

VOLUME ENTROPY AND INTEGRAL RICCI CURVATURES OVER CLOSED GEODESICS

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ABSTRACT. We obtain an upper bound of the volume entropy and the simplicial volume with integrals of Ricci curvature over closed geodesics and apply it to the real Schwarz lemma by Besson, Courtois and Gallot.

1. INTRODUCTION

Let M be an n -dimensional compact Riemannian manifold. It is an interesting problem *to what extent the geometry and the topology of a closed manifold are determined by some integrals over all closed geodesics*. For example, the marked length spectrum MLS_g for a Riemannian metric g is a function which maps a free homotopy class $\langle \gamma \rangle$ of the fundamental group to the length of the shortest geodesic curve in $\langle \gamma \rangle$. In [CD], Croke and Dairbekov showed that if $\text{MLS}_{g_1} \geq \text{MLS}_{g_0}$, then the area $A(g_i)$ for g_i satisfies that $A(g_1) \geq A(g_0)$ in the case of $n = 2$. If $A(g_1) = A(g_0)$, then g_1 is isometric to g_0 . For $n > 2$, if $g_1 = f^2(x)g_0$ for some smooth positive function f , similar results are obtained. In [DS], Dairbekov and Sharafutdinov showed that if M is an Anosov manifold and a smooth function f on M integrates to zero over every closed geodesic, then f must be zero. Furthermore, if M is an Anosov manifold and a smooth 1-form integrates to zero around every closed geodesic, then f is an exact form.

In this paper, we obtain an upper bound of the volume entropy and the simplicial volume with integrals of Ricci curvature over closed geodesics instead of a pointwise curvature bound or an integral curvature on the whole space and then apply it to the real Schwarz lemma by Besson, Courtois and Gallot in [BCG1], [BCG2], [BCG3]. The volume entropy $h(M)$ of M is defined as follows:

$$h(M) = \lim_{R \rightarrow \infty} \frac{\log(\text{vol}(B(x, R)))}{R},$$

where $B(x, R)$ is the R -ball centered at x in \tilde{M} . If the Ricci curvature is bounded below, we obtain an upper bound of the volume entropy by the Bishop-Gromov volume comparison theorem.

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We assume that $K_M < 0$ and obtain an upper bound of the entropy depending on integrals of Ricci curvature over closed geodesics. First, we introduce the following notation for the integral of Ricci curvature over a geodesics: Let $g(x)$ be the smallest eigenvalue of the Ricci tensor at $x \in M$ and let $u_+ = \max(0, u)$ be the positive part of u . We denote the function $((-g(x) + (n - 1)\lambda)_+)$ by $\rho(x)$. We can consider the function ρ on M as a function on the unit tangent bundle UM , i.e. $\rho(v_x) = \rho(x)$. Let ξ_t be the geodesic flow of M . If $K_M < 0$, then the geodesic flow on UM is ergodic. We define a function $\bar{\rho}_+$ as follows:

$$(1.1) \quad \bar{\rho}_+(w_y) = \lim_{R \rightarrow \infty} \frac{1}{R} \int_0^R \rho(\xi_t(w_y))^p dt.$$

By the Birkhoff ergodic theorem [KH], we have

$$(1.2) \quad \bar{\rho}_+(w_y) = \int_{v_x \in UM} \rho(v_x)^p d\mu = \int_{S^{n-1}} \int_M \rho(x)^p dv d\theta$$

for almost every $w_y \in UM$, where μ is the Liouville measure which is invariant with respect to the geodesic flow [C]. Hence $\bar{\rho}_+$ is constant almost everywhere. However $\bar{\rho}_+$ is not only nonconstant but also discontinuous in general.

If w_y and v_x in UM are contained in the same orbit of the geodesic flow, i.e. w_y and v_x are velocity vectors of the same geodesic, then $\bar{\rho}_+(w_y) = \bar{\rho}_+(v_x)$. Hence $\bar{\rho}_+$ can be considered as a function from the set of all closed geodesics Γ to \mathbb{R} . Then

$$\bar{\rho}_+(\gamma) = \frac{1}{\text{Length}(\gamma)} \int_{\gamma} \rho^p.$$

We define $k_{\Gamma}(\lambda, p)$ as follows:

$$k_{\Gamma}(\lambda, p) = \sup\{\bar{\rho}_+(\gamma) \mid \gamma \in \Gamma\}.$$

If $\text{Ric}_M \geq (n - 1)\lambda$, then $k_{\Gamma}(\lambda, p) = 0$.

Our main theorem is as follows.

Theorem 1.1. *Let M be an n -dimensional compact Riemannian manifold with the sectional curvature $K_M < 0$. For $p > \frac{1}{2}$, the volume entropy of M satisfies that*

$$h(M) \leq (n - 1)\sqrt{|\lambda|} + \sqrt{n - 1}k_{\Gamma}(\lambda, p)^{\frac{1}{2p}}.$$

We don't need to assume $2p > n$ unlike as in [PW1]. From Theorem 1.1, we obtain an upper bound on the simplicial volume. Gromov proved the following inequality between the volume entropy and the simplicial volume $\|M\|$ of M ([G2], [BCG2]):

$$h(M)^n C_n n! \text{vol}(M) \geq \|M\|,$$

so it was obtained that if $\text{Ric}_M \geq -1/(n - 1)$, then

$$\|M\| \leq C_n n! \text{vol}(M),$$

where $C_n = \Gamma(\frac{n}{2})/(\sqrt{\pi}\Gamma(\frac{n+1}{2}))$. As an immediate consequence of [G2], [BCG2], we have the following corollary:

Corollary 1.2. *Let M be an n -dimensional compact Riemannian manifold with $K_M < 0$. The simplicial volume $\|M\|$ of M satisfies that*

$$\|M\| \leq C_n n! \left((n - 1)\sqrt{|\lambda|} + \sqrt{n - 1}k_{\Gamma}(\lambda, p)^{\frac{1}{2p}} \right)^n \text{vol}(M).$$

From the real Schwarz lemma [BCG3], we obtain the following corollary:

Corollary 1.3. *Let M and N be n -dimensional compact negatively curved manifolds for $n \geq 3$.*

(a) *If $K_N \leq \lambda < 0$, then any continuous map $f : M \rightarrow N$ can be deformed to a smooth map F such that*

$$\text{vol}(F(A)) \leq \left(1 + \frac{k_\Gamma(\lambda, p)^{\frac{1}{2p}}}{\sqrt{(n-1)|\lambda|}}\right)^n \text{vol}(A).$$

(b) *If $\text{vol}(M) \leq \text{vol}(N)$ and*

$$K_N < -\left(\sqrt{|\lambda|} + \frac{k_\Gamma(\lambda, p)^{\frac{1}{2p}}}{\sqrt{n-1}}\right)^2,$$

then there exists no continuous map from M to N of nonzero degree.

In Section 2, we review the real Schwarz lemma in [BCG3]. In Section 3, we will prove Theorem 1.1 and Corollary 1.3.

2. REAL SCHWARZ LEMMA

We review the real Schwarz lemma [BCG3].

Theorem 2.1. *Let M, N be real complete Riemannian manifolds such that $3 \leq \dim(M) \leq \dim(N)$. Let us assume that there exists some constant $C \neq 0$ such that $K_N \leq -C^2$ and $\text{Ric}_M \geq -(n-1)C^2$. Then any continuous map $f : M \rightarrow N$ may be deformed to a family of C^1 maps F_δ for $\delta \rightarrow 0$ such that $\text{vol}(F_\delta(A)) \leq (1+\delta)\text{vol}(A)$ for any measurable set A in M . Moreover, assume that M and N are compact.*

(a) *If $\text{vol}(M) = |\text{deg}f|\text{vol}(N)$, then M and N have constant sectional curvature $-C^2$ and we may choose F_δ 's such that they converge to a Riemannian covering F as $\delta \rightarrow 0$.*

(b) *If M, N are homotopically equivalent and $K_M < 0$, then any homotopy equivalence f may be deformed to a smooth map F such that $\text{vol}(F(A)) \leq \text{vol}(A)$ for any open subset A in M , the equality being attained if and only if F is an isometry on A .*

Immediately, we obtain the following corollary:

Corollary 2.2. *Let M, N be n -dimensional compact Riemannian manifolds for $n \geq 3$. Let $K_N \leq -C^2$ and $\text{Ric}_M \geq -(n-1)\Lambda^2$ for $\Lambda < C$. If $\text{vol}(N) \geq \text{vol}(M)$, then there exists no continuous map from M to N of nonzero degree.*

Proof. Assume that $K_N \leq -C^2$, $\text{Ric}_M \geq -(n-1)\Lambda^2$ for $\Lambda < C$ and $\text{vol}(N) \geq \text{vol}(M)$. Rescaling the metric of M such that $\text{Ric}_M \geq -(n-1)C^2$, then $\text{vol}(M) < \text{vol}(N)$. From Theorem 2.1, if there is a continuous map from M to N of nonzero degree, then $\text{vol}(N) \leq \text{vol}(M)$, which is a contradiction. Hence there is no continuous map from M to N of nonzero degree. □

3. PROOF OF MAIN THEOREM

We will follow the notation in [PW1]. Around $x \in M$, we use polar coordinates and write the volume element as $d\text{vol} = \omega dt \wedge d\theta_{n-1}$, where $d\theta_{n-1}$ is the standard volume element on the unit sphere S^{n-1} . We define ω to be 0 beyond the cut locus. Then we have

$$(3.1) \quad \omega' = h\omega,$$

where h is the mean curvature of the distance spheres around x , and h satisfies that

$$(3.2) \quad h' + \frac{h^2}{n-1} \leq -\text{Ric}(\partial_t, \partial_t),$$

where ∂_t is the unit gradient of the distance function.

In the n -dimensional space form S_λ^n of constant curvature λ , we obtain ω_λ and h_λ for S_λ similarly as ω and h , respectively. Then

$$(3.3) \quad h'_\lambda + \frac{h_\lambda^2}{n-1} = -(n-1)\lambda.$$

Let $\psi = \psi(t, \cdot) = \max\{0, h(t, \cdot) - h_\lambda(t, \cdot)\}$. Since $\frac{\omega'}{\omega} = h$, integrating this equation, we obtain that

$$(3.4) \quad \log\left(\frac{\omega(R)}{\omega(r)}\right) = \int_r^R h dt \leq \int_r^R h_\lambda dt + \int_0^R \psi dt.$$

Then we have

$$(3.5) \quad \omega(R) \leq e^{\int_0^R \psi dt} e^{\int_r^R h_\lambda dt} \omega(r).$$

As $r \rightarrow 0$, $\omega_\lambda(r)/\omega(r) \rightarrow 1$, so we have

$$(3.6) \quad \lim_{r \rightarrow 0} e^{\int_r^R h_\lambda dt} \omega(r) = \omega_\lambda(R).$$

Then for $\lambda < 0$,

$$(3.7) \quad \omega(R) \leq e^{\int_0^R \psi dt} \omega_\lambda(R) \leq A_n e^{(n-1)\sqrt{|\lambda|R} + \int_0^R \psi dt}$$

for a constant A_n and for $\lambda = 0$,

$$(3.8) \quad \omega(R) \leq e^{\int_0^R \psi dt} \omega_\lambda(R) \leq \frac{\omega_{n-1}}{n} R^n e^{\int_0^R \psi dt},$$

where ω_{n-1} is the volume of a standard $(n-1)$ -sphere.

We will prove the following lemma similarly as Lemma 2.2 in [PW1]:

Lemma 3.1. *For any $p > \frac{1}{2}$, we have*

$$\int_0^r \psi^{2p} dt \leq (n-1)^p \int_0^r \rho^p dt.$$

Proof. We have

$$(3.9) \quad \psi' + \frac{\psi^2}{n-1} + \frac{2\psi h_\lambda}{n-1} \leq \rho.$$

Multiply by ψ^{2p-2} and integrate to get

$$(3.10) \quad \int_0^r \psi' \psi^{2p-2} dt + \frac{1}{n-1} \int_0^r \psi^{2p} dt + \frac{2}{n-1} \int_0^r h_\lambda \psi^{2p-1} dt \leq \int_0^r \rho \psi^{2p-2} dt.$$

Since $(\psi^{2p-1})' = (2p-1)\psi' \psi^{2p-2}$,

$$(3.11) \quad \int_0^r \psi' \psi^{2p-2} dt = \frac{1}{2p-1} \psi^{2p-1} \Big|_0^r \geq 0.$$

Inserting this in (3.11), we obtain that

$$(3.12) \quad \frac{1}{n-1} \int_0^r \psi^{2p} dt + \frac{2}{n-1} \int_0^r h_\lambda \psi^{2p-1} dt \leq \int_0^r \rho \psi^{2p-2} dt,$$

which implies that

$$(3.13) \quad \begin{aligned} \frac{1}{n-1} \int_0^r \psi^{2p} dt &\leq \int_0^r \rho \psi^{2p-2} dt \\ &\leq \left(\int_0^r \rho^p dt \right)^{\frac{1}{p}} \left(\int_0^r \psi^{2p} dt \right)^{1-\frac{1}{p}}. \end{aligned}$$

Dividing by $(\int_0^r \psi^{2p} dt)^{1-\frac{1}{p}}$, then we obtain that

$$(3.14) \quad \frac{1}{n-1} \left(\int_0^r \psi^{2p} dt \right)^{\frac{1}{p}} \leq \left(\int_0^r \rho^p dt \right)^{\frac{1}{p}}.$$

Consequently, we have

$$(3.15) \quad \int_0^r \psi^{2p} dt \leq (n-1)^p \int_0^r \rho^p dt,$$

which completes the proof of Lemma 3.1. □

In the proof of Lemma 2.2 in [PW1], it is an essential condition that $p > n/2$, but our lemma holds for $p > \frac{1}{2}$.

If the convergence (1.1) is uniform with respect to R , then we obtain an upper bound of the volume of R -balls for sufficiently large R by (3.5) and Lemma 3.1. The sequence (1.1) does not converge everywhere but almost everywhere, so generally we cannot have the uniform convergence. But if we assume that $K_M < 0$, then we can prove the following lemmas.

Lemma 3.2. *For every $v_x \in UM$, we have*

$$\lim_{R \rightarrow \infty} \frac{1}{R} \int_0^R \rho(\xi_t(v_x))^p dt \leq k_\Gamma(\lambda, p).$$

Proof. For any $v_x \in UM$, let \tilde{v} and \tilde{x} be the lifting of v_x and x to \tilde{M} , respectively, and let γ_v be the geodesic ray from \tilde{x} such that $\gamma'_v(0) = \tilde{v}$. Let D_n be the fundamental domain in the universal covering space that contains $\gamma_v(n)$, and let $x(n)$ be the deck transformation of x in D_n . Let γ_n be the geodesic segment from x to $x(n)$. Then it is projected to a closed curve by the covering map Π . Let $\tilde{\gamma}_n$ be the shortest closed geodesic homotopic to $\Pi(\gamma_n)$. Then the tangent vectors of $\tilde{\gamma}_n$ make a closed orbit of the geodesic flow in UM . Let \tilde{x}_n be $\tilde{\gamma}_n(0)$ and $\tilde{\gamma}_n$ be the lifting of $\tilde{\gamma}_n$ to the universal covering space such that $\tilde{\gamma}_n(0) \in D_0$. There exists a geodesic segment α_n from $\tilde{x}_n = \tilde{\gamma}_n(0)$ to $\gamma_v(n)$. Since $\tilde{\gamma}_n$ is a closed geodesic, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n \rho(\xi_t(\tilde{\gamma}'_n(0)))^p dt = \lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n \rho(\tilde{\gamma}'_n(t))^p dt \leq k_\Gamma(\lambda, p)$$

from our assumption. We have $d(x(n), \tilde{\gamma}_n(n)) \leq 2D$ for the diameter D of M since $\tilde{\gamma}_n$ and $\Pi(\gamma_n)$ are homotopic. Since $K_M < 0$,

$$\int_0^n d(\alpha_n(t), \tilde{\gamma}_n(t)) dt \leq C_1(M)$$

for some constant $C_1(M)$ depending on M . (Note that the areas of any triangles are bounded if $K_M < 0$.) Also since ρ^p is a Lipschitz function on M , we obtain

that

$$\int_0^n |\rho(\alpha'_n(t))^p - \rho(\tilde{\gamma}'_n(t))^p| dt \leq C_2(M),$$

$$\int_0^n |\rho(\alpha'_n(t))^p - \rho(\gamma'_v(t))^p| dt \leq C_2(M),$$

where $C_2(M)$ is a constant depending on M . Then we have

$$\int_0^n |\rho(\gamma'_v(t))^p - \rho(\tilde{\gamma}'_n(t))^p| dt \leq 2C_2(M)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n \rho(\xi_t(v_x))^p dt &= \lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n \rho(\gamma'_v(t))^p dt \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n \rho(\tilde{\gamma}'_n(t))^p dt \\ &\leq k_\Gamma(\lambda, p), \end{aligned}$$

which completes the proof. \square

Lemma 3.3. *For any fixed $\epsilon > 0$, there exists $R > 0$ such that*

$$(3.16) \quad \frac{1}{R} \int_0^R \rho(\xi_t(v_x))^p dt < (1 + \epsilon)k_\Gamma(\lambda, p)$$

for every $v_x \in UM$.

Proof. Let

$$\mathcal{R}(r, v_x) = \frac{1}{r} \int_0^r \rho(\xi_t(v_x))^p dt.$$

Then $\mathcal{R}(r, v_x)$ is a continuous function for $r > 0$ and $v_x \in UM$. So the set

$$V = \{(r, v_x) \mid \mathcal{R}(r, v_x) < (1 + \epsilon)k_\Gamma(\lambda, p)\}$$

is an open set for a small $\epsilon > 0$. From Lemma 3.2, for every $v_x \in UM$, there exists $T > 0$ such that if $r \geq T$, then

$$\frac{1}{r} \int_0^r \rho(\xi_t(v_x))^p dt < (1 + \epsilon)k_\Gamma(\lambda, p).$$

Let

$$T(v_x) = \inf\{T > 0 \mid \frac{1}{r} \int_0^r \rho(\xi_t(v_x))^p dt < (1 + \epsilon)k_\Gamma(\lambda, p) \text{ for any } r > T\}.$$

Then the set

$$V_\infty = \{(r, v_x) \mid r > T(v_x)\}$$

is a component of V containing infinity. So V_∞ is also open. If $T(v_x)$ is not upper semi-continuous, then V_∞ is not open, so T is upper semi-continuous. Since UM is compact and T is upper semi-continuous, $T(UM)$ is bounded. So there exists R such that

$$(3.17) \quad \frac{1}{R} \int_0^R \rho(\xi_t(v_x))^p dt < (1 + \epsilon)k_\Gamma(\lambda, p)$$

for every $v_x \in UM$. \square

Proof of Theorem 1.1. By the Hölder inequality, we have

$$\begin{aligned}
 \frac{1}{R} \int_0^R \psi dt &\leq \frac{1}{R} \left(\int_0^R \psi^{2p} dt \right)^{\frac{1}{2p}} R^{\frac{1}{q}} \\
 (3.18) \qquad &\leq \frac{1}{R} ((n-1)^p \int_0^R \rho^p dt)^{\frac{1}{2p}} R^{\frac{1}{q}} \\
 &\leq \sqrt{n-1} \left(\frac{1}{R} \int_0^R \rho^p dt \right)^{\frac{1}{2p}}
 \end{aligned}$$

for $1/(2p) + 1/q = 1$. For a fixed $\epsilon > 0$, we have

$$\begin{aligned}
 \frac{1}{R} \int_0^R \psi dt &\leq \sqrt{n-1} \left(\frac{1}{R} \int_0^R \rho^p dt \right)^{\frac{1}{2p}} \\
 (3.19) \qquad &\leq \sqrt{n-1} \left((1+\epsilon) k_\Gamma(\lambda, p) \right)^{\frac{1}{2p}}
 \end{aligned}$$

from Lemma 3.3. For $\lambda < 0$, we obtain from (3.7) that

$$\begin{aligned}
 \int_{S^{n-1}} \int_0^R \omega dt d\theta_{n-1} &\leq \int_{S^{n-1}} \int_0^R A_n e^{(n-1)\sqrt{|\lambda|R} + \int_0^R \psi ds} dt d\theta_{n-1} \\
 (3.20) \qquad &= A_n R e^{(n-1)\sqrt{|\lambda|R}} \int_{S^{n-1}} e^{\int_0^R \psi ds} d\theta_{n-1} \\
 &\leq A_n R e^{(n-1)\sqrt{|\lambda|R}} e^{\sqrt{n-1}R((1+\epsilon)k_\Gamma(\lambda, p))\frac{1}{2p}} \omega_{n-1}
 \end{aligned}$$

for some constant A_n . Hence

$$\begin{aligned}
 \lim_{R \rightarrow \infty} \frac{1}{R} \log(\text{vol}(B(y, R))) &= \frac{1}{R} \log \left(\int_{S^{n-1}} \int_0^R \omega dt d\theta_{n-1} \right) \\
 (3.21) \qquad &\leq (n-1)\sqrt{|\lambda|} + \sqrt{n-1} \left((1+\epsilon) k_\Gamma(\lambda, p) \right)^{\frac{1}{2p}}.
 \end{aligned}$$

For $\lambda = 0$, by (3.8) we have

$$\begin{aligned}
 \int_{S^{n-1}} \int_0^R \omega dt d\theta_{n-1} &\leq \int_{S^{n-1}} \int_0^R \frac{\omega_{n-1}}{n} R^n e^{\int_0^R \psi ds} dt d\theta_{n-1} \\
 (3.22) \qquad &= R^{n+1} \frac{\omega_{n-1}}{n} \int_{S^{n-1}} e^{\int_0^R \psi ds} d\theta_{n-1} \\
 &\leq R^{n+1} \frac{\omega_{n-1}}{n} e^{\sqrt{n-1}R((1+\epsilon)k_\Gamma(\lambda, p))\frac{1}{2p}} \omega_{n-1}.
 \end{aligned}$$

Hence we have

$$\lim_{R \rightarrow \infty} \frac{1}{R} \log(\text{vol}(B(y, R))) \leq \sqrt{n-1} \left((1+\epsilon) k_\Gamma(\lambda, p) \right)^{\frac{1}{2p}}.$$

(3.23)

Since ϵ is arbitrary, we have from (3.21) and (3.23) that

$$\begin{aligned}
 h(M) &= \lim_{R \rightarrow \infty} \frac{1}{R} \log(\text{vol}(B(y, R))) \\
 (3.24) \qquad &\leq (n-1)\sqrt{|\lambda|} + \sqrt{n-1} k_\Gamma(\lambda, p)^{\frac{1}{2p}}
 \end{aligned}$$

for $\lambda \leq 0$, which completes the proof of Theorem 1.1. □

Proof of Corollary 1.3. For the proof of Corollary 1.3, we only need to apply the proof of the real Schwarz lemma [BCG3] to our case. Actually, the condition that $\text{Ric}_M \geq -(n-1)C^2$ is only used to bound the volume entropy of M by $(n-1)C$. By [BCG3], there is an equivariant smooth map $\tilde{F} : \tilde{M} \rightarrow \tilde{N}$ such that

$$|\det(\tilde{F}_*)| \leq \left(\frac{h(M)}{(n-1)\sqrt{|\lambda|}} \right)^n.$$

From Theorem 1.1, we have

$$|\det(\tilde{F}_*)| \leq \left(\frac{\sqrt{(n-1)|\lambda|} + k_\Gamma(\lambda, p)^{\frac{1}{2p}}}{\sqrt{(n-1)|\lambda|}} \right)^n,$$

which completes the proof of Corollary 1.3(a). With Corollary 2.2, we can obtain Corollary 1.3(b). Precisely, if there is a continuous map from M to N of nonzero degree, then there is an equivariant smooth map $\tilde{F} : \tilde{M} \rightarrow \tilde{N}$ such that

$$\begin{aligned} |\det(\tilde{F}_*)| &\leq \left(\frac{h(M)}{h(N)} \right)^n \\ &\leq \left(\frac{(n-1)\sqrt{|\lambda|} + \sqrt{n-1}k_\Gamma(\lambda, p)^{\frac{1}{2p}}}{h(N)} \right)^n < 1 \end{aligned}$$

since $K_N < -\left(\sqrt{|\lambda|} + \frac{k_\Gamma(\lambda, p)^{\frac{1}{2p}}}{\sqrt{n-1}}\right)^2$. Then we have $\text{vol}(N) < \text{vol}(M)$, which is a contradiction to $\text{vol}(M) \leq \text{vol}(N)$. \square

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