = b$ and let $ab = \alpha a + \beta b$. If $\alpha = 0$ or $\beta = 0$ we get that Fb or Fa is a one-dimensional ideal of A, respectively. Hence we are done by Lemma 3.1 and Lemma 3.2. Therefore we may assume that $\alpha \neq 0$ and $\beta \neq 0$. The operators of left and right multiplication by a and b in the basis $\{a, b\}$ are identified with the following matrices:

$$L_a = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}, \ L_b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ R_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ R_b = \begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix}.$$

Hence

$$[L_a, L_b] = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \ [R_a, R_b] = \begin{pmatrix} 0 & 0 \\ -\beta & 0 \end{pmatrix}$$

and

$$[R_a, R_b]([L_a, L_b](b)) = -(\alpha\beta)b.$$

For $n = 4m + 1$ we construct the polynomial

$$g_m(x_1, x_2, x_3) = ([R_{x_1}, R_{x_2}][L_{x_1}, L_{x_2}])^m(x_3).$$

In our notation it can be written as

$$g_m(x_1, x_2, x_3) = \cdots (((\bar{x}_1(\bar{x}_2 x_3))\tilde{x}_2)\tilde{x}_1 \cdots .$$

Since $g_m(a, b, b) = (-\alpha\beta)^m b \neq 0$, the complete linearization of $g_m(x_1, x_2, x_3)$ generates an irreducible module not contained in $\mathrm{Id}(A)$. This module corresponds to the partition $\lambda = (2m + 1, 2m)$. In order to consider the other cases when $n = 4m + 2, 4m + 3, 4m + 4$, we need only to multiply on the left the polynomial g_m by L_{x_2}, $[L_{x_1}, L_{x_2}]$, $R_{x_2}[L_{x_1}, L_{x_2}]$, respectively. By Lemma 2.2 and Lemma 2.1 we obtain the desired conclusion, and the proof of the theorem is complete. □

The purpose of the following proposition is twofold. We construct a family of two-dimensional algebras $\{A_\alpha\}_{\alpha\in\mathbb{Q}}$ whose sequence of codimensions is $c_n(A_\alpha) = n + 1$, $n \geq 1$. This will show in particular that in the previous theorem the inequality $c_n(A) \leq n + 1$ cannot be improved. On the other hand we prove that any two distinct algebras of the family have distinct ideals of identities. This indicates that in the case of two-dimensional algebras, the sequence of cocharacters does not carry enough information of the corresponding T-ideals.

Let α be a rational number such that $\alpha > 0$ and let A_α be the two-dimensional algebra with basis $\{a, b\}$ and multiplication table

$$a^2 = 0, \ ab = a, \ ba = 0, \ b^2 = a + \alpha b.$$

It is clear that Fa is a two-sided ideal of A_α with trivial multiplication.

Proposition 3.1. *Let A_α be the two-dimensional algebra described above. Then $c_n(A_\alpha) = n + 1$ and $Id(A_\alpha) \neq Id(A_\beta)$, if $\alpha \neq \beta$.*

Proof. By Lemma 3.1 the n-th cocharacter of A_α satisfies

$$\chi_n(A) = p \cdot \chi_{(n)} + q \cdot \chi_{(n-1,1)}, \tag{5}$$

where $p \leq 2$ and $q \leq 1$. Hence $c_n(A_\alpha) = p \deg \chi_{(n)} + q \deg \chi_{(n-1,1)} = p + q(n-1)$, and, in order to prove the first statement of the proposition, we need to show that $p = 2$ and $q = 1$.

Consider the partition $\lambda = (n) \vdash n$ and the two polynomials of degree n in the variable x: $l_n(x) = (\cdots((xx)x)\cdots x)$, the left-normed product of x with itself n-times, and $r_n(x) = (x \cdots (x(xx))\cdots)$, the right-normed product of x with itself n-times. It is easily checked that

$$l_n(b) = (1 + \alpha + \alpha^2 + \cdots + \alpha^{n-2})a + \alpha^{n-1}b \quad \text{and} \quad r_n(b) = \alpha^{n-2}a + \alpha^{n-1}b.$$

It follows that the algebra A does not satisfy the identity $\gamma_1 l_n(x) + \gamma_2 r_n(x) \equiv 0$, for any scalars $\gamma_1, \gamma_2 \in F$. It is clear that the linearizations of $l_n(x)$ and $r_n(x)$ generate two irreducible modules corresponding to the partition (n), and they are linearly independent mod. $\mathrm{Id}(A)$. This proves that in (5), $p = 2$.

Consider now the polynomial

$$t_n(x_1, x_2) = (\cdots((\bar{x}_1\bar{x}_2)x_1)\cdots x_1).$$

It is easily checked that $t_n(b, a) = -a \neq 0$. Also the complete linearization of $t_n(x_1, x_2)$ generates an irreducible S_n-module corresponding to the partition $(n-1, 1)$, not contained in $\mathrm{Id}(A_\alpha)$. Hence in (5) we must have $q = 1$, and the first part of the proposition is proved.

Now let $\alpha, \beta \in \mathbb{Q}$ and consider the polynomial

$$f_\alpha(x_1, x_2, x_3) = \alpha \sum_{\sigma \in S_2} (-1)^\sigma ((x_{\sigma(1)} x_{\sigma(2)}) x_3) - \sum_{\sigma \in S_2} (-1)^\sigma (x_{\sigma(1)} (x_3 x_{\sigma(2)})).$$

By making the substitution $x_1 = b$, $x_2 = a$ and $x_3 = b$ it is easily checked that $f_\alpha(x_1, x_2, x_3)$ is an identity of A_β if and only if $\beta = \alpha$. Hence if $\alpha \neq \beta$ we have $\mathrm{Id}(A_\alpha) \neq \mathrm{Id}(A_\beta)$. □

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