PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 11, November 2007, Pages 3547–3552
S 0002-9939(07)08859-4
Article electronically published on July 3, 2007

ON THE ALGEBRAIC CLOSURE IN RINGS

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(Communicated by Joseph A. Ball)

ABSTRACT. The "algebraic closure" of a subset $K \subseteq A$ of a ring is an algebraic analogue of topological closure.

Suppose A is a ring, or additive category, with identity 1 and invertible group A^{-1}. Then we make the following

Definition 1. The algebraic closure of a subset $K \subseteq A$ is the set

$$\text{(1.1)}\quad \mathrm{cl}_{\mathrm{alg}}(K) \equiv \mathrm{cl}^A_{\mathrm{alg}}(K) = \{a \in A : \forall\, b, c \in A\ \exists\, a' \in K\ ,\ 1 - b(a - a')c \in A^{-1}\}\ .$$

Equivalently

$$\text{(1.2)}\qquad a \in \mathrm{cl}_{\mathrm{alg}}(K) \Longleftrightarrow \forall\, b \in A\ \exists a' \in K\ , 1 - b(a - a') \in A^{-1}$$

for if this holds, then for arbitrary $b, c \in A$ there is $a' \in A$ with

$$1 - cb(a - a') \in A^{-1} \Longleftrightarrow 1 - b(a - a')c \in A^{-1}\ .$$

For example if A is a Banach algebra, then the norm closure is a subset of the algebraic closure: using the geometric series [10]

$$\text{(1.3)}\qquad \|z\|\|a - a'\|\|w\| < 1 \Longrightarrow 1 - z(a - a')w \in A^{-1}\ .$$

The reverse inclusion is not clear: for example

$$\text{(1.4)}\qquad \mathrm{cl}_{\mathrm{alg}}\{a\} = a + \mathrm{Rad}(A),$$

the algebraic closure of a singleton is the coset modulo the Jacobson radical.

The algebraic closure has almost all of the properties of topological closure, and is compatible with the algebraic operations:

Theorem 2. *For arbitrary $K, H \subseteq A$ there is inclusion*

$$\text{(2.1)}\qquad K \subseteq \mathrm{cl}_{\mathrm{alg}}(K)\ ,$$

implication

$$\text{(2.2)}\qquad K \subseteq H \Longrightarrow \mathrm{cl}_{\mathrm{alg}}(K) \subseteq \mathrm{cl}_{\mathrm{alg}}(H),$$

and inclusion (necessarily equality)

$$\text{(2.3)}\qquad \mathrm{cl}_{\mathrm{alg}}\mathrm{cl}_{\mathrm{alg}}(K) \subseteq \mathrm{cl}_{\mathrm{alg}}(K)\ .$$

Received by the editors September 28, 2005 and, in revised form, July 10, 2006.
2000 *Mathematics Subject Classification.* Primary 46L05.
Key words and phrases. ring, algebraic closure, invertibility.
The first author was supported by Grant no. 144003 of Ministry of Science, Republic of Serbia.

There is also inclusion

$$\mathrm{cl_{alg}}(K) + \mathrm{cl_{alg}}(H) \subseteq \mathrm{cl_{alg}}(K + H) \tag{2.4}$$

and

$$\mathrm{cl_{alg}}(K) \cdot \mathrm{cl_{alg}}(H) \subseteq \mathrm{cl_{alg}}(K \cdot H)\ . \tag{2.5}$$

Proof. (2.1) and (2.2) are clear; towards (2.3), if $x \in \mathrm{cl_{alg}}\mathrm{cl_{alg}}(K)$, then for arbitrary $z, w \in A$ there is $x' \in \mathrm{cl_{alg}}(K)$ for which $1 - z(x - x')w = c \in A^{-1}$, and then $x'' \in K$ for which $1 - c^{-1}z(x' - x'')w \in A^{-1}$. Now

$$1 - z(x - x'')w = 1 - z(x - x')w - z(x' - x'')w = c(1 - c^{-1}z(x' - x'')w) \in A^{-1}\ .$$

Towards (2.4) and (2.5), if $x \in \mathrm{cl_{alg}}(K)$ and $y \in \mathrm{cl_{alg}}(H)$ and if $z, w \in A$, then, with $1 - z(x - x')w = c \in A^{-1}$,

$$\begin{aligned} 1 - z(x + y - x' - y')w &= 1 - z(x - x')w - z(y - y')w \\ &= (1 - z(x - x')w)(1 - c^{-1}z(y - y')w) \in A^{-1}A^{-1} = A^{-1} \end{aligned}$$

provided $1 - c^{-1}z(y - y')w \in A^{-1}$ and, with $1 - z(x - x')yw = d \in A^{-1}$,

$$\begin{aligned} 1 - z(xy - x'y')w &= 1 - z(x - x')yw - zx'(y - y')w \\ &= (1 - z(x - x')yw)(1 - d^{-1}zx'(y - y')w) \in A^{-1} \end{aligned}$$

provided $1 - d^{-1}zx'(y - y')w \in A^{-1}$. □

One of the Kuratowski closure axioms however seems to fail:

Example 3. For $K, H \subseteq A$, inclusion

$$\mathrm{cl_{alg}}(K \cup H) \subseteq \mathrm{cl_{alg}}(K) \cup \mathrm{cl_{alg}}(H) \tag{3.1}$$

is liable to fail.

Proof. In the ring $A = \mathbf{C}^2$ take

$$K = \{1\} \times \mathbf{C}\ ,\ \ H = \{-1\} \times \mathbf{C} \tag{3.2}$$

and claim

$$(2, 3) \in \mathrm{cl_{alg}}(K \cup H) \setminus \big(\mathrm{cl_{alg}}K \cup \mathrm{cl_{alg}}H\big)\ . \tag{3.3}$$

To see that $(2, 3) \notin \mathrm{cl_{alg}}K$ notice

$$(1, 1) - (1, 1)((2, 3) - (1, t)) = (0, t - 2) \notin A^{-1}\ .$$

To see that $(2, 3) \notin \mathrm{cl_{alg}}H$ notice

$$(1, 1) - (1/3, 1)((2, 3) - (-1, t)) = (0, t - 2) \notin A^{-1}\ .$$

Finally to see that $(2, 3) \in \mathrm{cl_{alg}}(K \cup H)$ suppose $(r, s) \in A$ is arbitrary. If $r \neq 1$, then for arbitrary $t \in \mathbf{C}$

$$(1, 1) - (r, s)((2, 3) - (1, t)) = (1 - r, 1 - s(t - 3)) \in A^{-1} \text{ if } 1 - s(3 - t) \neq 0\ .$$

If $r = 1$, then for arbitrary $t \in \mathbf{C}$

$$(1, 1) - (r, s)((2, 3) - (-1, t)) = (-2, 1 - s(3 - t)) \in A^{-1} \text{ if } 1 - s(3 - t) \neq 0\ .$$

□

The algebraic closure of the invertibles participates in a curious minuet with the left and the right invertibles:

Theorem 4. *There is equality*

$$A^{-1}_{\text{left}} \cap \text{cl}_{\text{alg}} A^{-1} = A^{-1} = A^{-1}_{\text{right}} \cap \text{cl}_{\text{alg}} A^{-1} \ , \tag{4.1}$$

and inclusion

$$\text{cl}_{\text{alg}} A^{-1} \subseteq A^{-1} + A^{-1} + A^{-1} \ . \tag{4.2}$$

Necessary and sufficient for

$$0 \in \text{cl}_{\text{alg}} A^{-1} \tag{4.3}$$

is inclusion

$$A \subseteq A^{-1} + A^{-1} \ . \tag{4.4}$$

Proof. Towards (4.1) we prove more, and claim

$$A^{-1}_{\text{left}} \cap \text{cl}_{\text{alg}} A^{-1}_{\text{right}} = A^{-1} = A^{-1}_{\text{right}} \cap \text{cl}_{\text{alg}} A^{-1}_{\text{left}} \ . \tag{4.5}$$

Towards the first equality suppose $a \in A^{-1}_{\text{left}} \cap \text{cl}_{\text{alg}} A^{-1}_{\text{right}}$ so that there are $a', b, b' \in A$ for which $a'a = 1 = bb'$ with $1 - a'(a - b) \in A^{-1}$. But this means $a'a = 1$ with $a'b \in A^{-1}$; hence, $b \in A^{-1}_{\text{left}}$ and so $b \in A^{-1}$. The argument for the second equality in (4.5) is the same.

Towards (4.2) suppose $a \in \text{cl}_{\text{alg}} A^{-1}$. Then for arbitrary $b \in A^{-1}$ there is $c \in A^{-1}$ for which

$$1 - b(a - c) = d \in A^{-1} \Longrightarrow a = b^{-1}(1 - d) + c \in A^{-1} + A^{-1} + A^{-1} \ .$$

Finally if (4.3) holds, then for arbitrary $b \in A$ there is $c \in A^{-1}$ for which $1 - b(-c) = d \in A^{-1}$, giving $b = (d - 1)c^{-1} \in A^{-1} + A^{-1}$, as in (4.4). Conversely for arbitrary $c, d \in A^{-1}$ we have

$$1 - (c + d)d^{-1} = -cd^{-1} \in A^{-1} \ ;$$

in the presence of (4.4) this is (4.3) □

The analogue of (4.1) for the topological closure is familiar in Banach algebras [8], [10]. From (4.1) it follows that everything in the algebraic closure of the invertibles is "consistent in regularity" [6], [7]:

$$\begin{aligned} \text{cl}_{\text{alg}} A^{-1} &\subseteq A^{-1} \cup \big(A \setminus \big(A^{-1}_{\text{left}} \cap A^{-1}_{\text{right}}\big)\big) \\ &= \{a \in A : ax \in A^{-1} \Longleftrightarrow xa \in A^{-1}\} \ . \end{aligned} \tag{4.6}$$

The last part of Theorem 4 generalizes to radical elements and to idempotents:

Theorem 5. *If*

$$1 + qA^{-1} \subseteq A^{-1}, \tag{5.1}$$

then the condition (4.3) *is necessary and sufficient for*

$$q \in \text{cl}_{\text{alg}} A^{-1} \ . \tag{5.2}$$

If instead $p = p^2 \in A$ is idempotent, then

$$0 \in \text{cl}^{A}_{\text{alg}} (pAp)^{-1} \Longleftrightarrow 0 \in \text{cl}^{pAp}_{\text{alg}} (pAp)^{-1} \Longrightarrow 1 - p \in \text{cl}_{\text{alg}} A^{-1} \ . \tag{5.3}$$

Proof. If $q \in A$ is in $\mathrm{cl_{alg}} A^{-1}$, then for arbitrary $b \in A$ there is $c \in A^{-1}$ for which $1 - b(q-c) = d \in A^{-1}$ and hence

$$be = dc^{-1} - c^{-1} \text{ with } e = 1 - qc^{-1} \ .$$

Now if (5.1) holds, then $e \in A^{-1}$ giving $b = (d-1)(ec)^{-1} \in A^{-1} + A^{-1}$, which is (4.3). Conversely for arbitrary $c, d \in A^{-1}$

$$1 - (c+d)(q + c^{-1}) = -(dc^{-1} + cq + dq) \in A^{-1} + A^{-1}q + A^{-1}q \ ,$$

which is included in A^{-1} by (5.1). In the presence of (4.3) this applies to arbitrary $b = c + d \in A$. Towards (5.3), if 0 is the algebraic closure in pAp of the invertible group $\big(pAp\big)^{-1}$, then for arbitrary $b \in A$ there is $c = pcp \in (pAp)^{-1}$ for which $p(1+bc) \in (pAp)^{-1}$, giving

$$1 + pbc = 1 - p + p(1+bc) \in A^{-1} \ ,$$

which says $0 \in \mathrm{cl}^A_{\mathrm{alg}}(pAp)^{-1}$. Conversely if this holds, then for arbitrary $b \in A$, in particular $pbp \in pAp$, there is $c = cpc \in (pAp)^{-1}$ and $d \in A$ for which $(1+pbc)d = 1 = d(1+pbc)$, giving

$$(p + pbc)pdp = p = pd(p + pbc)$$

and hence $p(1+bc) \in (pAp)^{-1}$. Also if for arbitrary $b \in A$ there is $c, d \in pAp$ with $1 - bc \in A^{-1}$ and $cd = p = dc$, then $a = 1 - p - c \in A^{-1}$ with $a^{-1} = 1 - p - d$ giving $1 - ba = 1 - bc \in A^{-1}$. □

The algebraic closure intervenes in generalized inverse theory:

Lemma 6. *With*

$$\text{(6.1)} \qquad \hat{A} = \{a \in A : a \in aAa\} \text{ and } \check{A} = \{a \in A : a \in aA^{-1}a\}$$

there is inclusion

$$\text{(6.2)} \qquad \hat{A}_\cap \mathrm{cl_{alg}} A^{-1} \subseteq \check{A} \ .$$

Necessary and sufficient for equality in (6.2) *is that*

$$\text{(6.3)} \qquad A^\bullet \equiv \{p \in A : p = p^2\} \subseteq \mathrm{cl_{alg}} A^{-1} \ .$$

Proof. If $a = aa'a \in \hat{A}$, so that $a'a = p = p^2$ is idempotent, and if

$$b \in A^{-1} \text{ with } 1 + (b-a)a' = c^{-1} \in A^{-1} \ ,$$

then

$$a = (cb)p \in \check{A} \ .$$

For equality in (6.2), observe [9], [10]

$$\check{A} = A^{-1}A^\bullet = A^\bullet A^{-1}.$$

□

In Banach algebras (6.3) is clear [9], [10]: generally

$$\text{(6.4)} \qquad 0 \notin \mathrm{int}\ \sigma(a) \Longrightarrow a \in \mathrm{cl_{alg}} A^{-1} \ ,$$

where $\sigma(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}\}$ is the usual spectrum.

We can extend the algebraic closure to tuples:

Definition 7. For arbitrary $K \subseteq A^n$

$$\text{(7.1)} \quad \mathrm{cl_{alg}} K = \{x \in A^n : \forall\ z, w \in A^n\ \exists\ x' \in K\ ,\ 1 - \sum_{j=1}^n z_j(x_j - x'_j)w_j \in A^{-1}\} \ .$$

We can also define left and right invertible tuples:

$$A^{-n}_{\text{left}} = \{a \in A^n : 1 \in \sum_{j=1}^{n} A a_j\} \text{ and } A^{-n}_{\text{right}} = \{a \in A^n : 1 \in \sum_{j=1}^{n} a_j A\} \ . \tag{7.2}$$

If we interpret A as an additive category, then in a sense we have already dealt with n-tuples, and more generally $n \times m$ matrices, over A, which just form another additive category $B = \text{Matrix}(A)$; thus most of Theorem 2 extends to n-tuples. For example the analogues of (2.1) and (2.2) for $K, H \subseteq A^n$ are immediate, while for the analogue of (2.3) we argue that if $x \in \text{cl}_{\text{alg}}\text{cl}_{\text{alg}}(K)$, then for arbitrary $z, w \in A^n$ there is $x' \in \text{cl}_{\text{alg}}(K)$ for which $1 - \sum_j z_j(x_j - x'_j)w_j = c \in A^{-1}$, and then $x'' \in K$ for which $1 - c^{-1}\sum_j z_j(x'_j - x''_j)w_j \in A^{-1}$. Now

$$\begin{aligned} 1 - \sum_j z_j(x_j - x''_j)w &= 1 - \sum_j (z_j(x_j - x'_j)w_j - z_j(x'_j - x''_j)w_j) \\ &= c(1 - c^{-1}\sum_j z_j(x'_j - x''_j)w_j) \in A^{-1} \ . \end{aligned}$$

Towards the extension of (2.4), if $x \in \text{cl}_{\text{alg}}(K)$ and $y \in \text{cl}_{\text{alg}}(H)$ and if $z, w \in A^n$ then, with $1 - \sum_j z_j(x_j - x'_j)w_j = c \in A^{-1}$,

$$\begin{aligned} 1- &\sum_j z_j(x_j + y_j - x'_j - y'_j)w_j \\ &= (\sum_j (1 - z_j(x_j - x'_j)w_j)(1 - c^{-1}\sum_j z_j(y_j - y'_j)w_j) \\ &\in A^{-1}A^{-1} = A^{-1} \text{ provided } 1 - c^{-1}\sum_j z_j(y_j - y'_j)w_j \in A^{-1} \ . \end{aligned}$$

Similarly, for an extended version of (2.5), with $1 - \sum_j z_j(x_j - x'_j)y_j w_j = d \in A^{-1}$,

$$\begin{aligned} 1- &\sum_j z_j(x_j y_j - x'_j y'_j)w_j \\ &= (1 - \sum_j z_j(x_j - x'_j)y_j w_j(1 - d^{-1}\sum_j z_j x'_j(y_j - y'_j)w_j) \\ &\in A^{-1} \text{ provided } 1 - d^{-1}\sum_j z_j x'_j(y_j - y'_j)w_j \in A^{-1} \ . \end{aligned}$$

(2.5) would also extend, in a category A, to K, H and a more general bilinear image $K * H$.

Declare that a ring A *has left stable range* $\leq n$ provided

$$\forall\ (a, b) \in A^n \times A\ ,\ (a, b) \in A^{-n-1}_{\text{left}} \Longrightarrow \exists\ c \in A^n\ ,\ a - cb \in A^{-n}_{\text{left}} \ . \tag{7.3}$$

Corach and Suarez [4], [5], and Blackadar [3], have considered this kind of situation when A is commutative, or a C^*-algebra.

Notice how the definition prefers the final element of an $n+1$ tuple; an alternative would say that if $d \in A^{n+1}$ was "left invertible", then there would exist an index $1 \leq j \leq n+1$ for which an analogue of (7.3) held. In commutative Banach algebras the condition that A has stable range ≤ 1 reduces to the topological closure of the invertible group being the whole of A. We offer here a curious hybrid result:

Theorem 8. *If*

$$A \subseteq \mathrm{cl}_{\mathrm{alg}} A^{-1}_{\mathrm{left}}, \tag{8.1}$$

then there is implication

$$(a, b) \in A^{-2}_{\mathrm{left}} \Longrightarrow (a - Ab)_{\cap} A^{-1}_{\mathrm{right}} \neq \emptyset\ . \tag{8.2}$$

Proof. Suppose

$$a'a + b'b = 1 \text{ with } a' \in \mathrm{cl}_{\mathrm{alg}} A^{-1}_{\mathrm{left}}\ ,$$

so that there are a'' , a''' in A for which

$$b'b = 1 - a'a = d - a''a \text{ with } d \in A^{-1} \text{ and } a'''a'' = 1$$

giving

$$a - cb = a'''d \in A^{-1}_{\mathrm{right}} \text{ with } c = -a'''b'\ .$$

□

When $b = 0$ this gives back (4.1). The same argument gives back a weaker version of (7.3): if $(a, b) \in A \times A^n$, then if (8.1) holds there is implication

$$(a, b) \in A^{-n-1}_{\mathrm{left}} \Longrightarrow (a - \sum_j Ab_j)_{\cap} A^{-1}_{\mathrm{right}} \neq \emptyset\ . \tag{8.3}$$

We remark that the condition (8.2) says that the element $a \in A$ is in another kind [1], [2] of "closure" of the semigroup A^{-1}_{right} : for the moment we find the precise relationship between this condition and ours elusive.

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